BOL. SOC. BRAS. MAT., VOL. 12 Nº 2 (1981), 83-94

On equivariant homotopy equivalences

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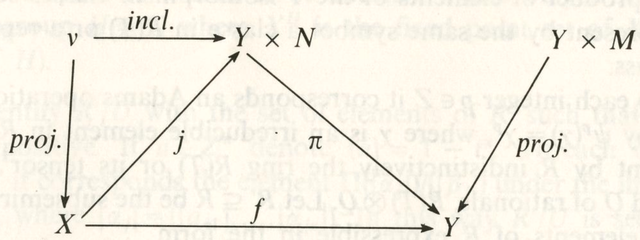
Let G be a compact Lie group. Ted Petrie set the following conjecture related to [7, pg. 20].

Conjecture 0. Let X and Y be closed smooth G-manifolds without boundary and let $f: X \to Y$ be a smooth G-map which represents a homotopy equivalence not necessarily having an equivariant homotopy inverse. Then there exist product G-fibre bundles $Y \times N$ and $Y \times M$ such that X is smoothly G-embedded in $Y \times N$ via an embedding j such that the composition $\pi \circ j$ with the projection $\pi: Y \times N \to Y$ is G-homotopic to f and such that the G-normal bundle ν of X in $Y \times N$ is G-isomorphic to the induced fibre bundle $f^*(Y \times M)$.

A description of the situation in the conjecture is given in the following diagram.



A G-homotopy equivalence $f: X \to Y$ with no G-homotopy inverse is called exotic and so X is called G-exotic homotopy Y. The quasi-equivalences of [7, pg. 21], whose classification for actions of a torus T, given in [9], we remark in our section 2, give exotic G-homotopic equivalences by Alexandroff compactification. In section 3 we contradict the possibility given by W. Y. Hsiang [6, pg. 108] of the inexistence of effective exotic actions of T^2 or T^3 in CP^n.

A G-homotopy equivalence is said to be atypical if it does not satisfy conjecture 0, in which case is necessarily exotic. In section 5 we show the existence of atypical G-homotopy equivalences constituted by the cases

* Supported by a grant from CNPq, Brasil.
Recebido em 20/02/81.

described in theorem 9 as indicated in remark 13. In fact theorem 9 shows the diversity of coordinates of the invariant $i_Y^* f_*(1)$ in equivariant K-theory [7, pg. 108], in contrast to their indifference in Petrie's exotic examples [7, pg. 125] and the effective ones of our section 3. We recall that [12, pg. 142] shows that these Petrie's examples $f: X \to Y$ preserve the classes $\widehat{A}(.)$ of Hirzebruch, [5], (Compare [4]), which suggests the preservation of the Pontrjagin classes. In section 6, proposition 14, we see that if $f: X \to Y$ is as in theorem 9, then f preserves the classes $\widehat{A}(.)$.

1. Cyclotomic toral representations.

The ring $R(T)$ of complex representations of the n-dimensional torus $T^n = S^1 \times \ldots \times S^1$ (n times), is obtained over the ring Z of integers by adjunction of n indeterminates $t_1, \ldots, t_n$ and their inverses, [1, pg. 77]. An irreducible element of this ring $Z[t_1, \ldots, t_n^{-1}]$, say $t_1^{a_1} \ldots t_n^{a_n}$, represents the classe of T-isomorphism of the complex vector space C endowed with the linear action of T given by $tz = (t_1, \ldots, t_n)\, z = t^a z = t_1^{a_1} \ldots t_n^{a_n} \cdot z$, where $t = (t_1, \ldots, t_n) \in T$, $z \in C$ and $a = (a_1, \ldots, a_n) \in Z^n$. The sum and the product of elements in $R(T)$ correspond respectively to the direct sum and to the tensor product of elements of the T-isomorphism classes so represented. We represent by the same symbol a classe in $R(T)$ or a representative of this class.

