Reformulation of the second Weierstrass – Erdmann condition

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1. Introduction.

We shall denote by $D^1[a,b]$ the class of the continuous functions y defined in the interval $[a,b]$, for which y' is piecewise continuous.

In the calculus of variations the following theorem is known, which provides a necessary condition for the existence of a weak relative minimum for functional such as:

$$(1) \qquad I(y) = \int_a^b F(x, y, y')\,dx, \quad F \in C^2.$$

Theorem. *If $\psi \in D^1[a,b]$ provides the functional* (1) *with a weak relative minimum, then ψ will fulfill the following integral equation:*

$$F(x,y,y') - y'F'_y(x,y,y') = \int_a^x F_x(\xi, y(\xi), y'(\xi))\,d\xi + C,$$

in which C is a constant (See V. $[a]$ and V. $[c]$).

The condition of minimum, in this theorem, refers to the existence of a weak neighborhood with center ψ and radius ε, defined thus:

$$V_1(\psi, \varepsilon) = \{y \in D^1[a,b], y(a) = a_1, y(b) = b_1 : d(\psi, y) < \varepsilon\},$$

where $d(\psi, y) = \sup_{a \le x \le b} |y'(x) - \psi'(x)|$.

From this theorem there results the so-called 2nd Weierstrass-Erdmann condition: if $\psi \in D^1[a,b]$ provides a weak relative minimum for the functional (1), then the function

$$F(x,y,y') - y'F_{y'}(x,y,y')$$

is continuous at the corner points of ψ.

Recebido em 29/6/81.

We would like to point out that this condition, which in the present case derives from the above theorem, is also deducted in the calculus of variations, using the variation of the functional (1), in the general case in which the end-points are variable (See V. [b] and V. [c]).

2. Counter-example.

We will show through an example that both the theorem presented in I and the 2nd Weierstrass-Erdmann condition are false.

Let us consider the following functional:

$$I(y) = \int_0^2 \left(\frac{y'^4}{4} - \frac{5y'^3}{12} + \frac{y'^2}{8} \right) dx$$

and the function: $\psi(x) = \begin{cases} 0 & , \ 0 \leq x \leq 1 \\ x - 1, & 1 \leq x \leq 2. \end{cases}$

Then we have:

$$F = \frac{y'^2}{4}\left(y' - \frac{5-\sqrt{7}}{6}\right)\left(y' - \frac{5+\sqrt{7}}{6}\right)$$

$$F_{y'} = y'(y'-1)\left(y' - \frac{1}{4}\right).$$

F presents an absolute minimum at point $y' = 1$ and $F(1) = -\dfrac{1}{24}$.

The value of the functional $I(y)$ calculated for any function $y \in D^1$, such that:

$$y'(x) = \begin{cases} \leq \dfrac{5-\sqrt{7}}{6}, & 0 \leq x < 1 \\ \text{any value}, & 1 < x \leq 2 \end{cases} \quad \text{is always: } I(y) \geq I(\psi) = -\frac{1}{24}.$$

Therefore, ψ provides a weak relative minimum for the functional. However the function $F - y'F_{y'}$ is not continuous at the corner point $x = 1$.

As a matter of fact:

$$F - y'F_{y'} = \begin{cases} 0 & 0 \leq x < 1 \\ -\dfrac{1}{24}, & 1 < x \leq 2. \end{cases}$$

3. Horizontal neighborhood.

We shall introduce a new neighborhood, through which it will be possible to reformulate the theorem presented in I and the 2nd Weierstrass-Erdmann condition.

Let us consider $\psi \in D^1[a, b]$, expressed parametricaly in the following way:

$$\begin{cases} x = t \\ y = \psi(t) \end{cases}, \quad a \leq t \leq b.$$

We will consider another curve, also expressed parametricaly,

$$\begin{cases} x = t + \beta(t) \\ y = \psi(t) \end{cases}, \quad a \leq t \leq b,$$

in which $\beta \in D^1[a, b]$, $\beta(a) = \beta(b) = 0$.

We shall admit that $1 + \dot{\beta}(t) > 0$, whatever the value of $t, a \leq t \leq b$, so that this curve will take the form $y(x)$. In the following definitions, the function y should always be considered generated by this setting of the above parameters.

