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The topological conjugacy problem for generalized Hénon mappings: some negative results.

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Abstract.

Mappings of the plane, introduced by M. Hénon and R. Lozi, are presented as perturbations of endomorphisms of the line. When some heteroclinic tangencies occur, which allow topological conjugacy between the corresponding endomorphisms, and arbitrarily close to the endomorphisms case, we prove the nonexistence of topological conjugacy between Hénon mapping and Lozi mapping. The proof, as well as related results, relies on the existence of a topological invariant previously introduced by J. Palis.

1. Introduction.

In [6], M. Hénon introduced a two-parameter family of diffeomorphisms of $\mathbb{R}^2$:

(1) $$H_{a,b} : (x, y) \to (1 - ax^2 + y, bx).$$

Numerical computations performed with this mapping led to the conjecture that, for any b with $|b| \in]0, 1[$, at least $|b|$ small enough, and some values of a depending on b, $H_{a,b}$ possesses a strange attractor (for recent works on the meaning one should attribute to these last words, see [15]). A proof of this conjecture (and more generally a clear understanding of what is going on in the dynamics of (1)) would be of prime interest for dynamical systems theory, and more particularly, for its applications to the description of the onset of turbulence. A similar conjecture has been proved recently by M. Misiurewicz [11] for the two-parameter family of homeomorphisms first investigated by R. Lozi [8]:

(2) $$L_{a,b} : (x, y) \to (1 - a\,|\,x\,| + y, bx).$$

More precisely, he proved the following

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Theorem 1 (Misiurewicz). *If a and b verify the following conditions:*

1. $0 < b < 1$, $a > 0$,
2. $2a + b < 4$,
3. $b < \dfrac{a^2 - 1}{2a + 1}$,
4. $a\sqrt{2} > b + 2$,

then $L_{a,b}$ has a strange attractor.

For $b \neq 0$, and when their non-wandering set is nonvoid, $H_{a,b}$ and $L_{a,b}$ can be written respectively as:

(3) $$H_{R,b} : (X, Y) \to (Y, RY(1 - Y) + bX),$$

and:

(4) $$L_{a,b} : (U, V) \to \left(V, \frac{a}{2}(1 - |1 - 2V|) + bU\right),$$

where use has been made of the changes of variables:

(5) $$\begin{cases} (X, Y) = (\alpha y + \beta, \alpha' x + \beta'), \\ \text{with:} \\ R = 1 - b + \varepsilon[(1-b)^2 + 4a]^{1/2}, \alpha' = \dfrac{R}{4} + \dfrac{b}{2} - \dfrac{1}{2}, \alpha = \dfrac{\alpha'}{b}, \beta = \beta' = \dfrac{1}{2}, \\ \varepsilon = \pm 1, \text{ only } \varepsilon = -1 \text{ is possible if } a = 0. \\ \text{Here, we shall take } a \neq 0 \text{ and } \varepsilon = 1. \end{cases}$$

and:

(6) $$\begin{cases} (U, V) = (\gamma y + \delta, \gamma' y + \delta'), \\ \text{with:} \\ \gamma' = \dfrac{a + b}{2} - \dfrac{1}{2}, \gamma = \dfrac{\gamma'}{b}, \delta = \delta' = \dfrac{1}{2}. \end{cases}$$

Both $H_{R,b}$ and $L_{a,b}$ can be considered, at least when $|b|$ is small enough, as perturbations of an endomorphism of $\mathbb{R}^2$. More generally, for an n parameter family:

(7) $$f_a : x \to f_a(x),$$

of endomorphisms of $\mathbb{R}$, it is interesting to consider the elements of the $(n+1)$ parameters family:

(8) $$F_{a,b} : (x, y) \to (y, f_a(y) + bx),$$

of homeomorphisms of $\mathbb{R}^2$, when b is small, as perturbations of the endomorphisms:

(9) $$F_{a,0} : (x, y) \to (y, f_a(y)),$$

whose dynamics in turn is easily deduced from the dynamics of f_a. (To my knowledge, this kind of remark first appeared in [2].)

Returning now to the maps:

(10) $$h_R : x \to Rx(1 - x),$$

and:

(11) $$\ell_a : x \to \frac{a}{2}(1 - |1 - 2x|),$$

which generate respectively $H_{R,b}$ and $L_{a,b}$, let us remark that, respectively for $0 \leq R \leq 4$ and $0 \leq a \leq 2$, these maps, restricted to $[0, 1]$, are endomorphisms of the interval. Most results on sensitive dependence to initial conditions for such maps of an interval into itself are expressed in terms of probabilistic invariant measures absolutely continuous with respect to Lebesque's (i.m.) [3, 7, 10] (see however [4, 15]). ℓ_a admits such i.m.'s for $1 \leq a \leq 2$ while the set of "good" values of R for h_R has a complicated (Cantor) structure. When h_R preserves an i.m., it is topologically conjugate to some ℓ_a [10]. As pointed out in [11], it is thus tempting to look for topological conjugacies relating some Hénon maps to some Lozi maps with (a, b) verifying the conditions in Theorem 1 (indeed, Misiurewicz mentioned that it may happen that such conjugacy does not exist: see also [16]).

