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Isometry classes of lattices of nonpositive curvature and uniformly bounded volume*

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Introduction.

In [24, p. 39] Margulis posed the following question: Given a positive integer n and numbers a, b with $0 \leq a < b$ is it true that for every positive number c there exist only finitely many homotopy equivalence classes of compact Riemannian manifolds M such that M has dimension n, volume at most c and sectional curvature between $-a$ and $-b$? The answer is yes if a is positive and $n \geq 4$ according to [15], [17] and [24], but the answer is no if $a = 0$ for all dimension $n \geq 3$ [15].

In this paper we consider the following simpler question: Let H be a complete, C^∞ connected and simply connected Riemannian manifold of nonpositive sectional curvature. For each positive number c what can one say about the set of compact quotient manifolds H/Γ whose volume is at most c? Our main result is

Theorem 1. *Let H be a complete, C^∞, connected and simply connected Riemannian manifold of nonpositive sectional curvature. Assume that the factors in the de Rham decomposition of H do not include a Euclidean space of any dimension or a hyperbolic space of dimension 2 or 3. For each positive number c let V_c denote the set of compact quotient manifolds of H whose volume is at most c. Then V_c contains only finitely many isometry classes for each positive number c.*

The restrictions on the de Rham factors of H cannot be eliminated. However if one considers diameter instead of volume, then one may eliminate the hypothesis that H have no 3-dimensional hyperbolic space as a de Rham factor. More precisely we have

Theorem 2. *Let H be a complete, C^∞, connected and simply connected Riemannian manifold of nonpositive sectional curvature. Assume that the*

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factors in the de Rham decomposition of H do not include a Euclidean space of any dimension or a hyperbolic space of dimension 2. For each positive number c let D_c denote the set of compact quotient manifolds of H whose diameter is at most c. Then D_c contains only finitely many isometry classes for each positive number c.

As an immediate consequence of Theorem 1 we obtain

Corollary 1. *Let H satisfy the hypotheses of Theorem 1. Then the set of numbers that occur as the volumes of compact Riemannian quotient manifolds of H is a closed and discrete subset of* $\mathbb{R}$.

One obtains a similar corollary from Theorem 2. Next, observe that the spectrum of the Laplace operator acting on the C^∞ functions of a compact Riemannian manifold M determines the volume of M. Hence from Theorem 1 we also obtain

Corollary 2. *Let H satisfy the hypotheses of Theorem 1. If M is a compact Riemannian quotient manifold of H, then there are at most finitely many isometry classes of Riemannian quotient manifolds of H that have the same spectrum as M relative to the Laplace operator acting on C^∞ functions.*

The conclusion of Corollary 2 may be true without any restrictions on the de Rham factorization of H, but we are unable to prove this. If H is a Euclidean space or a hyperbolic space of dimension 2 or 3, then the conclusion of Corollary 2 is known to be true [7], [25], [29].

Our final result is a proportionality theorem that generalizes a result of [5], which itself is an extension of the Hirzebruch proportionality principle [19].

Theorem 3. Let H be a complete, C^∞, connected and simply connected Riemannian manifold of nonpositive sectional curvature. Then there exists a constant $\alpha = \alpha(H)$ such that if M is any compact quotient manifold of H then $\chi(M) = \alpha \operatorname{vol}(M)$

Here $\chi(M)$ and $\operatorname{vol}(M)$ denote respectively the Euler characteristic and volume of M. From this result we immediately obtain the following corollaries. Part 2) of the first corollary strengthens the conclusion of the first corollary to Theorem 1.

Corollary 1. *Let H satisfy the hypothesis of Theorem 3, and suppose that H admits a compact quotient manifold of nonzero Euler characteristic. Then*

1) Any two compact quotient manifolds of H have nonzero Euler characteristic of the same sign.

2) There exists a positive constant β such that the volume of any compact quotient manifold of H is an integer multiple of β.

3) If M_1, M_2 are any compact quotient manifolds of H then the ratio of their volumes is a rational number.

4) If M_1, M_2 are any compact quotient manifolds of H then $\chi(M_1) = \chi(M_2)$ if and only if $\operatorname{vol}(M_1) = \operatorname{vol}(M_2)$. Moreover $\operatorname{vol}(M_1) = \operatorname{vol}(M_2)$ if M_1 and M_2 have isomorphic fundamental groups.

Corollary 2. *Let H satisfy the hypotheses of Theorem 3. Suppose that H does not admit the hyperbolic plane as a de Rham factor and does admit a compact quotient manifold M with nonzero Euler characteristic. Then*

1) The set of compact quotient manifolds M^* of H whose fundamental group is isomorphic to that of M contains only finitely many isometry classes.

2) For each positive number c let χ_c denote the set of compact quotient manifolds of H whose Euler characteristic has absolute value at most c. Then χ_c contains only finitely many isometry classes for each positive number c.

To obtain the conclusions of the two corollaries to Theorem 3 it is necessary to impose some restriction on H such as the existence of a compact quotient manifold with nonzero Euler characteristic. For example, Thurston [14], [30] shows that the set of volumes of compact quotients of hyperbolic 3-space forms a nondiscrete set of positive real numbers.

We give a brief outline of the proofs. By means of a decomposition result stated in Proposition 4.1 of [11] it suffices to prove Theorem 1 in the following cases:

1) H is a symmetric space of noncompact type
2) The isometry group of H is discrete
3) H is a nontrivial Riemannian product $H_1 \times B$ where H_1 is a symmetric space of noncompact type and the isometry group of B is discrete.

The result in case 1) is due to H.-C. Wang [31, Theorem 8.1]. In case 2) we use a result a Mumford [27] to prove the stronger result that for every positive number c there are only finitely many distinct lattice subgroups $\Gamma \subseteq I(H)$ such that the quotient space H/Γ is a compact smooth manifold with volume at most c. In case 3) we reduce to the previous 2 cases by showing (Lemma 3) that for every positive number c there exists a positive integer r such that for every compact quotient manifold H/Γ whose volume is at most c there exists a finite covering H/Γ^* of multiplicity at most r such that H/Γ^* is isometric to a Riemannian product $(H_1/A^*) \times (B/B^*)$.

