Generic properties of higher order endomorphisms

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1. Introduction.

Let M be a smooth compact manifold without boundary. Let $k \geq 1$ be an integer. We denote by M^k the product manifold $M \times \ldots \times M$ (k-times).

1.1. **Definition.** An *endomorphism of order k* (or k-endomorphism) is a map $f: M^k \to M$. A sequence $(x_n)_{n \geq 1}$, $x_n \in M\ \forall n \geq 1$, is an *orbit* of f if it satisfies the condition $f(x_n, \ldots, x_{n+k-1}) = x_{n+k}\ \forall n \geq 1$. Hence, the orbit $(x_n)_{n\geq 1}$ is determined by the k-tuple $(x_1, \ldots, x_k)$.

An endomorphism $f: M \circlearrowleft$ is a 1-endomorphism in our notation. If $k \geq 2$ we say that a k-endomorphism is a *higher order endomorphism.*

In the classical literature objects like higher order endomorphisms are called recurrences. See (*M*).

1.2. We will describe the dynamics of a k-endomorphism $f: M^k \to M$ using the endomorphism $\tilde{f}: M^k \circlearrowleft$ defined by $\tilde{f}(x_1, \ldots, x_k) = (x_2, x_k, f(x_1, \ldots, x_k))$, which we will call the lifting of f. There is a close relationship between the dynamics of f and that of $\tilde{f}$. In fact, let $(x_n)_{n \geq 1}$ be an orbit of f. Then the orbit of $\tilde{f}$ through $(x_1, \ldots, x_k)$ is $\{(x_n, \ldots, x_{n+k-1}), n \geq 1\}$. Moreover every orbit of $\tilde{f}$ has this form.

1.3. **Definition.** A *periodic orbit* of f is a p-tuple $\langle x_1, \ldots, x_p \rangle$ such that

$$f(x_1, \ldots, x_k) = x_{k+1}, f(x_2, \ldots, x_{k+1}) = x_{k+2}, \ldots, f(x_{p-k+1}, \ldots, x_p) = \\ = x_1, f(x_{p-k+2}, \ldots, x_p, x_1) = x_2, \ldots, f(x_p, x_1, \ldots, x_{k-1}) = x_k$$

for $k < p$. If $k \geq p$ the definition is analogous. In particular a fixed point of f is an element $x \in M$ such that $f(x, \ldots, x) = x$. The period of the periodic orbit $\langle x_1 \ldots, x_p \rangle$ is p if it is minimal with respect of the above conditions.

1.4. The correspondence between orbits of f and $\tilde{f}$ established in 1.2 preserves periodic orbits and their periods. At each periodic orbit

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$\langle x_1, \ldots, x_p \rangle$ of f with period p corresponds the periodic orbit $\{(x_1, \ldots, x_k), (x_2, \ldots, x_k, x_{k+1}), \ldots, (x_p, x_1, \ldots, x_{k-1})\}$ of $\tilde{f}$ with period p. Moreover, every periodic orbit of $\tilde{f}$ has this form. Furthermore this correspondence preserves invariant sets, w-limit sets and some kind af attractors.

1.5. **Definitions.** Let $f \in C^r(M^k, M), r \geq 1$, and let $\langle x_1, \ldots, x_p \rangle$ be a periodic orbit of f with period p. This orbit is hyperbolic if every eigenvalue of $D\tilde{f}^p(x_1, \ldots, x_k)$ has norm $\neq 0, 1$.