To each integer $p \in Z$ it corresponds an Adams operation ψ^p on $R(T)$ given by $\psi^p(\chi) = \chi^p$, where χ is an irreducible element in $R(T)$, [14]. We represent by R indistinctively the ring $R(T)$ or its tensor product with the field Q of rationals, $R(T) \otimes Q$. Let $R' \subseteq R$ be the subsemiring consisting of the elements of R expressible in the form

$$\prod_{j=1}^{r} (1 - t^{\alpha_j})(1 - t^{\beta_j})^{-1},$$

where $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$, $\beta_j = (\beta_{j1}, \ldots, \beta_{jn}) \in Z^n$ satisfy

$$\left| \prod_{j=1}^{r} \alpha_{ji} \right| = \left| \prod_{j=1}^{r} \beta_{ji} \right|, \quad \text{for} \quad i = 1, \ldots, n.$$

To each collection $P = \{p_1, \ldots, p_r\}$ of integers >1 relatively prime in pairs and irreducible $\chi \in R(T)$ we associate the unique pair of complex T-modules of dimension 2^{r-1} or 2^{r-1}-dimensional complex representation spaces of T,

$$M_P(\chi) = M_{p_1,\ldots,p_r}(\chi) \quad \text{and} \quad N_P(\chi) = N_{p_1,\ldots,p_r}(\chi)$$

determined by the relation, [9], $\psi^P(\chi) = (\psi^{p_i} - 1)\,(\chi) = M_P\,(\chi) - N_P\,(\chi)$.

For example, if $r = 2$ then

$$\psi^P(\chi) = (\psi^{p_1} - 1)(\psi^{p_2} - 1)(\chi) = (\psi^{p_1 p_2} - \psi^{p_1} - \psi^{p_2} + 1)(\chi) = M_P(\chi) - N_P(\chi),$$

where $M_P(t) = t^{p_1 p_2} + t$ and $N_P(t) = t^{p_1} + t^{p_2}$.

Next we give an equivalent characterization of the pair (M_P, N_P). Recall that the class λ_{-1} of a complex T-module, [14], is defined by $\lambda_{-1}(\chi) = 1 - \chi$, where χ is irreducible in $R(T)$, and by $\lambda_{-1}(\oplus\, \chi_i) = \Pi \lambda_{-1}(\chi_i)$, if χ_i runs through a finite family of complex T-modules. Then M_P and N_P satisfy

$$\phi_P(\chi) = \lambda_{-1}(M_P(\chi)) / \lambda_{-1}(N_P(\chi)) \in R'.$$

In fact $\phi_P(\chi)$ constitutes an integer polynomial. Furthermore, M_P and N_P are the unique complex T-modules of dimension 2^{r-1} such that $\phi_P = \lambda_{-1}(M_P)/\lambda_{-1}(N_P)$. If $p_1, \ldots, p_r$ are prime integers then $\phi_P = \phi_{p_1 \cdots p_r}$ is the cyclotomic polynomial of order $p_1 \cdots p_r$.

Note that the multiplicative subgroup U of units of R satisfies $U \subseteq R'$.

Lemma 1. *A basis for R'/U is constituted by the cyclotomic polynomials of composite numbers applied to effective irreducible complex T-modules.* (Recall that a T-space X is said to be effective if and only if $X^T \neq X^H$ for every subgroup $H \neq T$, where X^H is the fixed point set of X under the action of H).

Proof. We identify R'/U with the set of elements of R' such that the α_{ji} and β_{ji} are positive. If $a \in Z^n$ denote $(a) = 1 - t^a$. To each element $\Pi(\alpha_j)/(\beta_j) \in R'$ it corresponds the element $\Pi(|\alpha_j|)/(|\beta_j|)$ under the indicated identification, where $|\alpha_j| = (|a_{j1}|, \ldots, |\alpha_{jn}|)$. In this way R'/U is seen as a multiplicative subsemiring of the polynomial ring $Z[t]$. On the other hand the cyclotomic polynomial of the number $a = p_1^{r_1} \ldots p_\ell^{r_\ell}$, where $\ell > 1$, $p_1, \ldots, p_\ell$ are prime numbers >1 and $r_1, \ldots, r_\ell$ are positive integers, is $\phi_a(t) = \phi_{p_1,\ldots,p_\ell}(t^{a'})$, where $a' = p_1^{r_1 - 1} \cdot\, p_\ell^{r_\ell - 1}$ by Moebius formula, [8, pag. 207]. Since the irreducible polynomials in t constitute a basis for $Z[t]$ and the only irreducible polynomials in R'/U are the cyclotomic polynomials of composite numbers, the lemma is proved.

2. Quasi-equivalences.

A proper map $f: N \to M$ between vector spaces of the same dimension is said to have degree one if the map induced by f between the Alexandroff compactifications N^+ and M^+ of N and M respectively, which is a map between spheres of the same dimension, is a homotopy equivalence. This makes sense since f is proper.

A smooth T-map of degree one between complex T-modules of the same dimension is said to be a quasi-equivalence of complex T-modules. In [9] nontrivial quasi-equivalences of complex S^1-modules $f_{p_1,\ldots,p_r} = f_P : N_P \to M_P$ are constructed.