Under the hypotheses of Theorem 2 one may also reduce considera-tion to the same 3 cases listed above. The reason that one need not exclude 3-dimensional hyperbolic spaces as de Rham factors of H is apparent in case 1), where H is a symmetric space of noncompact type. Here by considering diameter instead of volume one may replace the rigidity theorem of Weil [32] in the argument of Wang [31, Theorem 8.1] by the strong rigidity theorem of Mostow [26] and the convergence of Macbeath [23] for uniform lattices in a Lie group. At the same time when one considers diameter there are extra technical difficulties that are not present when one considers volume. In particular if $\{M_n\}$ is a sequence of compact quotient manifolds of H such that $\{diam(M_n)\}$ is uniformly bounded above it is not immediately clear that if M_n^* is a finite covering of M_n of multiplicity $r_n \leq r$ for every n then $\{diam(M_n^*)\}$ is uniformly bounded above. This difficulty is handled by assertion 3) of Proposition 2.

The proof of Theorem 3 is again obtained by considering separately the three cases listed above. In the case that H is a symmetric space of noncompact type the result is an immediate consequence of Theorem 3.3 of [5].

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Preliminaries.

All Riemannian manifolds in this paper will be assumed to be complete, connected and C^∞ and to have nonpositive sectional curvature. H will denote a simply connected manifold of this type, sometimes referred to as a Hadamard manifold. For manifolds of nonpositive sectional curvature we shall assume the notation, definitions and basic facts found in the first few sections of [13] and in shorter form in section 1 of [12].

de Rham decompositions.

A Hadamard manifold H is said to be reducible if it can be expressed as the Riemannian product of two manifolds of positive dimension. H is irreducible if it is not reducible. Every Hadamard manifold H can be expressed as a Riemannian product $H_0 \times H_1 \times \ldots \times H_k$, where H_0 is a Euclidean space and each of the manifolds H_i for $i \geq 1$ is non Euclidean and irreducible. This decomposition is unique up to the order of the factors and is called the de Rham decomposition of H. (This decomposition is valid for an arbitrary complete, C^∞, connected and simply connected Riemannian manifold. For further details see [21].)

Isometries of H.

Let $I(H)$ denote the isometry group of H and $I_0(H)$ the connected component of $I(H)$ that contains the identity. $I(H)$ is a Lie group relative to the compact-open topology (see for example [18, pp. 166-170]). If $\{\phi_n\} \subseteq I(H)$ is a sequence such that $\{\phi_n(p)\}$ is a bounded sequence in H for some point p in H, then it follows from [18, p. 167] that some subsequence of $\{\phi_n\}$ converges to an isometry ϕ of H.

For every isometry ϕ of a Hadamard manifold H one has a displacement function $d_\phi : H \to \mathbb{R}$ given by $d_\phi(p) = d(p, \phi p)$. An isometry ϕ is elliptic, hyperbolic or parabolic if the function d_ϕ has respectively zero as its minimum value, a positive minimum value or no minimum value. An isometry ϕ of H is a Clifford translation if d_ϕ is constant in H. If $I(H)$ admits a Clifford translation ϕ, then H admits a Euclidean de Rham factor H_0 and ϕ acts as a translation of H_0 while acting as the identity on the product of all non Euclidean de Rham factors. [33, Theorem 1].

A group $\Gamma \subseteq I(H)$ is called a lattice if the quotient space H/Γ is a smooth Riemannian manifold of finite Riemannian volume that has the same dimension as H. A lattice Γ is respectively uniform or nonuniform if H/Γ is compact or noncompact. For a lattice Γ we define $vol(\Gamma)$ and the isometry class of Γ to be the volume of H/Γ and the isometry class of H/Γ respectively. If Γ is a uniform lattice we define $diam(\Gamma)$ to be the diameter of $H/\Gamma = sup\{d(p,q) : p, q \in H/\Gamma\}$.

A group $\Gamma \subseteq I(H)$ is said to satisfy the duality condition if for every open set 0 in T_1H (the unit tangent bundle of H) and every number $a > 0$ there exists a number $t \geq a$ and an isometry ϕ in Γ such that $[(\phi)_* T_t(0)] \cap 0$ is nonempty. Here $\{T_t\}$ denotes the geodesic flow in T_1H. If H/Γ is a smooth manifold with the same dimension as H, then Γ satisfies the duality condition if and only if every vector in $T_1(H/\Gamma)$ is nonwandering relative to the geodesic flow in H/Γ. In particular every lattice $\Gamma \subseteq I(H)$ satisfies the duality condition. For an equivalent formulation of the duality condition and for various consequences of the duality condition see [1], [8], [9], [11]. The definition of the duality condition that is given here is due to Ballmann [1].

Volume and fundamental domains.

Let H be any Hadamard manifold and let $\Gamma \subseteq I(H)$ be a discrete group; that is, Γ is closed in $I(H)$ and is a zero dimensional Lie group. By virtue of a fact mentioned above [18, p. 167] it follows that a discrete group $\Gamma \subseteq I(H)$ acts properly discontinuously on H: if $C \subseteq H$ is any compact subset then $\phi(C) \cap C$ is nonempty for only finitely many isometries ϕ in Γ. In general a discrete group $\Gamma \subseteq I(H)$ may contain elliptic isometries so that the quotient space H/Γ is not a smooth manifold.

For any discrete group $\Gamma \subseteq I(H)$ there exists a fundamental domain for Γ; that is, there exists an open set $R \subseteq H$ such that $\phi(R) \cap R$ is empty for every nonidentity element ϕ in Γ and $H = \bigcup_{\phi \in \Gamma} \phi(\bar{R})$. For example if $p \in H$ is a point not fixed by an elliptic isometry of Γ (such points exist) then we may construct the Dirichlet fundamental domain for Γ with center p, $R_p = \{q \in H : d(p, q) < d(\phi p, q)$ for all $\phi \neq 1$ in $\Gamma\}$. Clearly fundamental domains are not unique.