Let $H_p = \{f \in C^r(M^k, M)$; every periodic orbit of f with period $\leq p$ is hyperbolic$\}$. Let $KS^r(M^k, M)$ be the set of $f \in C^r(M^k, M)$ which satisfy the following properties:

a. The periodic orbits of f are all hyperbolic;

b. If $\langle x_1, \ldots, x_p \rangle$ is a periodic orbit of f with period p, let $W^s_{loc}(x_1, \ldots, x_k)$ be the local stable manifold of $\tilde{f}$ at $(x_1, \ldots, x_k)$. Then the set $W^s(x_1, \ldots, x_k) = \{x \in M; \exists\, n \in Z_+$ such that $\tilde{f}^{np}(x) \in W^s_{loc}(x_1, \ldots, x_k)\}$ is a 1-1 immersed submanifold of constant dimension.

c. Let $\langle x_1, \ldots, x_p \rangle$ and $\langle y_1, \ldots, y_q \rangle$ be periodic orbits of f with periods p and q respectively. Let $W^u_{loc}(y_1, \ldots, y_k)$ be the local unstable manifold of $\tilde{f}$ at $(y_1, \ldots, y_k)$. Then $\tilde{f}^{nq} \mid W^u_{loc}(y_1, \ldots, y_k)$ is transversal to $W^s(x_1, \ldots, x_k)$ on $W^u_{loc}(y_1, \ldots, y_k)$ for all $n \in Z_+$.

We regard the elements in $KS^r(M^k, M)$ as the natural extension of Kupka-Smale endomorphisms therefore we call them Kupka-Smale k-endomorphisms.

The following theorem extends a result of M. Shub (Sh). It constitutes the main result of this work.

1.6. **Theorem.** *H_p is open and dense in $C^r(M^k, M)$ for all $p \geq 1$. Moreover $KS^r(M^k, M)$ is residual in $C^r(M^k, M)$.*

The proof of 1.6 is given in the next section.

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2. The Proof of Theorem 1.6.

2.1. Let $\mathscr{L}(R^{kn})$ be the space of real $kn \times kn$ matrices identified as usual with R^{kn^2}. Let $\mathscr{L}_0(R^{kn})$ be the subspace of matrices L of the form

$$(2.2) \qquad L = \begin{bmatrix} 0_{n(k-1),n} & & I_{n(k-1),n(k-1)} & \\ a_1^1 \ldots a_n^1 & & a_{n+1}^1 \ldots a_{nk}^1 & \\ \vdots & & \vdots & \\ a_1^n \ldots a_n^n & & a_{n+1}^n \ldots a_{nk}^n & \end{bmatrix}$$

where $0_{n(k-1),n}$ is the zero matrix $n(k-1) \times n$ and $I_{n(k-1),n(k-1)}$ is the identity matrix $n(k-1) \times n(k-1)$.

Note that if $f: R^{kn} \to R^{kn}$ is a C^1 k-endomorphism of R^n then $D\tilde{f}(x)$ belongs to $\mathscr{L}_0(R^{kn})$ for all $x \in R^n$. Moreover the determinant of the matrix (2.2) is given by

$$(-1)^{n(k+1)} \det \begin{bmatrix} a_1^1 \cdots a_n^1 \\ \vdots \\ a_1^n \quad a_n^n \end{bmatrix}$$

2.3. **Lemma.** *The set H of p-tuples $(L_1, \ldots, L_p)$ in $[\mathscr{L}_0(R^{kn})]^p = \mathscr{L}_0(R^{kn}) \times \ldots \times \mathscr{L}_0(R^{kn})$ (p-times) such that $L_1 \ldots L_p$ is a hyperbolic isomorphism is open and dense in $[\mathscr{L}_0(R^{kn})]^p$ for all $p \geq 1$.*