Let us now recall an old conjugacy result relating h_R to ℓ_a [18]:

Theorem 2 (Ulan, Von Neumann). *h_4 is topologically conjugate to ℓ_2. The conjugacy is given by*

$$x \to \frac{2}{\pi} \text{ arc sin } \sqrt{x}$$

which allows one to get $\dfrac{1}{\pi} \dfrac{dx}{\sqrt{x(1 - x)}}$ *as i.m. for h_4.*

Since the surjectivity of h_4 and ℓ_2 restricted to $[0, 1]$ is hardly generalized to a property of some $H_{R,b}$ or $L_{a,b}$ for $b \neq 0$, we shall characterize the endomorphisms in a different way: $R = 4$ (respectively $a = 2$) is the lowest value of the parameter such that a point in $[0, 1]$ has the trivial fixed point 0 as limit of its forward images and the non trivial fixed point as limit of one sequence of its backward images under h_R (respectively ℓ_a). Otherwise speaking, adapting the terminology introduced by L. Block for one dimensional endomorphisms [1] (see also [9]), $R=4$ (respectively $a=2$) corresponds to a first heteroclinic tangency of the unstable manifold of the non trivial fixed point with the stable manifold of 0, when increasing R in h_R (respectively a in ℓ_a). The same first tangency, between the unstable manifold of the non trivial fixed point and the stable manifold of the origin; seems then a good candidate for the research of a conjugacy of a $H_{R,B}$ to an $L_{a,b}$. However, the main result of this paper is the following:

Theorem A. *On the line* $b>0, a=2-\frac{b}{2}$ *in* $a-b$ *parameters space, there is a sequence* (a_i, b_i) *converging to* $(2,0)$ *such that no* L_{a_i,b_i} *is topologically conjugate to a Hénon mapping.*

Note that the line $a=2-\frac{b}{2}$, $b>0$ in the $a-b$ parameters space, which corresponds to the heteroclinic tangency above mentioned, is also a frontier of the open domain defined in Theorem 1, when $b\leq\frac{14-\sqrt{136}}{5}$ (see Figure 1 in [11]). As is already the case for ℓ_2, the closure of the invariant "manifold" (see [11] for the use of this word for $L_{a,b}$) of the non trivial fixed point of $L_{2-b/2,b}$ is not a strange attractor since in any neighborhood of this closure, one can find points whose trajectory diverges. This closure appears however as a "strange invariant set," as well as $[0,1]$ when one considers ℓ_2 as a mapping from $\mathbb{R}$ to itself.

The paper is organized as follows: section 2 contains a geometric description of the mappings considered and of part of some of the invariant manifolds. In section 3, we prove Theorem A: the proof relies deeply on the existence of a topological invariant introduced by J. Palis in [14]. Related results are formulated in section 4 and further comments are reported in the conclusive part.

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2. Description of the mappings.

Let $c\in]0,1[$ and $f:\mathbb{R}\to\mathbb{R}$ be continuous, concave, differentiable except maybe at c, with $f'(x)>0$ for $x<c$, $f'(x)<0$ for $x>c$, $f(0)=$

$=f(1)=0$ and $f(c)$ close to 1. We suppose furthermore that there is a $x^*\in]c,1[$ with $f(x^*)=x^*$ and $f'(x^*)<-1$, which implies $f'(0)>1$: f admits two unstable fixed points. The two dimensional map:

$$F:(x,y)\to(y,f(y)), \tag{12}$$

maps $\mathbb{R}^2$ on the graph of the function $y=f(x)$. F admits $0=(0,0)$ and $M=(x^*,x^*)$ as unique fixed points. The following definition will be well adapted for all our needs:

Definition. Let g be an endomorphism of a metric space into itself. A local stable (respectively, unstable) manifold at a fixed point A is defined as:

$$W^S_{A,loc}=\{B\mid \lim_{n\to\infty}\mathrm{dist}(A,g^n(B))=0 \text{ and } \mathrm{dist}(A,g^n(B))<\varepsilon \text{ for all } n\geq 0\},$$

(respectively:

$$W^U_{A,loc}=\{B\mid \lim_{n\to\infty}\mathrm{dist}(A,g^{-n}(B))=0 \text{ and } \mathrm{dist}(A,g^{-n}(b))<\varepsilon \text{ for all } n\geq 0\})$$

for some $\varepsilon>0$. A global stable (respectively unstable) set (we shall say also, often abusively, manifold) at a fixed point A is defined as $W^S_A=\bigcup_{n=0}^{\infty} g^{-n}(W^S_{A,loc})$ (respectively, $W^u_A=\bigcup_{n=0}^{\infty} g^n(W^u_{A,loc})$). A further subscript will be added when necessary to indicate to which mapping one refers.