One now defines the volume of a discrete group $\Gamma \subseteq I(H)$ ($= vol(\Gamma)$) to be the volume of a fundamental domain for Γ. It is not difficult to show that this definition of $vol(\Gamma)$ does not depend on the fundamental domain chosen. If H/Γ is a smooth manifold then $vol(\Gamma)$ is the volume of H/Γ. For future reference we observe that if Γ^* is a finite index subgroup of a discrete group $\Gamma \subseteq I(H)$, then $vol(\Gamma^*) = vol(\Gamma) \cdot [\Gamma : \Gamma^*]$. To see this let $\xi_1, \ldots, \xi_r$ be a complete set of representatives for the right cosets of Γ^* in Γ. If R is a fundamental domain for Γ then it is easy to see that $R^* = \bigcup_{i=1}^{r} \xi_i(R)$ is a fundamental domain (not necessarily connected) for Γ^*. Clearly $vol(R^*) = \sum_{i=1}^{r} vol(\xi_i R) = r\, vol(R)$ since the sets $\xi_i(R)$ are pairwise disjoint. Hence $vol(\Gamma^*) = r\, vol(\Gamma)$.

Lattices of bounded volume.

In this section we prove Theorem 1 as stated in the introduction. We note that the restrictions in Theorem 1 on the de Rham decomposition of H are necessary. For each positive integer n there exist infinitely many nonisometric flat tori of dimension n that have the same volume. If H is the hyperbolic plane then for each integer $g \geq 2$ the compact surfaces of genus g and curvature $K \equiv -1$ are quotients of H with area $4\pi(g-1)$, but the set of isometry classes is the Teichmuller space of dimension

$6g-6$. If H is the hyperbolic 3-space, then by methods of Thurston [14], [30] one can construct a sequence of compact 3-manifolds with sectional curvature $K \equiv -1$ whose volumes are uniformly bounded above but whose diameters are unbounded.

For applications of Theorem 1 see the two corollaries stated in the introduction.

Proof of Theorem 1. Let H satisfy the hypotheses of Theorem 1. We may assume that H admits a compact quotient manifold for otherwise there is nothing to prove. By the discussion of section 1 it follows that $I(H)$ satisfies the duality condition. By Proposition 4.1 of [11] it follows that H is a Riemannian product $H_1 \times B$, where H_1 is a symmetric space of noncompact type and B is a Hadamard manifold whose full isometry group is discrete and satisfies the duality condition. Either of the factors H_1 or B may be absent. Using this decomposition of H the proof reduces to a separate consideration of the following cases: 1) B is absent, 2) H_1 is absent, 3) H_1 and B are both present.

In case 1), $H = H_1$ is a symmetric space of noncompact type and the result is due to H.-C. Wang [31, Theorem 8.1]. We remark that in the statement of Wang's result one must add the hypothesis that G admit no factor locally isomorphic to $SL(2, \mathbb{C})$. Theorem 8.1 of [31] is false for $G = SL(2, \mathbb{C})$ as the work of Thurston shows [14], [30].

We consider case 2) where $I(H)$ is discrete and satisfies the duality condition. It suffices in this case to prove the following.

Proposition 1. *Let H be a Hadamard manifold with $I(H)$ discrete. For every positive number c let V_c denote the set of uniform lattices Γ in $I(H)$ for which $vol(\Gamma) \leq c$. Then V_c is a finite set for every positive number c. Moreover if $\Gamma \subseteq I(H)$ is a uniform lattice, then Γ has finite index in $I(H)$.*

Proof. The index assertion is fairly clear. The quotient space H/Γ has finite volume if Γ is a uniform lattice, and hence the quotient space $H/I(H)$, a manifold with singularities, must also have finite volume. This is only possible if Γ has finite index in $I(H)$. For a more detailed argument see the proof of Proposition 2.2 of [12].

To prove the finiteness of V_c we need some preliminary results.

Lemma 1a. *For every point $p \in H$ and every positive number R there exist only finitely many isometries $\phi \in I(H)$ such that $d(p, \phi p) \leq R$.*

Proof. Since $I(H)$ is discrete this is an immediate consequence of Theorem 2.2 of [18, p. 167].

Lemma 1b. *Suppose that $I(H)$ contains a uniform lattice. Then there exists a positive number ε such that if $d(p, \phi p) < \varepsilon$ for some $p \in H$ and some $\phi \in I(H)$, then ϕ is either elliptic or the identity.*

Proof. Suppose the assertion is false. Then we can find sequences $\{\phi_n\} \subseteq I(H)$ and $\{p_n\} \subseteq H$ such that for every n, $d(p_n, \phi_n p_n) < 1/n$ and ϕ_n is neither elliptic nor the identity. Let $\Gamma \subseteq I(H)$ be a uniform lattice. Since the translates by elements of Γ of some compact set will cover H we may choose a sequence $\{\xi_n\} \subseteq \Gamma$ so that $q_n = \xi_n(p_n)$ is a bounded sequence in H. If $\phi_n^* = \xi_n \phi_n \xi_n^{-1}$ then $d(q_n, \phi_n^* q_n) = d(p_n, \phi_n p_n) < 1/n$ and ϕ_n^* is neither elliptic nor the identity for every n. By Lemma 1a only finitely many of the isometries ϕ_n^* are distinct since $\{q_n\}$ is bounded. Passing to a subsequence if necessary, we may assume that $\phi_n^* = \phi^*$ for every n and some $\phi^* \in I(H)$. By continuity it follows that ϕ^* fixes every cluster point of the sequence $\{q_n\}$. Hence ϕ_n^* and $\phi_n = \xi_n^{-1} \phi_n^* \xi_n$ are either elliptic or the identify for every n, contradicting the hypothesis on $\{\phi_n\}$. The Lemma is proved.