Proof. It is clear that H is open in $[\mathscr{L}_0(R^{kn})]^p$. For the proof of the density we take the space $\mathscr{P}_n$ of real polynomials $p(x) = x^{kn} + c_{kn-1} x^{kn-1} + \ldots + c_0$ identified with R^{kn} by the isormorphism $p \to (c_{kn-1}, \ldots, c_0)$. Note that if 0 is a zero of p then $p \in f_0^{-1}(0)$ where $f_0: R^{kn} \to R$ is given by $f_0(X_{kn-1}, \ldots, X_0) = X_0$. Moreover if 1 is a zero of p then $p \in f_1^{-1}(0)$ where $f_1: R^{kn} \to R$ is given by $f_1(X_{kn-1}, \ldots, X_0) = 1 + X_{kn-1} + \ldots + X_0$, and if -1 is a zero of p then $p \in f_2^{-1}(0)$ where $f_2: R^{kn} \to R$ is given by $f_2(X_{kn-1}, \ldots, X_0) = 1 + (-1)^{kn-1} X_{kn-1} + \ldots + (-1)^i X_i + \ldots + X_0$. Finally if p is a polynomial with a complex zero $\lambda = s + it$, $t \neq 0$ and $|\lambda| = 1$, then $p(x) = (x^2 - 2sx + 1)(x^{kn-2} + d_{kn-3} x^{kn-3} + \ldots + d_0)$. It is easy to see that $p \in f_3^{-1}(0)$ where $f_3: R^{kn} \to R$ is a non zero polynomial. Furthermore there exist polinomials $C_i: (R^{kn^2})^p \to R$, $0 \leq i \leq kn-1$, such that $p(x) = x^{kn} + C_{kn-1}(L_1, \ldots, L_p)\, x^{kn-1} + \ldots + C_0(L_1, \ldots, L_p)$ is the characteristic polynomial of $L_1 \ldots L_p$ for all $(L_1, \ldots, L_p) \in (R^{kn^2})^p$. We put $g_i: (R^{kn^2})^p \to R$, $g_i(L_1, \ldots, L_p) = f_i(C_{kn-1}(L_1, \ldots, L_p), \ldots, C_0(L_1, \ldots, L_p))$ for $i = 0, 1, 2, 3$. Then $V = g_0^{-1}(0) \cup g_1^{-1}(0) \cup g_2^{-1}(0) \cup g_3^{-1}(0)$ is an algebraic submanifold of R^{kn^2} satisfying the following condition: if $(L_1, \ldots, L_p) \notin V$ then every eigenvalue of $L_1 \ldots L_p$ has norm $\neq 0, 1$. In particular, $(R^{kn^2})^p - V \subset H$. We claim that for $j = 0, 1, 2, 3$ there exists a p-tuple $(L_1, \ldots, L_p) \notin g_j^{-1}(0)$. Let

$$L_n^k(a) = \begin{bmatrix} 0_{n(k-1),n} & I_{n(k-1),n(k-1)} \\ a\, I_{n,n} & 0_{n,n(k-1)} \end{bmatrix}$$

where $a \in R$. We have $(L_n^k(a))^k = aI$ and $\det(L_n^k(a) - xI) = (-1)^k(x^k - a)\det(L_{n-1}^k(a) - xI)$. If $0 < a < 1$ $L_n^k(a)$ is a contractive hyperbolic isomorphism, hence $(L_n^k(a), \ldots, L_n^k(a)) \notin g_0^{-1}(0) \cup g_1^{-1}(0) \cup g_2^{-1}(0)$. Furthermore $(L_n^k(0), \ldots, L_n^k(0)) \notin g_3^{-1}(0)$. Then $(R^{kn^2})^p - V$ is dense and so the lemma is proved.

2.4. Transversality theorems.

Let $\mathscr{A}$, X and Y be C^r Banach manifolds ($r \geq 1$) and $\rho\colon \mathscr{A} \to C^r(X, Y)$ a map. We say that ρ is a C^r-representation if the evaluation map $EV_\rho\colon \mathscr{A} \times X \to Y$ defined by $EV_\rho(a, x) = \rho(a)(x)$ is a C^r map.

The following two theorems are proved in [A].

2.4.1. **Theorem.** *Let $\mathscr{A}$, X and Y be C^1 Banach manifolds, W a closed submanifold of Y and K a compact subset of X. Let $\rho\colon \mathscr{A} \to C^1(X, Y)$ be a C^1 representation. Then*

$$\mathscr{A}_{KW} = \{a \in \mathscr{A};\ \rho(a) \pitchfork W \text{ on } K\}$$

is an open subset of $\mathscr{A}$.