The weak horizontal distance between two functions ψ and y, is:

$$d_1^{(h)}(\psi, y) = \sup_{a \leq t \leq b} \left| \frac{\dot{\psi}(t)}{1 + \dot{\beta}(t)} - \dot{\psi}(t) \right|.$$

The weak horizontal neighborhood with a ψ center and an $\varepsilon > 0$ radius, is the set:

$$V_1^{(h)}(\psi, \varepsilon) = \{ y \in D^1[a, b] : d_1^{(h)}(\psi, y) < \varepsilon \}.$$

The functional (1) presents a weak horizontal minimum in the function ψ if there exists $V_1^{(h)}(\psi, \varepsilon)$ such that:

$$I(y) \geq I(\psi), \ \forall \, y \in V_1^{(h)}(\psi, \varepsilon).$$

Theorem. *If $\psi \in D^1[a, b]$ provides a weak* horizontal *minimum for the functional* (1), *then ψ will fulfill the integral equation:*

$$F(x, y, y') - y' F_{y'}(x, y, y') = \int_a^x F_x(\xi, y(\xi), y'(\xi)) \, d\xi + C,$$

in which C is a constant.

From this theorem, derives the 2nd Weierstrass-Erdmann condition, reformulated as it follows:

If ψ provides a weak *horizontal* minimum for the functional (1), then the function $F - y'F_{y'}$ is continuous at the corner point of ψ.

Proof. Let us set ψ expressed parametricaly by $\begin{cases} x = t \\ y = \psi(t) \end{cases}$, $a \le t \le b$, and consider the one-parameter family $\begin{cases} x = t + \alpha\lambda(t) \\ y = \psi(t) \end{cases}$, $a \le t \le b$, in which:

$$\lambda \in D^1[a, b],\ \lambda(a) = \lambda(b) = 0 \text{ and } \alpha \in \mathbb{R}.$$

Once the function λ is determined, it is simple to show that there is a number $\sigma > 0$ such that, if $|\alpha| < \sigma$, then any curve of the family:

a. may be expressed in the form $y(x)$
b. belongs to $V_1^{(h)}(\psi, \varepsilon)$. (*)

Therefore, the functional (1), calculated in a curve of the family $J(\alpha)$, presents a relative minimum for $\alpha = 0$. Thus, since $J(\alpha)$ is derivable, one has $J'(0) = 0$. Through a well known technique, one obtains:

$$J'(0) = \int_a^b \{F(t, \psi(t), \dot{\psi}(t)) - \dot{\psi}(t) F_{y'}(t, \psi(t), \dot{\psi}(t)) - \int_a^t F_x(\tau, \psi(\tau), \dot{\psi}(\tau))\, d\tau\} \dot{\lambda}(t)\, dt.$$

Since the function λ is arbitrary, we come to the result:

$$F(x, \psi(x), \psi'(x)) - \psi'(x)\, F_{y'}(x, \psi(x), \psi'(x)) = \int_a^x F_x(\tau, \psi(\tau), \psi'(\tau))\, d\tau + C.$$

4. Example.

Let us set an example of a functional which, for a given function ψ, presents a weak horizontal minimum but does not present a weak "vertical" minimum.

Let there be a functional $I(y) = \int_0^2 y'(y' - 1)^2\, dx$ and the function ψ,

$$\psi(x) = \begin{cases} 1 \quad , & 0 \le x \le 1 \\ x - 1, & 1 \le x \le 2. \end{cases}$$

(*) It is important to notice that there is no $\sigma > 0$ such that, if $|\alpha| < \sigma$, then $y \in V_1(\psi, \varepsilon)$. This is the reason why the proof of Theorem I, which may be found in V. [c], is not correct.

We will consider y generated by the setting of parameter in III. One has:

$$y' = \frac{\dot{\psi}(t)}{1 + \dot{\beta}(t)} = \begin{cases} 0 \quad , & 0 \le t < 1 \\ \dfrac{1}{1 + \dot{\beta}(t)} > 0, & 1 < t \le 2. \end{cases}$$

Therefore, $I(y) > I(\psi) = 0$ and then ψ will provide a weak horizontal minimum. We can observe that

$$F - y'F_{y'} = 0, \quad 0 \le x \le 2$$

It should also be pointed out that:

$$F_{y'} = \begin{cases} 1, & 0 \le x < 1 \\ 0, & 1 < x \le 2. \end{cases}$$

Hence we conclude, through the 1st Weierstrass-Erdmann condition, that ψ does not provide the functional with a weak "vertical" minimum.

References

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[b] Gelfand, I. M., Fomin, S. V.: *Calculus of Variations*, translated by Richard A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963).

[c] Pars, L. A.: *An Introduction to the Calculus of Variations*, Heinemann London Melbourne Toronto (1962).

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