We now complete the proof of Proposition 1. Let $c > 0$ be given and let $\{\Gamma_k\}$ be a sequence of uniform lattices in $I(H)$ with $vol(\Gamma_k) \leq c$ for every k. It suffices to show that there are only finitely many distinct lattices Γ_k. Using a result of Mumford [27], we show first that $diam(\Gamma_k) \leq c'$ for every k and a suitable constant $c' > 0$. Recall that each element of a uniform lattice is hyperbolic and for any hyperbolic isometry ϕ the minimum locus of the convex function $d_\phi : p \to d(p, \phi p)$ is the union of all geodesics translated by ϕ, [3, Proposition 4.2]. By Lemma 1b, there exists a number $\varepsilon > 0$ such that $d(p, \phi p) \geq \varepsilon > 0$ for every point p of H and every nonidentity element ϕ in $\bigcup_{k=1}^{\infty} \Gamma_k$. From this fact and the remark above, it follows immediately that if L_k is the length of the smallest periodic geodesic in H/Γ_k, then $L_k \geq \varepsilon$ for every k. By the lemma of [27, p. 291] we have $diam(\Gamma_k) \leq A\, vol(\Gamma_k)/(L_k)^{n-1}$ for every k where n is the dimension of H and $A > 0$ is a constant independent of k. Hence $diam(\Gamma_k) \leq c' = Ac/\varepsilon^{n-1}$ for every k.

Let c' be the constant just defined. Fix a point $p \in H$ and for each integer k let $A_k = \{\phi \in \Gamma_k : d(p, \phi p) \leq 3c'\}$. By Lemma 1a, $\bigcup_{k=1}^{\infty} A_k$ is a finite set. In particular, each set A_k is finite and only finitely many of the sets A_k are distinct. The proof of the proposition will be complete when we show that A_k generates Γ_k for every k.

More generally, let Γ be any uniform lattice in $I(H)$ with $diam(\Gamma) = R > 0$, and let $A = \{\phi \in \Gamma : d(p, \phi p) \leq 3R\}$. We show that A generates Γ.

Let $\psi \in \Gamma$ be given and let $q = \psi p$. Let $\{p_1, \ldots, p_N\}$ be a sequence of points on the geodesic from p to q such that $p_1 = p$, $p_N = q$ and $d(p_i, p_{i+1}) \leq R$ for $1 \leq i \leq N-1$. Since $diam(\Gamma) = R$ we can find $\phi_i \in \Gamma$ so that $d(p_i, \phi_i p) \leq R$ for $1 \leq i \leq N$. It follows that $d(\phi_i p, \phi_{i+1} p) \leq 3R$ and hence $\xi_i = \phi_i^{-1} \phi_{i+1} \in A$ for $1 \leq i \leq N-1$. Moreover, $\phi_1 \in A$ since $p_1 = p$ and $\phi_N^{-1} \psi \in A$ since $d(p, \phi_N^{-1} \psi p) = d(\phi_N p, \psi p) = d(\phi_N p, p_N) \leq R$. Since $\psi = \phi_1 \xi_1 \xi_2 \ldots \xi_{N-1} \phi_N^{-1} \psi$ it follows that A generates Γ. The proof of Proposition 1 is now complete.

We now consider the third and final case of the theorem, where H is a nontrivial Riemannian product $H_1 \times B$ with H_1 a symmetric space of noncompact type and B a Hadamard manifold such that $I(B)$ is discrete and satisfies the duality condition. Clearly, we want to reduce to the first two cases already considered. This will be made easier by

Lemma 2. *Let H be a Hadamard manifold that admits no Euclidean de Rham factor. Let $\{\Gamma_n\} \subseteq I(H)$ be a sequence of uniform lattices such that $vol(\Gamma_n) \leq c$ for all n and some positive number c. For each n let Γ_n^* be a normal subgroup of Γ_n such that the index of Γ_n^* in Γ_n is uniformly bounded above. If the lattices Γ_n^* belong to only finitely many isometry classes, then the lattices Γ_n belong to only finitely many isometry classes.*

Remark. For the proof of the lemma the restriction that the index $[\Gamma_n : \Gamma_n^*]$ be uniformly bounded above is unncessary. However, the lemma will only be applied and useful in the case that this index condition is satisfied.

Proof of Lemma 2. Let $\{\Gamma_n\}$ and $\{\Gamma_n^*\}$ be sequences of uniform lattices in $I(H)$ with the properties stated above. Let M_n^*, M_n denote $H/\Gamma_n^*, H/\Gamma_n$ respectively. The covering $M_n^* \to M_n$ is regular since Γ_n^* is a normal subgroup of Γ_n, and hence M_n is obtained from M_n^* by identifying all points in each orbit of some finite group of isometries of M_n^*. Suppose that the manifolds M_n^* belong to only finitely many isometry classes represented by compact manifolds $\tilde{M}_i = H/\tilde{\Gamma}_i$, $1 \leq i \leq r$. Observe that the center of each lattice Γ_i is the identity; by the work of [22], [33] or Proposition 2.3 of [9] any central element of $\tilde{\Gamma}_i$ would be a Clifford translation and by Theorem 1 of [33] the existence of a nonidentity Clifford translation would imply the existence of a Euclidean de Rham factor of H. It now follows from Corollary 3 of [22, p. 225] or Theorem 5.3 of [12] that the isometry group of $\tilde{M}_i$ is finite for each $1 \leq i \leq r$. Since each M_n^* is isometric to some $\tilde{M}_i$, it follows that only finitely many groups occur as subgroups of the isometry groups $I(M_n^*)$, $n \geq 1$. Therefore the manifolds $M_n = H/\Gamma_n$ belong to only finitely many isometry classes, which proves Lemma 2.

Proceeding now to the proof of the theorem in case 3), we let $\{\Gamma_n\}$ be a sequence of uniform lattices in $I(H)$ with $vol(\Gamma_n) \leq c$ for every n and some positive number c. It suffices to show that the lattices $\{\Gamma_n\}$ belong to only finitely many isometry classes. We first prove

Lemma 3. *Let H be a nontrivial Riemannian product $H_1 \times B$, where H_1 is a symmetric space of noncompact type and B is a Hadamard manifold whose isometry group is discrete and satisfies the duality condition. Then for every number $c > 0$ there exists a number $r > 0$ such that if $\Gamma \subseteq I(H)$ is a uniform lattice with $vol(\Gamma) \leq c$ then there exists a normal subgroup Γ^* of Γ such that $[\Gamma : \Gamma^*] \leq r$ and Γ^* is a direct product $A^* \times B^*$, where $A^* \subseteq I_0(H_1)$ and $B^* \subseteq I(B)$. In particular, H/Γ^* is isometric to the Riemannian product $(H_1/A^*) \times (B/B^*)$.*

Assuming for the moment that Lemma 3 has been proved, we complete the proof of the theorem in case 3). By Lemma 3, we can choose uniform lattices $\{\Gamma_n^*\}$ in $I(H)$ such that for every n, Γ_n^* is a normal subgroup of Γ_n, $[\Gamma_n : \Gamma_n^*]$ is uniformly bounded above and Γ_n^* is a direct product $A_n^* \times B_n^*$, where $A_n^* \subseteq I_0(H_1)$ and $B_n^* \subseteq I(B)$. Moreover, there exists a constant $c^* > 0$ such that $vol(\Gamma_n^*) \leq c^*$ for every n since $vol(\Gamma_n) \leq c$ and $[\Gamma_n : \Gamma_n^*]$ is uniformly bounded. By Lemma 2, it suffices to prove that the lattices $\{\Gamma_n^*\}$ belong to only finitely many isometry classes.