2.4.2. **Theorem.** *Let $\mathscr{A}$, X and Y be C^r Banach manifolds ($r \geq 1$), $\rho\colon \mathscr{A} \to C^r(X, Y)$ a C^r-representation and W a C^r submanifold of Y. Suppose that the following four conditions are true.*

(i) *X has dimension n and W has codimension q in Y;*
(ii) *$\mathscr{A}$ and X have countable topological base;*
(iii) *$r > m\acute{a}x\ \{0, n - q\}$;*
(iv) *$EV_\rho \pitchfork W$.*

Then $A_W = \{a \in \mathscr{A};\ \rho(a) \pitchfork W\}$ is residual in $\mathscr{A}$.

2.5. Let p be a positive integer. We consider the map $G_p\colon C^r(M^k, M) \to C^r(M^k, M^{2k})$ defined by $G_p(f)(x) = (x, \tilde{f}^p(x))$. Let $EV_p\colon C^r(M^k, M) \times M^k \to M^{2k}$ be the evaluation $EV_p(f, x) = (x, \tilde{f}^p(x))$.

Let $\Delta = \{(x, x);\ x \in M^k\}$ be the diagonal set of $M^k \times M^k$.

2.6. **Lemma.** *G_p is a C^r-representation. Furthemore if $x \in M^k$ is a periodic point of $\tilde{f}$ with period p then $EV_p \pitchfork \Delta$ at (f, x).*

Proof. Let $f \in C^r(M^k, M)$, Let f^* TM be the pull-back of TM by f. Let $\Gamma^r(f^* TM)$ be the space of C^r sections of f^* TM with the topology of uniforme convergence of the first r derivatives. We know that $\Gamma^r(f^* TM)$

is the tangent space of $C^r(M^k, M)$ at f. A section $h \in \Gamma^r(f^* TM)$ can be written locally as $h(x) = (x, f(x), h_{f(x)}) \in M^k \times M \times T_{f(x)} M$. In the same way we consider $\Gamma^r(f^* TM^k)$. Given $h \in \Gamma^r(f^* TM)$ we define $\tilde{h} \in \Gamma^r(f^* TM^k)$ by the local expression $\tilde{h}(x) = (x, \tilde{f}(x), 0, \ldots, 0, h_{f(x)}) \in M^k \times M^k \times T_{\tilde{f}(x)} M^k$. In what follows we will adopt the abbreviated notation $h(x) \in T_{f(x)} M$ and $\tilde{h} = (\underbrace{0, \ldots, 0}_{(k-1)-times}, h)$. Note that $EV_p = \mathrm{EV}_0(j \times id)$, where $j\colon C^r(M^k, M) \to C^r(M^k, M^k)$ is defined by $j(f) = \tilde{f}$, $id\colon M^k \to M^k$ is the identity map and $EV\colon C^r(M^k, M^k) \times M^k \to M^{2k}$ is defined by $EV(g, x) = (x, g^p(x))$. The derivative $T(j \times id)(f, x)\colon \Gamma^r(f^* TM) \times T_x M^k \to \Gamma^r(\tilde{f}^* TM^k) \times T_x M^k$ is given by $T(j \times id)(f, x)(h, w) = (\tilde{h}, w)$ and the derivative $\mathrm{TEV}(g, x)\colon \Gamma^r(\tilde{f}^* TM) \times T_x M^k \to TM^{2k}$ is given by $\mathrm{TEV}(g, x)(k, w) = (w, \sum_{i=0}^{p-1} Tg^i(g^{p-i}(x))\, k(g^{p-i-1}(x)) + Tg^p(x)\, w)$.

Hence $\mathrm{TEV}_p(f, x)(h, w) = (w, \sum_{i=0}^{p-1} \mathrm{T}\tilde{f}^i(\tilde{f}^{p-i}(x))\, \tilde{h}(\tilde{f}^{p-i-1}(x)) + \mathrm{T}\tilde{f}^p(x)\, w)$.

This gives the derivative of EV_p, and it is clear that EV_p is of class C^r(Cf[A] ch 2).