For example if $n=2$, let a, b be natural numbers such that $-ap+bq=1$. Then we can take $f_{p,q}(z_1,z_2) = (\bar{z}_1^a z_2^b,\ z_1^q + z_2^q)$.

Corollary 2, [9]. *If $T=S^1$ for each $\phi\in R'$ there exists a quasi-equivalence of complex S^1-modules $f: N\to M$, where N and M satisfy $\lambda_{-1}(M)/\lambda_{-1}(N)=\phi$. Furthermore, the common dimension of N and M may be taken to be arbitrarily large with $N^{S^1}=M^{S^1}=\{0\}$.*

Remark 3. Consider the set Γ of ordered pairs (M,N) of nontrivial complex T-modules such that $\dim M = \dim N$ and such that $\lambda_{-1}(M)/\lambda_{-1}(N)\in R(T)$ takes value ± 1 when $t=1$, i.e. it converges in absolute value to 1 when $t\to 1$. Let $\Omega(M,N)=\lambda_{-1}(M)/\lambda_{-1}(N)$. Ω may be interpreted as a semiring homomorphism. In fact Ω is an epimorphism. Then for each $(M,N)\in\Gamma$ we apply corollary 2 and lemma 1 to obtain a pair $(M',N')\in\Gamma$ and a T-quasi-equivalence $f: N'\to M'$ such that $\phi=\Omega(M',N')$. Furthermore $M'=M\oplus P_M$ and $N'=N\oplus P_N$, where $(P_M,P_N)\in\Gamma$ and $\Omega(P_M,P_N)$ is a unit in R', i.e. $P_M=P_N$ up to units. This is the sufficiency of the main theorem of [9].

For example let $T=S^1$, $M=t^1\oplus t^{12}$, $N=t^3\oplus t^4$. Then

$$\Omega(M,N)=\frac{(1-t)(1-t^{12})}{(1-t^3)(1-t^4)}=\frac{(1-t^{12})(1-t^2)}{(1-t^6)(1-t^4)}\cdot\frac{(1-t^6)(1-t)}{(1-t^3)(1-t^2)}= \\ =\phi_{2,3}(t^2)\cdot\phi_{2,3}(t).$$

In this case $P_M=P_N=t^2\oplus t^6$. Then we can take $f=(f_{2,3}\circ q)\oplus f_{2,3}$, where $q(z_1,z_2)=(z_1^2,z_2^3)$.

3. Effective nonlinear T-actions on CP^r

Let Δ be a complex $T\times S^1$-module on which $\{1\}\times S^1$ acts freely, i.e. for example Δ is C^{r+1} with $(t,\eta)\in T\times S^1$ acting on $(z_0,\ldots,z_r)\in C^{r+1}$ by $(t,\eta)\circ(z_0,\ldots,z_r)=(z_0\eta\Pi t_i^{a_{i0}},\ldots,z_r\eta\Pi t_i^{a_{ir}})$, that is $\Delta=\theta\otimes\eta$ for some complex T-module θ. Note that the unitary sphere $S(\Delta)$ is $T\times S^1$-invariant. Define $P(\Delta)=P(\theta)=S(\Delta)/\{1\}\times S^1$, the T-linear CP^r associated to Δ or space of complex lines of θ. This is a smooth T-manifold.

Question 4. *For positive integers s and r, are there effective nonlinear actions of T^s on CP^r?*

Remark 5. It is easy to see that there are not effective actions of T^k on CP^r for $k>r$. If T^r acts effectively on CP^r for $r>1$ then the T-action is T-quasi-linear in the sense that T-actions induced on the tangent bundle and the Hopf bundle are essentially the same, as in the linear case, [4]. Kai Wang announced similar result for T^{r-1} acting on CP^r, [16].

Remark 6. Known affirmative answer to the question above happened for $s=1$ and $r=4k-1$, [7, pg. 125], [12, pg. 148].

W. Y. Hsiang in [6, pg. 108], cites the S^1-nonlinear CP^r of [12] but expresses that quasi-linearity seems to be possible for $s\geq 2$ ou 3.

Theorem 7. *There exist effective nonlinear T^s-actions on $CP^{4\ell-1}$, for $1\leq s\leq\ell$.*

This solves partially the problem of existence of question 4 with the restrictions imposed in remarks 5 and 6.