Remark. $W^S_{loc,A}$ is the point A in the case of f as defined above. W^S_A and W^u_A are the usual stable and unstable manifolds for F when F is a diffeomorphism.

The following proposition is straightforward (see [1] for more general considerations):

Proposition 1. *For a map* $f:\mathbb{R}\to\mathbb{R}$ *as defined above, one has:*

$$W^S_{x^*}=\{y\in\mathbb{R}\mid f^n(y)=x^* \text{ for some } n>0\},$$
$$W^u_{x^*}=[f^2(c),f(c)] \text{ if } f(c)\leq 1 \text{ and }]-\infty,f(c)] \text{ if } f(c)>1,$$
$$W^S_0=\{y\in\mathbb{R}: f^n(y)= \text{ for same } n>0\}\ (W^S_0 \text{ reduces to } \{0,1\} \text{ if } f(c)<1),$$
$$W^u_0=]-\infty,f(c)]=f(\mathbb{R}).$$

One deduces then (see also [9]):

Proposition 2. *For the map $F:\mathbb{R}^2 \to \mathbb{R}^2$, one has:*

$$W^S_M = \{\text{horizontal lines through each point of } W^S_{x^*,f}\},$$
$$W^u_M = \text{Graph}\,[f\,|_{W^u_{x^*,f}}],$$
$$W^S_0 = \{\text{horizontal lines through each point of } W^S_{0,f}\},$$
$$W^u_0 = \text{Graph}\,[f\,|_{]-\infty,f(c)]}].$$

Before proving theorem A, we shall describe, for the homeomorphisms:

$$(13) \qquad (b>0, \text{small}) \qquad F_b:(x,y)\to(y, f(y)+bx),$$

the invariant sets relative to the fixed points 0 and M_b which may be viewed as perturbations of the fixed points 0 and M for F (see also [9]).

It is convenient to choose a small $\zeta>0$ and to consider the successive images, under F_b, of the square $Q=Q_1Q_2Q_3Q_4$ where $Q_1=(-\zeta,-\zeta)$ and $Q_3=(1+\zeta, 1+\zeta)$.

$F_b(Q)$ is the intersection with the vertical strip $-\zeta\le x\le 1+\zeta$, of the parabolic strip $f(x)-b\zeta\le y\le f(x)+b(1+\zeta)$. Using the change of coordinates:

$$(14) \qquad (X,Y)=(x,[y-f(x)]/b),$$

which brings $F_b(Q)$ on Q, one gets:

$$(15) \qquad (X,Y)\to(RX(1-X)+bY, Y),$$

as new expression for F_b: the same as before but with the roles of coordinates exchanged. This allows to construct the sucessive iterates of Q in a quite comprehensive way [17], the first steps being illustrated in figure 1 (see also [17]). For n small enough, $F^n_b(Q)$ is roughly made of 2^n "parabolas" which can be numbered from 1 to 2^n, starting from the upper one. We are now in position to describe W^u_{M,F_b} as well as necessary for our purpose. This description will be made in four steps.

Step 1. Using proposition 2, W^u_{M,F_b} contains an arc, to which M belongs, near the graph of $y=f(x)$. This arc, like M, lies in the first parabola of $F^2_b(Q)$. The part of this arc with positive abscissa and to the left of M is mapped by F_b onto that part of the arc joining M to the right end of the arc. This right part is in turn mapped by F_b onto the part joining M to the left end. With the notations of Figure 1, one gets:

$$\widehat{P_2P_3}=F_b(\widehat{P_1P_2})=\widehat{P_2M}\cup\widehat{MP_3}.$$

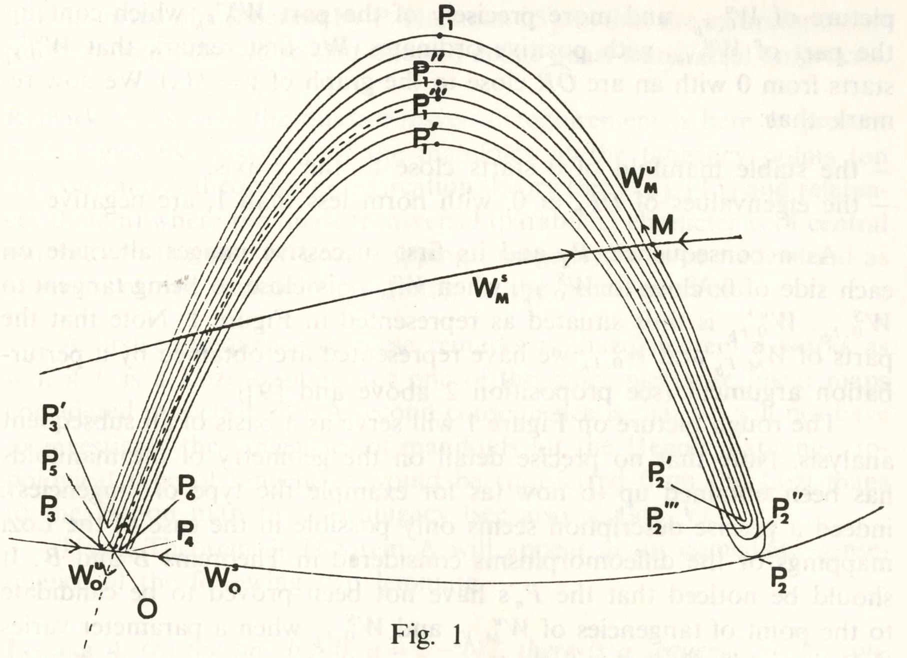

Fig. 1

Step 2. Since $f(c)\simeq 1$, the projection of $\widehat{P_3P_1}$ on the y-axis covers, at least almost completely, the interval $-\varepsilon\le y\le 1+\varepsilon$. As a consequence, $F_b(\widehat{P_1P_3})$ will run at least almost all along $F_b(Q)$. More precisely, $F_b(\widehat{P_1P_3})=$ $=\widehat{P_2P_4}$ will be in the second parabola of $F^2_b(Q)$.

Step 3. $\widehat{P_4P_2}$ is below $\widehat{P_3P_2}$. Ordering from left to right, one gets successively $\widehat{P_3P_1}$, $\widehat{P_4P'_1}$, $\widehat{P'_1P_2}$ and $\widehat{P_1P_2}$. The image of $\widehat{P_2P_4}$ will be made of two parabolas, images of $\widehat{P_2P'_1}$ and $\widehat{P'_1P_4}$:

$$\widehat{P_3P_5}=F_b(\widehat{P_2P_4})=\widehat{P_3P'''_1}\cup\widehat{P''_1P'_2}\cup\widehat{P'_2P'''_1}\cup\widehat{P'''_1P_5}=\widehat{P_3P'_2}\cup\widehat{P'_2P_5}.$$

Step 4. $\widehat{P_3P_2}$, $\widehat{P_3P'_2}$, $\widehat{P_3P'_2}$ and $\widehat{P_4P_2}$ are respectively in the first, second, third and fourth parabolas of $F^4_b(Q)$. We now look at the image of $\widehat{P_3P_5}$: it will be made of four parabolas, images of $\widehat{P_3P''_1}$, $\widehat{P''_1P'_2}$, $\widehat{P'_2P'''_1}$ and $\widehat{P'''_1P_5}$.

$$\widehat{P_4P_6}=F_b(\widehat{P_3P_5})=\widehat{P_4P''_2}\cup\widehat{P''_2P'_3}\cup\widehat{P'_3P'''_2}\cup\widehat{P'''_2P_6}.$$

The part of W^u_{M,F_b} we have constructed at this point has eight arcs pertaining to the eight parabolas of $F^3_b(Q)$ and ordered as $\widehat{P_3P_2}$, $\widehat{P'_3P''_2}$, $\widehat{P'_3P'''_2}$, $\widehat{P_3P'_2}$, $\widehat{P_5P'_2}$, $\widehat{P_6P'''_2}$, $\widehat{P_4P''_2}$ and $\widehat{P_4P_2}$. We shall also need a rough picture of $W^u_{0.F_b}$, and more precisely of the part $W^{u+}_{0.F_b}$ which contains the part of $W^u_{0.loc}$ with positive ordinates. We first remark that $W^{u+}_{0.F_b}$ starts from 0 with an arc $\widehat{OR}$ close to the graph of $y = f(x)$. We now remark that:

– the stable manifold of 0 starts close to the x axis,
– the eigenvalues of DF_b at 0, with norm less than 1, are negative.

As a consequence, P_2 and its first successive images alternate on each side of 0, close to $W^S_{0.F_b}$, when $W^u_{M.F_b}$ is close to being tangent to $W^S_{0.F_b}$. $W^{u+}_{0.F_b}$ is thus situated as represented in Figure 1. Note that the parts of $W^S_{M.F_b}$ and $W^S_{0.F_b}$ we have represented are obtained by a perturbation argument (see proposition 2 above and [9]).

The rough picture on Figure 1 will serve as a basis of all subsequent analysis. Note that no precise detail on the geometry of the manifolds has been obtained up to now (as for example the type of tangencies): indeed a precise description seems only possible in the case of the Lozi mappings or the diffeomorphisms considered in Theorems *B* and *B'*. It should be noticed that the P_n's have not been proved to be candidate to the point of tangencies of $W^u_{M.F_b}$ and $W^S_{0.F_b}$ when a parameter varies if F_b is not further specified.