Let $x = (x_1, x_k) \in M^k$ be a periodic point of $\tilde{f}$ with period p. Let us prove that $\mathrm{EV} \pitchfork \Delta$ at (f, x).

First suppose $p \geq k$. The elements $\tilde{f}^{p-i-1}(x)$, $0 \leq i \leq p-1$, are pairwise different. Let $h_j \in \Gamma^r(f^* TM)$, $0 \leq j < k$, defined by

$$\begin{cases} \tilde{h}_j(\tilde{f}^{p-i-1}(x)) = 0 \quad \forall\ i \neq j, \\ \tilde{h}_j(\tilde{f}^{p-j-1}(x)) = (\underbrace{0, \ldots, 0}_{(k-1)-times}, v_j) \in \mathrm{T}_{\tilde{f}^{p-j}(x)} M^k, \end{cases}$$

where v_j will be determined bellow, and let $w_j \in \mathrm{T}_x M^k$ be defined by

$$\begin{cases} w^0 = (0, \ldots, 0) \in \mathrm{T}_x M^k \\ w^j = (0, \ldots, 0, w_{k-j+1}, 0, \ldots, 0) \in \mathrm{T}_x M^k,\ j \neq 0, \end{cases}$$

where w_{k-j+1} is the $(k - j + 1) -$ th coordinate lying in a base of $\mathrm{T}_{x_{k-j\pm 1}} M$. We have $\mathrm{TEV}(f, x)(h_j, w^j) = (w^j, \mathrm{T}\tilde{f}^j(\tilde{f}^{p-j}(x))\, \tilde{h}_j(\tilde{f}^{p-j-1}(x)) + \mathrm{T}\tilde{f}^p(x)\, w^j)$. It easy to see that this vector has the $(2k - j) - th$ coordinate in the form $a + v_j$, and then we choose v_j such that every coordinate of $a + v_j$ is non zero. This proves the lemma for $p \geq k$. If $p < k$ the proof is similar.

2.7. Let $f \in C^r(M^k, M)$. A periodic point x of $\tilde{f}$ with period p is elementary if 1 is not an eigenvalue of $F\tilde{f}^p(x)$.

2.7.1. **Remark.** Let $f \in C^r(M^k, M)$. Then $G_p \pitchfork \Delta$ iff every fix point of $\tilde{f}^p$ is elementary.

2.7.2. **Remark.** Let $f \in C^r(M^k, M)$. We have the following properties:

i) If $G_p(f) \pitchfork \Delta$ at x then $EV \pitchfork \Delta$ at x.

ii) If x is a hyperbolic periodic point of f then $G_p(f) \pitchfork \Delta$ at x for all $p \geq 1$.

2.8. Let $T_p = \{f \in C^r(M^k, M);\ G_p(f) \pitchfork \Delta\}$. Then $T_p = \{f \in C^r(M^k, M);$ every fixed point of $\tilde{f}^p$ is elementary$\}$ and $H_p \subset T_p$ for all $p \geq 1$.

2.9. **Proposition.** *T_p and H_p are open and dense in $C^r(M^k, M)$ for all $p \geq 1$.*

Proof. It is clear that T_p and H_p are open in $C^r(M^k, M)$ for all $p \geq 1$. We will prove the density of T_p and H_p simultaneously by induction on p. Using 2.4.2 and 2.3 we can see that T_1 and H_1 are dense. Suppose that T_p and H_p are dense for some p. First note that $T_{p+1} \cap H_p$ is dense in H_p. In fact, let $EV: H_p \times M^k \to M^{2k}$ be the evaluation $EV(f, x) = (x, \tilde{f}^{p+1}(x))$. If $EV(f, x) \in \Delta$, then x is a periodic point of $\tilde{f}$ with period $k \leq p+1$. If $k = p+1$, Lemma 2.6 implies that $EV \pitchfork \Delta$ at (f, x); if $k < p+1$, x is a hyperbolic periodic point and by 2.7.2 we have that $EV \pitchfork \Delta$ at (f, x). Then $EV \pitchfork \Delta$. This proves that $T_{p+1} \cap H_p$ is dense in H_p. In particular T_{p+1} is dense in $C^r(M^k, M)$. Finally Lemma 2.3 can be used to prove that H_{p+1} is dense in $T_{p+1} \cap H_p$ and hence in $C^r(M^k, M)$. The proposition is proved.