In fact, let χ be an irreducible complex T-module. Let $M=\chi\oplus\chi^{pq}$, where pq is a composite number. Then the equivariant exterior power of M, $\Lambda(M)=\chi^0\oplus\chi^{pq+1}\oplus\chi^1\oplus\chi^{pq}$ has their levels $\Lambda^0(M)=\chi^0$, $\Lambda^1(M)=\chi^1\oplus\chi^{pq}$ and $\Lambda^2(M)=\chi^{pq+1}$. Let ϕ be any T^s-complex module of dimension ℓ and let $\theta=\phi\otimes\Lambda$. Then a T^s-nonlinear $CP^{4\ell-1}$, say X, is obtained together with an equivariant homotopy equivalence $f: X\to P(\theta)$ without equivariant homotopy inverse, being the construction of f for $T=S^1$ due to [7, pg. 125].

We will remark at the end of this section a construction method for this fact.

Now, if $P(\theta)$ has effective T-action, also X has effective T-action. Assume $\theta=(\Pi t_i^{a_{i1}})\oplus\ldots\oplus(\Pi t_i^{a_{i,4\ell}})$. We will see in lemma 8 that for $P(\theta)$ to be T-effective, the associated matrix

$$\begin{bmatrix} a_{11}\ldots a_{1,4\ell} \\ a_{s1}\ldots a_{s,4\ell}\end{bmatrix}$$

must contain a maximal square matrix whose determinant is ± 1.

For convenience denote $Z_0=S^1$. Let $G=Z_{\alpha_1}\times\ldots\times Z_{\alpha_s}$, where $0<\alpha_j<\infty$. We represent elements in G by s-tuples $t=(t_1,\ldots,t_s)$, where $t_i\in Z_{\alpha_i}$. Let $Z^v(G)$ be the additive group of $s\times v$-matrices of integers modulo α_j in the j^{th} row for $j=1,\ldots,s$. Let $R^v(G)$ be the set of v-dimensional complex G-modules and let $M_G^v: Z^v(G)\to R^v(G)$ given by

$$M_G^{\nu}(\{\gamma_{jk}\}) = \sum_{k=0}^{\nu-1} (t_j^{\gamma_{jk}}).$$

Then M_G^{ν} is a bijection and $R^{\nu}(G)$ gets a group structure by means of M_G^{ν} in such a way that M_G^{ν} is a group isomorphism. Let $\gamma = \{\gamma_{jk}\} \in Z^{\nu}(G)$ and let W be a G-invariant subset of the projective space $P(M_G^{\nu}(\gamma))$ of complex lines of the G-module $M_G^{\nu}(\gamma)$. Denote the k-column of γ by γ_k, for $k = 0, \ldots, \nu - 1$.

Lemma 8. *If there exists a point $[z_0 ; \ldots ; z_\nu] \in W$ (in homogeneous coordinates) with $s+1$ nonzero coordinates $z_0, z_{k_1}, \ldots, z_{k_s}$, where $0 < k_1 < \ldots < k_s < \nu - 1$, such that $|det(\gamma_{k_1} - \gamma_0, \ldots, \gamma_{k_s} - \gamma_0)| = 1$, then the action of G on W is effective.*

Proof. We may assume $\gamma_0 = 0$. Suppose that the action is not effective. Then there exists $t \in G - \{1\}$ such that $tw = w$ for every $w \in W$ and such that, if we denote by $[t] \subseteq G$ the subgroup topologically generated by t, then $[t]$ is the image of an injective homomorphism ϕ: $G' \to G$, where $G' = \prod_{i=1}^{\sigma} Z_{\beta_i}$ and $0 \leq \beta_i < \infty$. This determines a matrix $\delta = \{\delta_{k\ell}\} \in Z^s(G')$ such that $\phi(\tau_1, \ldots, \tau_\sigma) = \left(\prod_{k=1}^{s} t_k^{\delta_{k1}}, \ldots, \prod_{k=1}^{s} t_k^{\delta_{ks}} \right)$. If W' is the G'-space obtained from W by means of ϕ, then $W' = P(M_G^{\sigma}, (\delta \cdot \gamma))$, where $\delta \cdot \gamma \in Z^{\sigma}(G')$ is obtained by multiplying two integer matrices representing respectively δ and γ and taking the reduction of the result in $Z^{\sigma}(G')$, contradicting that $|det(\gamma_{k_1}, \ldots, \gamma_{k_s})| = 1$, absurd.

Example. Let $\ell = 2$, $\chi = t_1 t_2^2$, $pq = 6$, $\phi = 1 + t_1 t_2$. Then the matrix associated to θ is

$$\begin{bmatrix} 0 & 7 & 1 & 6 & 1 & 8 & 2 & 7 \\ 0 & 14 & 2 & 12 & 1 & 15 & 3 & 13 \end{bmatrix}$$

in which we find for example the maximal square matrix formed by the third and seventh columns having determinant -1.