Suppose that there are infinitely many isometry classes represented by the sequence $\{\Gamma_n^*\}$. Passing to a subsequence, we may assume further that no two lattices of the sequence $\{\Gamma_n^*\}$ are isometric. Since H_1 is a symmetric space of noncompact type and A_n^* is a discrete subgroup of $I_0(H_1)$ for every n it follows from the corollary to Theorem 1 of [20] or from Corollary 11.9 of [28, p. 178] that there exists $\delta > 0$ such that $vol(A_n^*) \geq \delta$ for every n. Next observe that $vol(B/B_n^*) = vol(H/\Gamma_n^*)/vol(H_1/A_n^*) \leq c^*/\delta$ for every n since H/Γ_n^* is a Riemannian product. By Proposition 1, only finitely many of the uniform lattices $B_n^* \subseteq I(B)$ are distinct. Passing to a subsequence, we may assume that $B_n^* = B^* \subseteq I(B)$ for every n. Then $vol(H_1/A_n^*) = vol(H/\Gamma_n^*)/vol(B/B^*) \leq c^*/vol(B/B^*)$ for every n. By case 1), that is, Theorem 8.1 of [31], there are only finitely many isometry classes of manifolds H_1/A_n^*. Passing to a further subsequence we may assume that H_1/A_n^* is isometric to H_1/A^* for every n and some uniform lattice $A^* \subseteq I(H_1)$. Finally, H/Γ_n^* is isometric to $(H_1/A^*) \times (B/B^*)$ for every n, contradicting the hypothesis that no two of the manifolds H/Γ_n^* are isometric.

We conclude the proof of the theorem by proving Lemma 3. Let $c > 0$ be given. We first define the constant r whose existence is asserted

by the lemma. Since H_1 is a symmetric space of noncompact type it follows by the argument just mentioned above that there exists a number $\delta > 0$ such that $vol(H_1/D) \geq \delta$ for every discrete subgroup D of $I_0(H_1)$. Let σ be the volume of the quotient space $B/I(B)$. This makes sense since $I(B)$ is discrete. Finally let $r = (c/\delta\sigma)^2 [I(H_1) : I_0(H_1)]^3$. We assert that r is the desired constant.

Now let $\Gamma \subseteq I(H)$ be a uniform lattice with $vol(\Gamma) \leq c$. Define $A^* = \Gamma \cap I_0(H_1)$, $B^* = \Gamma \cap I(B)$ and $\Gamma^* = A^* \times B^*$. We assert that Γ^* is a normal subgroup of Γ with $[\Gamma : \Gamma^*] \leq r$.

To prove that Γ^* is a normal subgroup of Γ we consider the de Rham decomposition $H = H_1^* \times \ldots \times H_k^*$ of H. This decomposition is unique up to the order of the factors and hence the factor B in the decomposition $H = H_1 \times B$ is the Riemannian product of those de Rham factors H_i^* such that $I(H_i^*)$ is discrete. It follows that every isometry of H preserves the factors of the decomposition $H = H_1 \times B$ and in particular $I(H) = I(H_1) \times I(B)$. It is now clear that A^*, B^* are normal subgroups of Γ since $I_0(H_1) = I_0(H)$ and $I(B)$ are normal subgroups of $I(H)$. Hence $\Gamma^* = A^* \times B^*$ is a normal subgroup of Γ.

We begin the proof that $[\Gamma : \Gamma^*] \leq r$. Since $I(H) = I(H_1) \times I(B)$, we may define projection homomorphisms $p_1 : I(H) \to I(H_1)$ and $p_2 : I(H) \to I(B)$. Let $\tilde{A} = p_1(\Gamma) \subseteq I(H_1)$ and $\tilde{A}_0 = \tilde{A} \cap I_0(H_1)$. Let $\tilde{B} = p_2(\Gamma) \subseteq I(B)$. Note that A^*, B^* as defined above are normal subgroups of $\tilde{A}_0$, $\tilde{B}$ respectively since A^*, B^* are normal subgroups of Γ. The group $\tilde{B}$ is discrete since $I(B)$ is discrete and hence $\tilde{A}$ is discrete by Theorem 4.1 of [12]. It follows that $\tilde{A} \times \tilde{B}$ is a discrete subgroup of $I(H)$ that contains Γ.

Next we show that $[\tilde{A} \times \tilde{B} : \Gamma] \leq (c/\delta\sigma) \cdot [I(H_1) : I_0(H_1)]$, where δ, σ are the constants defined above. Since $\tilde{A}_0 \subseteq I_0(H_1)$, we have $vol(H_1/\tilde{A}_0) \geq \geq \delta > 0$ by the way in which δ was chosen. Observe that $[\tilde{A} : \tilde{A}_0] \leq \leq [I(H_1) : I_0(H_1)]$ since the map $\alpha\tilde{A}_0 \to \alpha I_0(H_1)$ is a well defined injective homomorphism of $\tilde{A}/\tilde{A}_0$ into $I(H_1)/I_0(H_1)$. Hence

$$vol(H_1/\tilde{A}) = vol(H_1/\tilde{A}_0)/[\tilde{A} : \tilde{A}_0] \geq \delta/[I(H_1) : I_0(H_1)].$$