2.10. Fix once for all a Riemannian metric on M^k. Let $B_\varepsilon(x)$ be the closed ball centered at x with radius $\varepsilon > 0$. We will define inductively open and dense subsets $G_p \subset H_p$ and continuous functions $E_p^i: G_p \to R_+$, $1 \leq i \leq p$, such that:

i) $G_{p+1} \subset G_p, p \geq 1$;

ii) $E_p^i = E_{p+1}^i$, $0 < i \leq p$;

iii) If $f \in H_p$ and x is a saddle point of f with period $j \leq p$, put $L_f^s(x) = B_\varepsilon(x) \cap W_{loc}^s(f, x)$ and $L_f^u(x) = B_\varepsilon(x) \cap W_{loc}^u(f, x)$, where $\varepsilon = E_p^j(f)$. If $f \in G_p$, then $\tilde{f}^{kj}(L_f^u(x)) \subset W_{loc}^u(f, x)$;

iv) If $f \in G_{p+1}$ and x is a saddle point of f with period $j \leq p$, then $\tilde{f}^j(L_f^u(x))$ has no periodic point with period $\leq p+1$ with exception of x;

v) Let $f \in G_p$ and let x and y be saddle points of f with period $j \leq p$ and $\ell \leq p$ respectively. Then $[L_f^s(x) \cup \tilde{f}^j(L_f^u(x))] \cap [L_f^s(y) \cup \tilde{f}^\ell(L_f^u(y))] = \varnothing$ if $x \neq y$ and $\tilde{f}^j(L_j^u(x)) \cap L_f^s(x) = \{x\}$ if $x = y$.

We begin with $G_1 = H_1$. It is easy to define a continuous function $E_1^1: G_1 \to R_+$ satisfying iii) and v).

Suppose that we have G_j and E_j^i, $1 \leq i \leq j \leq p$ for some $p \geq 1$. Denote by G_{p+1} the subset of $f \in G_p \cap H_{p+1}$ with satisfy iv). We claim that G_{p+1} is dense in $G_p \cap H_{p+1}$. Let $f \in G_p \cap H_{p+1}$. We can suppose that the following property is satisfied: if x is a saddle point of f with period $\leq p$ then for all $y = (y_1, \ldots, y_k) \in W_{loc}^u(f, x)$ there exists a non zero vector $(0, \ldots, 0, v) \in T_{y_1}M \times \ldots \times T_{y_k}M$ such that $(0, \ldots, 0, v) \notin T_y W_{loc}^u(f, x)$. Hence we can remove every saddle point y with period $p+$ from $\tilde{f}^j(L_k^u(x))$, for all saddle point x with period $\leq p$, by perturbing f locally in the direction of $(0, \ldots, 0, v)$. This procedure gives us an element of G_{p+1} arbitrarily close to f. Hence G_{p+1} is dense in $G_p \cap H_{p+1}$. Furthermore it is clear that G_{p+1} is open and there exist continuous functions E_{p+1}^i satisfiyng ii), iii) and v).

2.11. **Definition.** Given $p \in Z_+$ let $p: C^r(M^k, M) \to C^r(M^k, M)$ defined by $p(f) = \tilde{f}^p$. Let $V \subset M^k$ be a compact submanifold (perhaps with boundary). We consider the restriction map $R = R_V: C^r(M^k, M^k) \to C^r(V, M^k)$, defined by $R(g) = g \mid V$, and the evaluation $EV_{R_0P} = EV_P \mid C^r(M^k, M) \times V$. Clearly EV_{R_0P} is of class C^r.