The end of this section was suggested by conversations mantained with W. Iberkleid.

The construction used in theorem 7 may be obtained elegantly as an application of the homotopy covering theorem of Palais, [3, pg. 93] and the generalized Poincaré conjecture, [10, pg. 109]. In fact let $\theta_\pm = \phi \otimes \Lambda^\pm(M)$, where $\Lambda^+(M)$, (resp. $\Lambda^-(M)$) is the equivariant exterior power of M in event, (resp. odd) dimensions, i.e. $\Lambda^-(M) = \chi^1 \oplus \chi^{pq}$,

$\Lambda^+(M) = \chi^0 \oplus \chi^{pq+1}$. Then there exists a structure of quaternionic vector space H^n on $\theta^\pm$ so that $\rho : S(\theta) \to D(\mathbb{R} \times M)$ (with image in the equivariant unit disk of $\mathbb{R} \times M$), given by $\rho(u_+, u_-) = (|u_+|^2 - |u_-|^2, 2\langle u_+, u_- \rangle)$, where $u_\pm \in H^n \theta_\pm$, is a T-map. ρ may be considered as the orbit map of $S(H^n \times H^n)$ with the diagonal action of $Sp(n)$. There are exactly two types of orbits for this action, namely $Sp(n)/Sp(n-2)$ over $S(\mathbb{R} \times M) \subseteq D(\mathbb{R} \times M)$, and $Sp(n)/Sp(n-1)$ over the interior of $D(\mathbb{R} \times M)$. Let ϕ: $\chi^p \oplus \chi^q \to M$ be given by $\phi = f_{p,q} \cdot |f_{p,q}|^{-1}$. Then ϕ is a T-map, $D_\phi = \phi^{-1}(D(\mathbb{R} \times M))$ is a 5-disk [7, pg. 122], and ϕ restricted to the boundary ∂D_ϕ of D_ϕ is homotopic to the identity map $id | \partial D^5$ of the boundary S^4 of D^5 by a homotopy that extends radially over D^5, starting at ϕ.

Let Y be the pullback space of ρ and $\phi | D_\phi$. Then the lifting of ϕ, $\bar{\phi} : Y \to S(\theta)$ is a $Sp(n)$-map homotopic to the identity map of Y by the Palais theorem mentioned above. By the generalized Poincaré conjecture, Y is diffeomorphic to $S^{8\ell - 1}$. Let $b : S^1 \to Sp(n)$ be given by $b(t) = diag(t, \ldots, t)$. The image of S^1 through b acts freely on both sides of $\bar{\phi}$, so that by taking the orbit spaces by the resulting S^1-actions we obtain a homotopy equivalence $f : X \to P(\theta)$ with Y diffeomorphic to $P(C^{4\ell})$. Note that f is a T-map. However f does not have equivariant homotopy inverse, as was established in [7, pg. 128], technic that we describe subsequently in relation to questions and facts suggested in the beginning of this paper.

4. Equivariant homotopy irreversibility.

Given a closed boundaryless smooth T-manifold Y, let $S_T(Y)$ be, as in [7, pg. 102], the family of classes $[X, f]$ of T-homotopy equivalences of smooth T-maps $f : (X, X^T) \to (Y, Y^T)$. In addition we assume that $H^1(Y; Z) = 0$ and that $H^3(Y; Z_2) = 0$. Under these circunstances [10, pg. 116] establishes the existence of a Thom isomorphism in equivariant K-theory, $\psi_Y : K_T^*(Y) \to K_T^*(\tau Y)$, where τY is the tangent bundle of Y. Let $h^*(.) = K_T^*(.) \otimes Q$. Then [7, pg. 116] establishes the existence of a homomorphism $f_* : h^*(X) \to h^*(Y)$ adjoint to the usual induced homomorphism $f^* : h^*(Y) \to h^*(X)$ with respect to nondegenerate bilinear forms ([7, pg. 98]), $Id_T^X(x \cdot x')$ and $Id_X^Y(y \cdot y')$, where Id_T^Y is the composition $(Ind_Y \circ \psi_Y) \otimes 1_Q$ with $Ind_Y : K_T^*(\tau Y) \to R(T)$, the Atiyah-Singer index homomorphism, $x, x' \in h^*(X)$, $y, y' \in h^*(Y)$ and 1_Q being the identity map of the rational numbers.