3. Proof of Theorem A.

The main tool of the proof is a topological invariant introduced by J. Palis in [14]. He proved the following:

Theorem 3 (Palis). *Let F, F' be two C^2 diffeomorphisms on two manifolds, with hyperbolic fixed (or periodic) points p and q, p' and q' of saddle type. Suppose that $W^u(q)$ and $W^s(p)$, $W^u(q')$ and $W^s(p')$ have one orbit γ, γ' of quasi-transversal intersection, respectively. Denote by ρ, ρ' the eingenvalues of $DF(q)$, $DF'(q')$ with norm less than one and μ, μ' the eigenvalues of $DF(p)$, $DF'(p')$ with norm greater than one. If f and f' are conjugate (even only in neighborhoods of γ and γ') then:*

$$\log|\mu|/\log|\rho| = \log|\mu'|/\log|\rho'|. \tag{16}$$

We need two remarks in order to apply this theorem to our problem.

Remark 1. The C^2-diffeomorphism character of the maps is mainly used to insure: a) the existence of smooth (C^1) linearization in the neighborhood of the saddle points, independently of any non-ressonance condition, in virtue of a theorem by P. Hartman [5], b) that the invariant manifolds are well behaved so that one can define quasi-transversal tangencies.

Remark 2. As well, the quasi-transversal requirement is here to prevent pathologies like crossing of the manifolds at the tangency points (on the contrary to theorems in bifurcation theory (see [12], [13] and references therein) where the quasi-transversal (parabolic) character is of central interest.) In the case of Lozi mappings, tangency is to be understood as a vertex of one manifold pertaining to the other manifold.

A first consequence of these remarks is that Theorem 3 works as well if *F* is a Lozi mapping (or one of the other piecewise linear maps considered in section IV). A second consequence is that we will not have to investigate the tangencies of manifolds for the Hénon mappings: topological type of tangencies would be transferred from the Lozi maps to the Hénon map by a conjugacy (see also section V).

Using Theorem 3, Theorem A will appear as an immediate consequence of the following two lemmata.

Lemma 4. *On the line $b > 0$, $a = 2 - b/2$, there is a sequence (a_i, b_i) converging to $(2, 0)$ such that, for the Lozi mapping W^u_M is tangent to W^s_M.*

Let us call $\log|\mu|/\log|\rho|$ in Theorem 3 the Palis invariant of type (p, q). Then we can formulate the:

Lemma 5. *The Lozi maps and the Hénon maps cannot have simultaneously their Palis invariants of types $(0, M)$ and (M, M) respectively equal when $b \neq 0$.*

Proof of Lemma 4. We need to know some pieces of the invariant manifolds of 0 and M for $L_{a,b}$. We start with the manifolds of M.
– M itself is given by:

$$(x_M, y_M) = \left(\frac{a}{1+a-b}, \frac{a}{1+a-b}\right). \tag{17}$$

– In the neighborhood of M, $F_{a,b}$ reads:

$$(x, y) \to (y, a - ay + bx), \tag{18}$$

and the Jacobian matrix is:

$$\begin{bmatrix} 0 & 1 \\ b & -a \end{bmatrix}. \tag{19}$$

with eigenvalues:

$$\lambda_M^u = \frac{-a-\sqrt{a^2+4b}}{2}\,; \qquad \lambda_M^s = \frac{-a+\sqrt{a^2+4b}}{2}, \tag{20}$$

and associated eigenvectors given respectively by:

$$\begin{pmatrix}1\\ \lambda_M^u\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}1\\ \lambda_M^s\end{pmatrix}. \tag{21}$$

– The first part of W_M^u we construct is a segment of the straight line (D) containing M:

$$(\mathrm{D}):\ y = \lambda_M^u x + (1-\lambda_M^u)x_M. \tag{22}$$

This segment is delimited by P_1, obtained by the intersection of (D) with the vertical line $x = 1/2$, and by P_2 obtained as the image of P_1. By setting that $P_1 \in W_0^s$, one gets the condition:

$$a = 2 - \frac{b}{2}, \tag{23}$$

for the first tangency of W_M^u and W_0^s. Indeed, the same kind of arguments as those developed at the end of this proof of Lemma 4 show that $P_2 \in W_0^s$ is a good condition to impose in order to obtain this first tangency. Using (23), one gets:

$$x_M = y_M = \frac{4-b}{6-3b}, \tag{24}$$

$$\lambda_M^u = -2;\ \lambda_M = \frac{b}{2}. \tag{25}$$

$$(x_{P_1}, y_{P_1}) = \left(\frac{1}{2}, \frac{2}{2-b}\right); \quad (x_{P_2}, y_{P_2}) = \frac{2}{2-b}, \frac{b}{b-2}. \tag{26}$$

One computes then easily the points $P_3, P_1', P_4, P_1'', P_2', P_1''', P_4, P_1^{(4)}, P_2''$, $P_1^{(5)}, P_3'$. Note that all $P_n's$ are $(n-1)^{th}$ images of P_1 and that $P_n^{(m)\prime}s$ are $(n-1)^{th}$ images of $P_1^{(m)}$: the $P_1^{(m)\prime}s$ are on a vertical segment and the $P_n^{(m)\prime}s$ are on the $(n-1)^{th}$ image of this segment, which become fairly complicated when n is large enough.