2.12. **Lemma.** *Let V and W be closed C^r submanifolds of M^k such that $V \pitchfork W$. Let $p \geq k$ be an integer and let $f \in C^r(M^k, M)$.*

a) *Let $x \in V$ such that $\tilde{f}^j(x) \neq \tilde{f}^i(x)$, $0 \leq j < i \leq p-1$. Then $EV_{R_0P} \pitchfork W$ at (f, x), where $R = R_V$.*

b) *Suppose that $\tilde{f}$ has maximum rank on W and that $\tilde{f}(W) \subset W$. Let $x \in V$ be such that if $\tilde{f}^j(x) = \tilde{f}^\ell(x)$ then $\tilde{f}^i(x) \in W$ for all $i \geq j$. It follows $EV_{R_0P} \pitchfork W$ at (f, x), where $R = R_V$.*

Proof. a) We have

$$TEV_{R_0P}(f, x)(h, v) = \sum_{i=0}^{p-1} T\tilde{f}^i(\tilde{f}^{p-1}(x))\, \tilde{h}(\tilde{f}^{p-i-1}(x)) + T\tilde{f}^p(x)\, v,$$ where $\tilde{h} = (0, \ldots, 0, h)$, $h \in \Gamma^r(f^* TM)$, and $v \in T_x V$ (cf. 2.5). Let $(f, x) \in C^r(M^k, M) \times V$ such that $EV_{R_0P}(f, x) \in W$. For $0 \leq j \leq k-1$ let $\tilde{h}_j \in \Gamma^r(f^* TM^k)$ be defined by

$$\begin{cases} \tilde{h}_j(\tilde{f}^{p-i-1}(x)) = 0 & \text{for all } i \neq j, \\ \tilde{h}_j(\tilde{f}^{p-j-1}(x)) = (0, \ldots, 0, v) \in T_{\tilde{f}^{p-j}(x)} M^k, \end{cases}$$

where v lies in a bases of the last factor of $T_{\tilde{f}^{p-j}(x)}M^k = T_{y_1} \times \ldots \times T_{y_k}M$. Then $TEV_{R_0P}(f, x)(h_j, 0) = T\tilde{f}^j(\tilde{f}^{p-j}(x))(0, \ldots, 0, v)$, and so we obtain a base of $T_{\tilde{f}^p(x)}M^k$.

b) Let $EV_{R_0P}(f, x) \in W$. If $x \in W$ we have $TEV_{R_0P}(f, x)(0, v) = T\tilde{f}_x^p v$ for all $v \in T_x V$. Hence $EV_{R_0P} \pitchfork W$ at (f, x). Now suppose that $x \notin W$. Let $s = max\{n \in Z;\ 0 \leq n \leq p-1$ and $\tilde{f}^n \notin W\}$. Note that if $\pi: C^r(M^k, M) \times V \to C^r(M^k, M)$ is the canonical projection, then

$$\mathrm{EV}_{R_0p} = \mathrm{EV}_{R_0(p-s-1)} \circ (\pi, \mathrm{EV}_{R_0(s+1)}). \text{ Hence } \mathrm{TEV}_{R_0p}(f,x)(h,v) =$$
$$= \mathrm{TEV}_{R_0(p-s-1)}(f, \tilde{f}^{s+1}(x)) \circ T(\pi, \mathrm{EV}_{R_0(s+1)})(f,x)(h,v) =$$
$$= \mathrm{TEV}_{R_0(p-s-1)}(f, \tilde{f}^{s+1}(x))(h, \mathrm{TEV}_{R_0(s+1)}(f,x)(h,v) =$$
$$= \sum_{i=0}^{p-s-2} T\tilde{f}^{i}(\tilde{f}^{p-i}(x))\, \tilde{h}(\tilde{f}^{p-i-1}(x)) +$$
$$+ T\tilde{f}^{p-s-1}(\tilde{f}^{s+1}(x))\, \mathrm{TEV}_{R_0(s+1)}(f,x)(h,v).$$

We have by a) that $\mathrm{EV}_{R_0(s+1)} \pitchfork W$ at (f,x). Since $\tilde{f}^{s+1}(x) \in W$ and $\tilde{f}$ has maximal rank on W, we can show that $\mathrm{TEV}_{R_0p}(f,x)$ is onto. This proves the lemma.