Since $\tilde{B} \subseteq I(B)$ we have $vol(B/\tilde{B}) \geq vol(B/I(B)) = \sigma > 0$. Hence

$$vol(H/\tilde{A} \times \tilde{B}) = vol(H_1/\tilde{A}) \cdot vol(B/\tilde{B}) \geq \delta\sigma/[I(H_1) : I_0(H_1)].$$

Finally,

$$[\tilde{A} \times \tilde{B} : \Gamma] \; vol(H/\Gamma)/vol(H/\tilde{A} \times \tilde{B}) \leq vol(H/\Gamma) \cdot [I(H_1) : I_0(H_1)]/\delta\sigma \leq c[I(H_1) : I_0(H_1)]/\delta\sigma.$$

We conclude the proof that $[\Gamma : \Gamma^*] \leq r$. Observe that $[\tilde{A}_0 : A^*] \leq \leq [\tilde{A} \times \tilde{B} : \Gamma]$ and $[\tilde{B} : B^*] \leq [\tilde{A} \times \tilde{B} : \Gamma]$ since the maps $aA^* \to \alpha\Gamma$ and

$\beta B^* \to \beta\Gamma$ are well defined injective maps of the groups $\tilde{A}_0/A^*$ and $\tilde{B}/B^*$ into the left coset space $\tilde{A} \times \tilde{B}/\Gamma$ respectively. Finally,

$$[\Gamma:\Gamma^*] \leq [\tilde{A} \times \tilde{B}:\Gamma^*] = [\tilde{A}:A^*]\cdot[\tilde{B}:B^*] = [\tilde{A}:\tilde{A}_0]\cdot[\tilde{A}_0:A^*][\tilde{B}:B^*] \leq \leq [I(H_1):I_0(H_1)]\cdot[\tilde{A} \times \tilde{B}:\Gamma]^2 \leq (c/\delta\sigma)^2[I(H_1):I_0(H_1)]^3 = r.$$

Lattices of Bounded Diameter.

If we consider diameter instead of volume, we may improve Theorem 1 by eliminating the restriction on 3-dimensional hyperbolic factors of H. The restrictions on Euclidean or 2-dimensional hyperbolic factors of H still remain.

In this section, we prove Theorem 2 as stated in the introduction. For the proof of Theorem 2, the next result is useful and has some interest in its own right as well.

Proposition 2. *Let H be a Hadamard manifold with no Euclidean factor in its de Rham decomposition. Then*

1) *For each positive number ε there exists a positive number c^* such that if $\Gamma \subseteq I(H)$ is a uniform lattice with $d(p, \phi p) \geq \varepsilon$ for all $p \in H$ and all $\phi \in \Gamma$, then $diam(\Gamma) \leq c^*\, vol(\Gamma)$.*

2) *For each positive number c there exists a positive number ε such that if $\Gamma \subseteq I(H)$ is a uniform lattice with $diam(\Gamma) \leq c$, then $d(p, \phi p) \geq \varepsilon$ for all $p \in H$ and all $\phi \in \Gamma$.*

3) *For each positive number c there exists a positive number c^* such that if $\Gamma \subseteq I(H)$ is a uniform lattice with $diam(\Gamma) \leq c$ and if $\Gamma^* \subseteq \Gamma$ is a subgroup of finite index, then $diam(\Gamma^*) \leq c^*[\Gamma:\Gamma^*]$.*

Remark. Assertion 2) of Proposition 2 is clearly false if H is allowed to have a Euclidean de Rham factor. Let M^* be a fixed compact manifold of nonpositive sectional curvature, and let M_n be the Riemannian product of M^* with a circle of length ε_n where $\varepsilon_n \to 0$ as $n \to +\infty$. The diameters of $\{M_n\}$ are uniformly bounded above and the manifolds $\{M_n\}$ have the same universal Riemannian covering space H. Clearly, however, no constant $\varepsilon > 0$ has the desired properties of Assertion 2) for all the manifolds M_n simultaneusly.

Proof. 1) This assertion follows immediately from the lemma of [27, p. 291] which asserts that $diam(\Gamma) \leq (A/L^{n-1})\, vol(\Gamma)$, where $\Gamma \subseteq I(H)$ is a uniform lattice, $A > 0$ is a constant independent of Γ and L is the length of the smallest closed geodesic in H/Γ. We set $c^* = A/\varepsilon^{n-1}$.

2) Since H has no Euclidean de Rham factor, we may express H as a Riemannian product $H_1 \times B$ by Proposition 4.1 of [9], where H_1 is a symmetric space of noncompact type and B is a Hadamard manifold whose isometry group is discrete and satisfies the duality condition. We again consider separately the cases i) H_1 is absent, ii) B is absent, iii) H_1 and B are both present.

In case i), we have $H = B$ and the result follows from Proposition 1 and the following well known result whose proof we include for completeness.

Lemma 4. *Let H be an arbitrary Hadamard manifold. For every positive number c there exists a positive number c^* such that if $\Gamma \subseteq I(H)$ is a uniform lattice with $diam(\Gamma) \leq c$, then $vol(\Gamma) \leq c^*$.*

Proof. Let $c > 0$ be given and let $\Gamma \subseteq I(H)$ be a uniform lattice with $diam(\Gamma) \leq c$. It follows that all values of the sectional curvature of H are assumed on 2-planes tangent to a compact subset of H. In particular, $-b^2$ is a lower bound for the sectional curvature of H for some constant $b > 0$. It follows by a comparison theorem [2, p. 256] that the volume of a closed ball of radius c in H is at most c^*, the volume of a closed ball of radius c in a hyperbolic space of the same dimension and curvature $K \equiv -b^2$. However, the volume of H/Γ is at most the volume of a closed ball of radius c in H since $diam(\Gamma) \leq c$. This proves the lemma.