– The part of W_M^s which we shall use here is the part of the line (D'):

$$(D'):\ y = \frac{b}{2}x + \frac{b^2-6b-8}{12-6b}, \tag{27}$$

between M and the intersection of (D') with the line $y = 1/2$. Note that for b small enough, (D') is below the segment $\overline{P_1'P_1}$.

– We need also a small part of the invariant manifolds of 0. We first note that $L_{a,b}$ reduce to:

$$(x, y) \to (y, ay + bx), \tag{28}$$

in the $y \leq 0$ half space. With (23), the Jacobian matrix is there:

$$\begin{pmatrix}0 & 1\\ b & a\end{pmatrix} = \begin{pmatrix}0 & 1\\ b & 2-b/2\end{pmatrix}, \tag{29}$$

with eigenvalues:

$$\lambda_0^u = \frac{a+\sqrt{a^2+ub}}{2} = 2, \quad \lambda_0^s = \frac{a-\sqrt{a^2+ub}}{2} = -\frac{b}{2}, \tag{30}$$

and eigenvectors:

$$\begin{pmatrix}1\\ \lambda_0^u\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\ \lambda_0^s\end{pmatrix}. \tag{31}$$

The piece of W_0^u we will consider is the segment of (Δ):

$$(\Delta):\ y = 2x, \tag{32}$$

between 0 and the intersection of (Δ) with the line $x = 1/2$.

The piece of W_0^s of interest is the segment of (Δ'):

$$(\Delta'):\ y = -\frac{b}{2}x, \tag{33}$$

between the intersection of (Δ') with the line $y = 1/2$ and the image of this last point under $F_{2-b/2,b}$.

The part of invariant sets we have constructed is as represented on Fig. 1, except of course that $\widehat{P_3P_1}, \ldots, \widehat{P_2P_1}$ are now straight lines. The proof of Lemma 4 relies on this picture and on the two following remarks.

Remark 1. For $b = 0$, $a = 2$, all $P_n's$ and all $P_n''s$, for $n \geq 3$ coincide with 0: as a consequence all $P_n''s$, for $n < N$ given, remain as close as one wants from 0 from $a = 2 - b/2$ and b sufficiently small.

Remark 2. $F_{a,b}$ is linear in the half plane $y \leq 1/2$. The dynamics there is the usual linear dynamics governed by a saddle point (0 in this case): in particular, the successive images P'_n, $n > 2$ of P'_1 will run almost along the piece of W^u_0 we have constructed, as long as they remain in the half space $y \leq 1/2$. On the other hand, the line $y = 1/2$ is mapped under $F_{a,b}$ on the line $x = 1/2$: if P'_n is on $y = 1/2$, its image is on $x = 1/2$, and and more precisely on the segment $\widehat{P_1 P'_1}$, thus above the part of W^s_M we have constructed.

Now take any $\tilde{b}'_0 > 0$ small enough: one can then find n_0 such that, with obvious notations:

$$y_{P'_{n_0-1}}(\tilde{b}'_0) < \frac{1}{2} \quad \text{and} \quad y_{P'_{n_0}}(\tilde{b}'_0) \geq \frac{1}{2}\,. \tag{34}$$

Thus there is a $b'_0 \leq \tilde{b}'_0$ with:

$$y_{P'_{n_0}}(b'_0) = \frac{1}{2}\,, \tag{35}$$

and a sequence $\{b'_i\}$ converging to 0 with:

$$y_{P'_{n_0+i}}(b'_i) = \frac{1}{2}\,. \tag{36}$$

One deduces then the existence of a sequence $\{b_i\}$, with:

$$b_i \in \,]b'_i,\ b'_{i+1}[\,, \tag{37}$$

defined by:

$$y_{P'_{n_0+i}}(b'_i) \in \overline{(D')}, \tag{38}$$

where $\overline{(D')}$ is the piece of W^s_M we have constructed. This sequence corresponds to homoclinic tangencies of the invariant manifolds W^u_M and W^s_M, in the sense that a vertex of W^u_M belongs to W^s_M.

Proof of Lemma 5.