This way $f_*(1)$ becomes an invariant of the setting $[X, f] \in S_T(Y)$ satisfying (if X_α are the connected components of X^T and $i_Y : Y^T \to Y$ is the usual inclusion) that the element $i_Y^*(f_*(1)) \in h^*(Y^T) = \prod_{\alpha=1}^{r} h^*(X_\alpha)$

has its coordinate in $h^*(X_\alpha)$ equal to $\lambda_{-1}(\nu Y_\beta)/\lambda_{-1}(\nu X_\alpha)\cdot 1$ up to units in $R(T)$, where $f(X_\alpha)\subseteq Y_\beta$, Y_β is a connected component of Y^T and νX_α (resp. νY_β) is the equivariant normal bundle of X_α (resp. Y_β) in X (resp. Y).

According to [2] we know that i_Y^* is a monomorphism of $R(T)$-modules (Atiyah-Segal localization lemma) and since f restricted to the fixed point sets is a T-homotopy equivalence, we conclude that the R-algebra $h^*(Y)$ is an R-submodule via f^* of $h^*(X)$ with coordinates taken in the larger R-module $h^*(X^T)=h^*(Y^T)$. See [7, pg. 116] and [4, Thm. 1.1].

Observe that in section 3 we obtained a homotopy equivalence $f: X\to P(\theta)$ given by a smooth T-map. The restriction of f to X^T is a bijection $f^T: X^T\to P(\theta)^T$ so that $i_Y^*(f_*(1))=\phi_{p,q}(\chi)\cdot 1\in h^*(X^T)=$ direct sum of 4ℓ copies of the ring R. Thus f fails to be a T-homotopy equivalence, as we claimed in section 3 to warrant nonlinearity of the T-action obtained over X, since $\phi_{p,q}$ is not a unit in $R(T)$.

5. Diversity of coordinates of $f_*(1)$ for X connected.

The examples of nontrivial settings $[X, f]$ in the work of T. Petrie are characteristic in that all the coordinates of $i_Y^*(f_*(1))$ coincide up to units. Yet in our effective generalization in section 3 this fact always happens, as in the Alexandroff compactifications of the T-quasi-equivalences of our first two sections. When does it fail to happen?

Theorem 9. *Given $P_1,\dots,P_m$ in $R'(S^1)$, pairwise different, given an irreducible complex T-module χ and positive integers $a_1,\dots,a_m$ there exists a closed boundaryless smooth T-manifold Y such that $H^1(Y;Z)=0$, $H^3(Y;Z_2)=0$ and such that Y^T is isolated and a map $f: X\to Y$ representing and element $[X, f]\in S_T(Y)$ such that $2a_j$ coordinates of $i_Y^*(f_*(1))\in K_T^*(Y^T)=$ $=\Pi R(T)$ are of the form $P_j(\chi)$ up to units, for $j=1,\dots,m$, being the number of nonunit coordinates of $i_Y^*(f_*(1))$ equal to $2\sum_{i=1}^m a_i$.*

The proof of theorem 9 depends on the following.

Lemma 10. *Let N and M be nontrivial complex S^1-modules of real dimension $n+1$ and let $f: N^+\to M^+$ be a smooth S^1-map of degree one. Given a regular value x of f such that $S^1(x)$ is a principal orbit, ([3, pg. 179]) there exists a smooth S^1-map $g: N^+\to M^+$ which is S^1-homotopic to f such that x is a regular value of g and such taht $g^{-1}(x)$ consists of a point. Thus $g^{-1}(S^1(x))$ is a principal orbit.*