The proof of Assertion 2) of the proposition in case ii) is an obvius simplification of the proof in case iii) so we consider only case iii), where $H = H_1 \times B$ and both factors H_1, B are present. Suppose that the assertion is false for some positive number c. Then we can find a sequence $\{\Gamma_n\}$ of uniform lattices in $I(H)$, a sequence $\{p_n\}$ in H and a sequence $\{\phi_n\}$ of nonidentity elements in $I(H)$ such that for every integer n we have $diam(\Gamma_n) \leq c$, $\phi_n \in \Gamma_n$ and $d(p_n, \phi_n p_n) < 1/n$. By Lemma 4, there exists a constant $c^* > 0$ such that $vol(\Gamma_n) \leq c^*$ for every n. By Lemma 3 of the previous section, we can find uniform lattices Γ_n^* such that for every n, Γ_n^* is a normal subgroup of Γ_n, $[\Gamma_n:\Gamma_n^*]$ is uniformly bounded above and Γ_n^* is a direct product $A_n^* \times B_n^*$, where $A_n^* \subseteq I_0(H_1)$ and $B_n^* \subseteq I(B)$. By Theorem 1 of [20] or Theorem 11.8 of [28], there exists a neighborhood U of the identity in $I_0(H_1)$ and a sequence $\{g_n\} \subseteq I_0(H_1)$ such that $(g_n A_n^* g_n^{-1}) \cap U$ is the identity for every n. Hence, we may assume to begin with that $A_n^* \cap U$ is the identity for every n by replacing Γ_n with $g_n \Gamma_n g_n^{-1}$.

Now fix a point $p \in H$. The fact that $diam(\Gamma_n) \leq c$ for every n means that $H = \bigcup_{\phi \in \Gamma_n} \phi(B_c(p))$, where $B_c(p)$ denotes the closed spherical ball in

H of center p and radius c. Hence, by replacing each ϕ_n by a suitable conjugate in Γ_n, we may assume that $\{p_n\} \subseteq B_c(p)$. By Theorem 2.2 of [18, p. 167], we may assume by passing to a subsequence that $\{\phi_n\}$ converges to an isometry ϕ of H. It follows by continuity that ϕ fixes any cluster point p of $\{p_n\}$. Since $[\Gamma_n : \Gamma_n^*]$ is uniformly bounded above, we may assume that $[\Gamma_n : \Gamma_n^*] \equiv N$ for every n by passing to a further subsequence. In particular $\phi_n^N \to \phi^N$ as $n \to +\infty$; ϕ^N fixes some point $p \in H$ and $\phi_n^N \in \Gamma_n^*$ for every n.

By definition of Γ_n^*, there exist sequences $\{\alpha_n\} \subseteq I_0(H_1)$ and $\{\beta_n\} \subseteq I(B)$ such that $\alpha_n \in A_n^*$ and $\phi_n^N = \alpha_n \times \beta_n$ for every n. It follows that $\phi^N = \alpha \times \beta$, where $\alpha \in I_0(H_1)$, $\beta \in I(B)$, $\{\alpha_n\} \to \alpha$ and $\{\beta_n\} \to \beta$ as $n \to \infty$. Since ϕ^N fixes a point p, we conclude that either α and β are both elliptic isometries of H or one of them (and possibly both) is the identity. In any case, we can find an integer $k > 0$ such that $\beta^k = 1$ and $\alpha^k \in U$, where U is the neighborhood of the identity in $I_0(H_1)$ that was chosen above. We use the discreteness of $I(B)$ to conclude that $\beta^k = 1$ for some $k \geq 1$. It follows that $\alpha_n^k \in U$ for large n and $\beta_n^k \to 1$ as $n \to +\infty$. Hence $\beta_n^k = 1$ for all sufficiently large n since $I(B)$ is discrete. Since $A_n^* \cap U$ is the identity for every n by hypothesis, we conclude that $\alpha_n^k = 1$ for all sufficiently large n. Finally $\phi_n^{kN} = \alpha_n^k \times \beta_n^k = 1$ for all sufficiently large n. This contradicts the fact that $\phi_n \in \Gamma_n$ for every n and every nonidentity element of Γ_n has infinite order in Γ_n by a fixed point theorem of Cartan [18, p. 75]. This completes the proof of Assertion 2) of Proposition 2.

We prove Assertion 3) of Proposition 2. Let $c > 0$ be given. Let $\varepsilon > 0$ be a number satisfying the properties of Assertion 2) of Proposition 2. For this choice of ε let $c^* > 0$ be a number satisfying the properties of Assertion 1) of Proposition 2. By Lemma 4, we may choose a constant $c' > 0$ such that $vol(\Gamma) \leq c'$ whenever $\Gamma \subseteq I(H)$ is a uniform lattice with $diam(\Gamma) \leq c$. Finally, let $\tilde{c} = c^*c'$.

We assert that if $\Gamma \subseteq I(H)$ is a uniform lattice with $diam(\Gamma) \leq c$ and if $\Gamma^* \subseteq \Gamma$ is a subgroup of finite index, then $diam(\Gamma^*) \leq \tilde{c}[\Gamma : \Gamma^*]$. This will prove Assertion 3). Let Γ, Γ^* be given as above. From Assertion 2) and the way in which ε was chosen, we see that $d(p, \phi p) \geq \varepsilon$ for all $p \in H$ and all $\phi \in \Gamma$. Hence $d(p, \phi p) \geq \varepsilon$ for all $p \in H$ and all $\phi \in \Gamma^*$. It follows from Assertion 1) that

$$diam(\Gamma^*) \leq c^* \, vol(\Gamma^*) = c^* \, vol(\Gamma)[\Gamma : \Gamma^*] \leq c^*c'[\Gamma : \Gamma^*] = \tilde{c}[\Gamma : \Gamma^*].$$

We now prove Theorem 2. By Proposition 4.1 of [11], it suffices to consider the following cases: 1) $I(H)$ is a discret group that satisfies the duality condition, 2) H is a symmetric space of noncompact type, 3) H is a nontrivial Riemannian product $H_1 \times B$, where H_1 is a symmetric space

of noncompact type and $I(B)$ is a discrete group that satisfies the duality condition. The result in case 1) follows from Lemma 4 and Proposition 1.

We now consider case 2). Let $c > 0$ be given and let $\{\Gamma_n\}$ be a sequence of uniform lattices in $I(H)$ such that $diam(\Gamma_n) \leq c$ for every n. It suffices to show that the lattices $\{\Gamma_n\}$ belong to only finitely many isometry classes. Let $\Gamma_n^* = \Gamma_n \cap I_0(H)$ for every n. Then $[\Gamma_n : \Gamma_n^*] \leq [I(H) : I_0(H)]$ and Γ_n^* is a normal subgroup of Γ_n for every n. By Lemma 2, it suffices to show that the lattices Γ_n^* belong to only finitely many isometry classes.