The Palis invariants of types $(0, M)$ and (M, M) for $L_{a,b,a} = \dfrac{2-b}{2}$ are both equal to:

$$p = Log^2 / Log\,(b/2). \tag{39}$$

For $H_{R,B}$, the fixed points are 0 and M given by:

$$(n_M, y_M) = \left(\frac{R+B-1}{R},\ \frac{R+B-1}{R}\right). \tag{40}$$

The Jacobian matrix is:

$$\begin{pmatrix} 0 & 1 \\ B & R-2Ry \end{pmatrix}, \tag{41}$$

with eigenvalues:

$$\lambda = \frac{R - 2Ry \pm \sqrt{(R-2Ry)^2 + 4B}}{2}\,. \tag{42}$$

For R near 4 and B small, this gives:

$$\lambda^u_M = \frac{1}{2}\,(2 - R - 2B - \sqrt{(2-R-2B)^2 + 4B}), \tag{43}$$

$$\lambda^s_M = \frac{1}{2}\,(2 - R - 2B + \sqrt{(2-R-2B)^2 + 4B}), \tag{44}$$

$$\lambda^u_0 = \frac{1}{2}\,(R + \sqrt{R^2 + 4B}). \tag{45}$$

The Palis invariants of types $(0, M)$ and (M, M) are respectively:

$$p_1 = \frac{Log\ \lambda^u_0}{Log\ \lambda^s_M} \quad \text{and} \quad p_2 = \frac{Log - \lambda^u_M}{Log\ \lambda^s_M}\,. \tag{46}$$

In order to prove Lemma 5, one has only to check that one cannot have $p_1 = p_2$: supposing equality gives $\lambda^u_0 = -\lambda^u_M$ or:

$$2B - 2 + \sqrt{(2-R-4B)^2 + 4B} = \sqrt{R^2 + 4B}\,, \tag{47}$$

which implies:

$$R(B-1) = (1-B)\sqrt{R^2 + 4B}, \tag{48}$$

and thus is always impossible, since we have supposed $R > 0$. This concludes the proof of Lemma 5 and Theorem A.

4. Related results.

The arguments in the proof of Theorem A must be slightly adapted if one wants to prove the same kind of non conjugacy results between

$L_{a,b}$ and maps such that $\lambda_0^u \equiv \lambda_M^u$: one can then introduce a sequence (a_i', b_i') on $a = 2 - b/2$ such that, for $L_{a_i', b_i'}$, some $P'_{n(i)}$ belongs to the stable manifold of the periodic point of period 2.

In particular, we have considered the C^1 diffeomorphisms $G_{\varepsilon,a,b}$ associated with the one dimensional mappings (see [3]):

$$(49) \qquad g_{\varepsilon,a}: x \to \begin{cases} ax & , \quad x \leq \frac{1}{2} - \varepsilon \\ \frac{-a}{2}x^2 + \frac{a}{2}x + \frac{a}{2} + \frac{a\varepsilon}{2} - \frac{a}{8\varepsilon}, & \left| x - \frac{1}{2} \right| \leq \varepsilon \\ a(1-x) & , \quad x \geq \frac{1}{2} + \varepsilon \end{cases}$$

for $a > 0$, $\varepsilon > 0$ small. $g_{\varepsilon,a}$ is C' and C^0 – as close as one wants from ℓ_a. However, one has the following:

Theorem B. *On the line $b > 0$, $a = \frac{2-b}{2}$, there is a sequence (a_i', b_i') converging to (2,0) such that no $L_{a'_i, b'_i}$ is topologically conjugate to a $G_{\varepsilon, a, \beta}$ with ε small enough.*

The next result illustrates another aspect of how restrictive are the Palis invariants: if we consider the homeomorphisms $K_{e,d,b}$ associated with the one dimensional mappings:

$$(50) \qquad k_{e,d}: x \to \begin{cases} ex, \ x < c \\ \qquad\qquad c = \frac{d}{e+d} \in]0, 1[, \\ d(1-x), \ x \geq c \end{cases}$$

with $e > 0$, $d > 0$, we get the following:

Theorem C. *If (a_i, b_i) is the sequence defined in Theorem A, no L_{a_i, b_i} is topologically conjugate to a $k_{e,d,b}$ with $d \neq e$.*

This last theorem is proved using the same Palis invariants as Theorem A.

Before ending this section, let us mention that the methods used here, finding sequences of double tangencies, can be used to investigate other lines of the (a, b) parameter space of Lozi maps: for example, one gets similar results on the line $b > 0$, $a = 1/2\sqrt{3b^2 + 4 + \sqrt{(3b^2+4)^2 - 32b^3}}$ which corresponds to first tangencies of W_0^u with W_0^s when b is small enough.

5. Some Comments.

We have tried to replace the sequence of points in theorems A, B, C by a continuous piece of the line $a = 2 - b/2$ starting from $b = 0$, by considering mainly the signature invariants of Birkhoff. Such tentatives have been unsuccessful up to now. However, Theorem B can be somewhat improved as follows: replace $G_{\varepsilon,a,b}$ by $\tilde{G}_{\varepsilon,a,b}$ obtained by smoothing $G_{\varepsilon,a}$ near $1/2 - \varepsilon$ and $1/2 + \varepsilon$ (C^3 is sufficient, see [12] Theorem 3). For $\tilde{G}_{\varepsilon,a,b}$ (as well as for $G_{\varepsilon,a,b}$), the homoclinic tangencies and heteroclinic tangencies of interest can be proved to be non degenerate (of parabolic type). Then, using Newhouse's results ([12] and references therein) on homoclinic tangencies, we get the following:

Theorem B'. *On the line $b > 0$, $a = 2 - b/2$, there is a sequence (a_i', b_i') converging to $(2, 0)$ such that no $L_{a_i', b_i'}$ is topologically conjugate to a $\tilde{G}_{\varepsilon,\alpha,\beta}$, with ε small enough. For any such ε, there is a sequence $(\alpha_i'(\varepsilon), \beta_i'(\varepsilon))$ corresponding to the same tangencies of invariant manifolds of 0 and M as for $L_{a_i', b_i'}$, and for any $\delta > 0$, $\varepsilon > 0$ small enough, there are subsets $S_{\varepsilon,i,\delta}$ of the line of heteroclinic tangency in the (α, β) plane such that:*

- *$S_{\varepsilon,i,\delta}$ is contained in an interval of length δ centered at (α_i', β_i') and the Lebesgue measure $\mu S_{\varepsilon,i,\delta}$ is positive*
- *if $(\alpha, \beta) \in S_{\varepsilon,i,\delta}$, $G_{\varepsilon,\alpha,\beta}$ has infinitely many sinks.*

Note that in view of the results of [13], $S_{\mu,i,\delta}$ could be somewhat small, with $\lim_{\delta \to 0} \frac{\mu(S_{\mu,i,\delta})}{\delta} = 0$, if one could extend the results in [13] which are proved for diffeomorphisms with finitely many orbits in their limit set.

In [12], S. Newhouse announces the non degenerate character of some homoclinic tangencies of the Hénon mapping. He does not study the whole parameters range where such non degenerate tangencies occur but it is likely that a theorem similar to B' holds for the Hénon mapping.

Of course, the existence of sinks, by itself, is enough to prevent the conjugacy of a homeomorphism to a $L_{a,b}$, with (a, b) as in Misiurewicz's theorem reported in section 1, but:

- it is not proved that sinks can occur on the line of first heteroclinic tangency in an arbitrary neighborhood of $b = 0$, for diffeomorphisms of Hénon's type.
- Theorem C shows that, within the class of generalized Hénon mappings, sinks are not essential in the lack of conjugacy.

References

[1] Block, L.; *Homoclinic points of mappings of the interval*, Proc. A.M.S. 72, 576-580 (1978).

[2] Bridges, R., Rowlands, G.; *On the analytic form of some strange attractors*, Phys. Lett. 63A, 189-190 (1977).

[3] Collet, P., Eckmann, J. P.; *On the abundance of aperiodic behaviour for maps on the interval*, Commun. Math. Phys. 73, 115-160 (1980) and to appear.

[4] Guckenheimer, J.; *Sensitive dependence to initial conditions for one dimensional maps*, Commun. Math. Phys. 70, 133-160 (1979).

[5] Hartman, P.; *On local homeomorphisms of euclidian spaces*, Bol. Soc. Mat. Mexicana 5, 200-241 (1960).

[6] Hénon, M.; *A two-dimensional mapping with a strange attractor*, Commun. Math. Phys. 50, 69-77 (1976).

[7] Jakobson, M. V.; *Absolutely continuous invariant measures for one parameter families of one-dimensional maps*, Commun. Math. Phys. 81, 39-88 (1981).

[8] Lozi, R.; *Un attracteur etrange (?) du type de Hénon*, J. Phys. 39 (Coll C-5) 9 (1978).

[9] Marotto, F.; *Perturbations of stable and chaotic difference equations*, J. Math. Anal. Appl. 72, 716-729 (1979).

[10] Misiurewicz, M.; *Absolutely continuous measures for certain maps of an interval*, to appear in Publ Math. IHES 53.

[11] Misiurewicz, M.; *Strange attractors for the Lozi mappings*, Annals of the New York Academy of Sciences, 357, 348-358 (1980).

[12] Newhouse, S.; *Abundance of wild hyperbolic set and non-smooth stable sets for diffeomorphisms*, Pub. Math. I.H.E.S. 50, 101-151 (1979).

[13] Newhouse, S., Palis, J.; *Cycles and bifurcations*, *Asterisque* 31. 43-140 (1976).

[14] Palis, J.; *A differential invariant of topological conjugacy*, Asterisque 51, 335-346 (1978).

[15] Ruelle, D.; *Small random perturbations of dynamical systems and the definition of attractors*, Prepring IHES/P/81/23 (1981).

[16] Stein, P. R., Ulan, S. M.; *Non linear transformations studies on electronic computers*, Rozprawy Matematyczne 39, 401-484 (1964).

[17] Tresser, C., Coullet, P., Arneodo, A.; *Topological horseshoe and numerically observed chaotic behaviour in the Hénon mapping*, J. Phys. A, 13, L123-L127 (1980).

[18] Ulan, S. M., Von Neumann, J.; *On combination of stochastic and deterministic processes*, Bull. A.M.S., 53, 1120 (1947).

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