At this point we can complete the proof of theorem 1.6 using the following proposition of [Sh].

2.13. **Proposition.** *The set $D_p^1 = \{ f \in G_p$; if x is a periodic point of $\tilde{f}$ with period $j \le p$ then $\tilde{f}^{ij} \pitchfork L^s_f(x)$ for all positive integer i such $ij \le p\}$ is open and dense in $C^r(M^k, M)$ for all $p \ge 1$.*

2.14. **Proposition.** *The set $D_p^2 = \{ f \in G_p$; if x and y are saddle points of $\tilde{f}$ with periods $i \le p$ and $j \le p$ respectively then $(\tilde{f}^{\ell j} \mid L^u_f(y)) \pitchfork L^s_f(x)$ for all positive integer ℓ such that $\ell j < p\}$ is open and dense in $C^r(M^k, M)$ for all $p \ge 1$.*

Proof of 1.6. Follows from the inclusion $\bigcap_{p=1}^{\infty} (D_p^1 \cap D_p^2) \subset KS^r(M^k, M)$.

2.15. **Remark.** The map $(C^r(M^k, M) \to C^r(M^k, M^k)$, $f \to \tilde{f}$, is an embedding, therefore $C^r(M^k, M)$ can be considered as a submanifold of $C^r(M^k, M^k)$. Theorem 1.6 establishes that $KS^r(M^k) \cap C^r(M^k, M)$ is residual in $C^r(M^k, M)$, where $KS^r(M^k)$ is the space of C^r Kupka-Smale endomorphisms of M^k (see [Sh]).

3. A stability property.

Now we discuss another subject also studied in our thesis, which aims to describe stability properties of diffeomorphisms under small perturbations in the space of 2-endomorphisms.

Let $\phi : M \circlearrowleft$ be a C^r diffeomorphism and let $\tilde{f}(x,y) = (y, \phi(y))$ be the lifting of $(x,y) \to \phi(y)$. The graph of ϕ $V = \{(x, \phi(x));\ x \in M\}$ is an $\tilde{f}$-invariant C^r submanifold of M^2 and $\tilde{f} \mid V$ is a C^r diffeomorphism. The "horizontal leaves" $M_p = \{(x,y) \in M^2;\ x \in M\}$ where $p = (\phi^{-1}(y), y) \in V$ constitute a stable invariant family (see [H, P, S] for $\tilde{f}$ such that $\tilde{f}(M_p) = (y, \phi(y))$. Note that $\tilde{f}(M^2) = V$ and $\tilde{f} \mid V$ is C^r conjugate to ϕ. We have proved that there exists a neighborhood η of $\tilde{f}$ in $C^r(M^2, M^2)$ such that every $g \in \eta$ has an invariant submanifold $V_g \subset M^2$ satisfying the following properties: $g \mid V_g$ is a C^r diffeomorphism and $V_g \overset{C^r}{\to} V$ if $g \overset{C^r}{\to} f$. Moreover given a small neighborhood U of V_f in M^2 for all $g \in \eta$ we have that $g(M^2) \subset U$ and U is foliated by a continuous family $\{W^s_\delta(g,p)\}_{g \in V_g}$ of C^r stable discs pairwise disjoint and transversal to V_g. We have the following characterization

$$x \in W^s_\delta(g,p) \Leftrightarrow d(g^n(x), g^n(p)) \le C\rho^n \quad \forall\ n \ge 0,$$

for some constants $C > 0$ and $0 < \rho < 1$.

As a consequence it follows that if ϕ is structurally stable (for example if ϕ satisfies Axiom A and the strong transversality condition) then $g \mid V_g$ is conjugate to ϕ. In such case we can consider that ϕ describes the dynamics of g.

This problem in a more general context is the subject of a forthcoming paper.

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