Proof. We may assume that the inverse image of x under f is a collection of points $f^{-1}(x)=\{y_1,\dots,y_{2s+1}\}\subseteq N^+$ such that $\mathrm{sign}(df)\,y_j=(-1)^j$, for $j=1,\dots,2s+1$, ([11, pg. 27]). If $s=0$ the lemma holds. Assume $s>0$. Let $\rho: N^+\to N^+/S^1$ be the quotient map. We can choose a neighborhood V of $\{y_j/S^1\}_{j=2}^{2s+1}$ diffeomorphic to the n-disk D^n, contained in the subset of principal orbits and excluding the point y_1/S^1, in such a way that ρ is a fibration over V having a section $\sigma: V\to N^+$. We want to define $g: N^+\to M^+$ coinciding with f out of ρ^{-1} (interior of V) and sending $\rho^{-1}(V)$ out of $S^1(x)$. To attain this purpose it suffices to extend the restriction of f over $\sigma(\partial V)$ to a smooth map $\sigma(V)\to M^+$ that sends $\sigma(V)$ to the exterior of $S^1(x)$, that is out of a tubular neighborhood W of $S^1(x)$ of the type $W=S^1\times\mathrm{interior}(D^n)$, i.e. inside $M^+-W\cong D^2\times S^{n-1}$. But the degree of the restriction h of f to $\sigma(V)$ is zero, since by [11, pg. 27], $\deg(h,x)=\sum_{j=2}^{2s+1}\mathit{sign}(df)\,y_j=0$, and this value does not depend on the choice of the regular point y of h. Thus the restriction $h': \sigma(\partial V)\to M^+-W$ has degree zero, so it is homotopic to a constant map. This implies that h' can be extended to a smooth map $h'': \sigma(V)\to M^+-W$. Moreover, we may choose h'' in such a way that the map $g: N^+\to M^+$ defined $g(y)=$ $=f(y)$ if $y\in N^+-\rho^{-1}$ (interior of V) and by $g(ty)=th''(y)$ if $y\in\sigma(V)$ and $t\in S^1$, is a smooth S^1-map. To prove that g is S^1-homotopic to f it suffices to extend the map $\phi: \partial\{\sigma(V)\times[0,1]\}\to M^+$ defined by $\phi(y,0)=f(y)$ and $\phi(y,1)=g(y)$, for $y\in\sigma(V)$ and by $\phi(y,\tau)=f(y)$ for $y\in\sigma(\partial V)$ and $\tau\in[0,1]$, to a map $\overline{\phi}: \sigma(V)\times[0,1]\to M^+$. But the domain of ϕ is topologically S^n and its image is S^n so that ϕ is homotopic to a constant map, and so $\overline{\phi}$ exists as a continuous map. We define $\psi: N^+\times[0,1]\to M^+$ by $\psi(y,\tau)=f(y)$ for $y\in N^+-\rho^{-1}$ (interior of V) and $\tau\in[0,1]$ and by $\psi(ty,\tau)=t\phi(y,\tau)$ for $y\in\sigma(V)$, $t\in S^1$ and $t\in[0,1]$. Then ψ is an S^1-homotopy between $f=\psi_0$ and $g=\psi_1$.

Proof of theorem 9. Without loss of generality we may assume that $T=S^1$ and $\chi=t$. For $j=1,\dots,m$ let M_j and N_j be complex S^1-modules of real dimension $n+1$ and let $f_j: N_j^+\to M_j^+$ be representatives of classes $[N_j^+, f_j]\in S_{S^1}(M_j^+)$. According to lemma 10 the maps f_j may be taken so that there exists for $k=1,2$, smooth tubular S^1-neighborhoods, ([3, pg. 303]), P_{jk}, (resp. Q_{jk}) of principal orbits of N_j^+, (resp. M_j^+) and such that each f_j restricts to a map $g_{jk}: N_j^+$-interior $(P_{jk})\to M_j^+$-interior (Q_{jk}) that restricts respectively to a diffeomorphism $h_{jk}: \partial P_{jk}\to\partial Q_{jk}$ from the boundary of P_{jk} to the boundary of Q_{jk}, for $k=1,2$. We will construct by induction closed smooth S^1-manifolds X_j and Y_j without boundary and with exactly $2a_j$ fixed points and a smooth S^1-map $F_j: X_j\to Y_j$ representing a class $[X_j, f_j]\in S_{S^1}(Y_j)$, for $j=1,\dots,m$. We define $X_1=N_1^+$,

$Y_1 = M_1^+$ and $F_1 = f_1$. Then we define recursively, (see for example [3, pg. 50]).

$$X_{j+1} = (X_j\text{-interior}(P_{j2}) \bigcup_{\phi_j} (N_{j+1}^+\text{-interior}(P_{j+1,1}),$$

where $\phi_j : \partial P_{j2} \to \partial P_{j+1}$ is an S^1-diffeomorphism:

$$Y_{j+1} = (Y_j\text{-interior}(Q_{j2}) \bigcup_{\psi_j} (M_{j+1}^+\text{-interior}(Q_{j+1,1}),$$

where $\psi_j = h_{j2} \circ \phi_j \circ h_{j1}$. Also define $F_{j+1} : X_{j+1} \to Y_{j+1}$ by $F_{j+1} \mid X_j$-interior (P_{j2}) and by $F_{j+1} \mid N_{j+1}^+$-interior$(P_{j+1,1}) = g_{j+1,1}$. Note that F_{j+1} is a smooth S^1-map. Finally we take $X = X_m$, $Y = Y_m$ and $f = F_m$.

Observe that if $r = 2$ then the subjacent spaces of X and Y are $S^2 \times S^{n-1}$

Proposition 11. $[X, f] \in S_{S^1}(Y)$.

Proof. The fact that $f : X \to Y$ is a homotopy equivalence follows from the theorem of Whitehead, (see for example [15]), and the exactness of appropriate Mayer Vietoris sequences associated to our construction, such as for example in [3, pag. 51].

From [7, pags. 117-118] we conclude the following.

Proposition 12. *If we denote* $Y^{S^1} = \sum_{j=1}^{2m} q_j$, *where* q_{2k-1} *is the origin of* M_k *and* q_{2k} *is the point at infinity of* M_k^+ *then up to units*

$$i_Y^*(f_*(1)) = (\lambda_{-1}(M_1), \lambda_{-1}(M_1), \ldots, \lambda_{-1}(M_m), \lambda_{-1}(M_m)).$$

Now theorem 9 for $T = S^1$ is obtained from corollary 2 and the last two propositions.

Remark 13. The G-homotopy equivalences obtained in the theorem 9 are atypical, i.e. they do not satisfy the conjecture 0 if for example

$$P_1 = \lambda_{-1}(M_1)/\lambda_{-1}(N_1) \qquad \text{and} \qquad P_2 = \lambda_{-1}(M_2)/\lambda_{-1}(N_2)$$

are different in R'/U.

6. Atypical Settings and the Class $\hat{A}(\,.\,)$.

Proposition 14. *Let* $f : X \to Y$ *as in theorem 9. Then* $f^*\,(\hat{A}(Y)) = \hat{A}(X)$.

Proof. With the notation of theorem 9 suppose that $r = 2$ and $T = S^1$. The general case may be concluded from what follows, an induction

procedure and substitution of the group. Now for every complex S^1-module M of real dimension $n+1$ there exists a diffeomorphism: $\phi_M : S(\mathbb{R} \times M) \to M^+$ given by $\phi_M(r, \zeta) = (1+r)^{-1}\zeta$, where $r \in [-1, 1)$ and by $\phi_M(-1, 0) = +$. Then for $j = 1, 2$ the normal bundle of $N_j^+ \simeq S(\mathbb{R} \times N_j)$ in $\mathbb{R} \times N_j$ is a trivial bundle and from this we have embeddings $N_j^+ \subseteq N_1 \times N_2 \times \mathbb{R}^2 = E$ with trivial normal bundle of N_j^+ in E. By attaching a handle $S^1 \times S^{n-1} \times [0, 1] \subseteq E$, with effective S^1-action on S^1 and trivial S^1-action on $S^{n-1} \times [0, 1]$, so that $S^1 \times S^{n-1} \times \{0\}$, (resp. $S^1 \times S^{n-1} \times \{1\}$) is identified equivariant and diffeomorphically with ∂P_{12}, (resp. ∂P_{21}), we obtain an S^1-embedding of X in E which can be chosen smooth, [3, pag. 317]. Then there exists a bundle isomorphism from the normal bundle of N_1^+ in E restricted to ∂P_{12} onto the normal bundle of N_2^+ in E restricted to ∂P_{21} covering the S^1-diffeomorphism ϕ_1. Consider the correspondence $\psi : \partial P_{12} \to SO(n+3)$ taking each point $x \in \partial P_{12}$ to the linear transformation over x associated to the mentioned bundle ismorphism. Since ψ is homotopic to a constante then the normal bundle of X in E is trivial, ([14]). The same argument shows that Y can be embedded in $M_1 \times M_2 \times \mathbb{R}$, which implies that τY is stably trivial. Thus the induced bundle $f^*(\tau Y)$ is stably trivial. Consider the commutative S^1-diagram

$$\begin{array}{ccc} E & \longrightarrow & Y \times E \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle proj.} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

where $i : X \to E$ is the constructed embedding. Then $f \circ i : X \to Y \times E$ is also an S^1-embedding. The S^1-normal bundle of X in $Y \times E$ is $\nu_{Y \times E}X = \nu_E X \oplus f^*(\tau Y)$. From the observations above we have that $\nu = \nu_{Y \times E}X$ is a trivial bundle. Thus $\tau X \oplus \nu = f^*(\tau Y \oplus E')$, where $E' = Y \times E$ is a trivial bundle. We conclude that

$$\hat{A}(\tau X) \cdot \hat{A}(\nu) = \hat{A}(\tau X \oplus \nu) = f^*[\hat{A}(\tau Y) \cdot \hat{A}(E')] = f^*(\hat{A}(\tau Y)).$$

Since ν is trivial, it follows proposition 14.

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