Suppose that there are infinitely many isometry classes represented by the sequence $\{\Gamma_n^*\}$. Then by passing to a subsequence, we may assume that no two of the lattices in $\{\Gamma_n^*\}$ are isometric. I follows from Theorem 1 of [20] or Theorem 11.8 of [28] that there exists a neighborhood U of the identity in $I_0(H)$ and a sequence $\{g_n\} \subseteq I_0(H)$ such that $\Gamma_n^{**} = g_n \Gamma_n^* g_n^{-1}$ intersects U only in the identity for every n. Clearly $M_n^{**} = H/\Gamma_n^{**}$ is isometric to $M_n^* = H/\Gamma_n^*$ for every n so we may consider the sequence of lattices $\{\Gamma_n^{**}\}$ instead. By Chabauty's theorem [6] or [16, pp. 319-322], there exists a subsequence $\{\Gamma_{n_k}^{**}\}$ that converges to a lattice Γ^{**} as $k \to +\infty$. By Proposition 2 the lattices $\{\Gamma_{n_k}^{**}\}$ have uniformly bounded diameter and thus Γ^* is a uniform lattice. It now follows from a result of Macbeath [23], [16, p. 322] that the groups $\Gamma_{n_k}^{**}$ are isomorphic to Γ^{**} for sufficiently large k. By the rigidity theorem of Mostow [26], we see that the manifolds $M_{n_k}^{**} = H/\Gamma_{n_k}^{**}$ are isometric, which contradicts the hypothesis on $\{\Gamma_n^{**}\}$. Therefore, the lattices $\{\Gamma_n^*\}$ belong to only finitely many isometry classes, which completes case 2).

We conclude the proof of the theorem by considering case 3) where H is a nontrivial Riemannian product $H_1 \times B$. As in case 2), we let $c > 0$ be given and let $\{\Gamma_n\}$ be a sequence of uniform lattices in $I(H)$ with $diam(\Gamma_n) \leq c$ for every n. Again, it suffices to prove that the lattices $\{\Gamma_n\}$ belong to finitely many isometry classes. By Lemma 3, we can find a sequence of uniform lattices $\{\Gamma_n^*\}$ in $I(H)$ such that for each n, Γ_n^* is a normal subgroup of Γ_n, $[\Gamma_n : \Gamma_n^*]$ is uniformly bounded above and Γ_n^* is a direct product $A_n^* \times B_n^*$, where $A_n^* \subseteq I_0(H_1)$ and $B_n^* \subseteq I(B)$. By Lemma 2, it suffices to prove that the lattices $\{\Gamma_n^*\}$ belong to only finitely many isometry classes. Since $[\Gamma_n : \Gamma_n^*]$ is uniformly bounded above, it follows from Assertion 3) of Proposition 2 that we can find a constant $c^* > 0$ such that $diam(\Gamma_n^*) \leq c^*$ for every n.

By the definition of Γ_n^*, it is apparent that H/Γ_n^* is the Riemannian product $(H_1/A_n^*) \times (B/B_n^*)$. It follows that for every n we have $diam(A_n^*) \leq$ $\leq diam(\Gamma_n^*) \leq c^*$ and $diam(B_n^*) \leq diam(\Gamma_n^*) \leq c^*$. By Lemma 4 and Proposition 1, there are only finitely many distinct lattices B_n^*. By case 2) above, there are only finitely many isometry classes of lattices $\{A_n^*\}$.

Therefore, the lattices $\{\Gamma_n^*\}$ belong to only finitely many isometry classes, which completes the proof of Theorem 2.

Proportionality of Euler characteristic and volume

If Γ is a uniform lattice in a Hadamard manifold H then we define $\chi(\Gamma)$, the Euler characteristic of Γ, to be $\chi(H/\Gamma)$, the Euler characteristic of the quotient space H/Γ. The existence of a uniform lattice Γ with nonzero Euler characteristic in a Hadamard manifold H has strong consequences. In this section we prove Theorem 3 and its two corollaries. See the introduction for a precise statement of these results. We begin with the following

Lemma. *Let H be a Hadamard manifold that admits a uniform lattice Γ with nonzero Euler characteristic. Then H admits no Euclidean de Rham factor nor any nonEuclidean de Rham factor of odd dimension.*

Proof. Clearly H must have even dimension $2n$ for some integer $n \geq 1$ since $\chi(H/\Gamma) \neq 0$ by hypothesis. Let $\{E_1 \ldots, E_{2n}\}$ denote an orthonormal frame field defined in an open set $\cup$ of the quotient manifold $M = H/\Gamma$. Passing to a double cover if necessary we may assume that M is orientable. It is known (see for example [10]) that the Euler characteristic of M is given by $\chi(M) = \int_M \Delta$, where Δ is the $2n$-form given by

$$\Delta = \Sigma\varepsilon(i_1, i_2, \ldots i_{2n})\,\Omega_{i_1 i_2} \wedge \Omega_{i_3 i_4} \wedge \ldots \wedge \Omega_{i_{2n-1} i_{2n}}$$

Here $i = (i_1, i_2, \ldots, i_{2n})$ denotes an arbitrary permutation of $2n$ letters, $\varepsilon(i_1, i_2, \ldots i_{2n})$ denotes the sign of this permutation and $\{\Omega_{ij}\}$ denotes the skew symmetric matrix of curvature 2-forms on M determined by the frame field $\{E_1, \ldots E_{2n}\}$. By our hypothesis $\chi(M)$ is nonzero and hence Δ cannot be identically zero. In particular H cannot be a Euclidean space for this would imply that $\Omega_{ij} = 0$ for every i, j.

Suppose now that H can be written as a Riemannian product $H = H_1 \times H_2$ of two manifolds of positive dimensions k_1 and k_2. By the de Rham decomposition theorem [21] we can find for any point p in $M = H/\Gamma$ a neighborhood U of p isometric to a Riemannian product $U_1 \times U_2$, where for $i = 1{,}2$ U_i is an open subset of H_i and carries the metric from H_i. By making U smaller if necessary we may choose an adapted frame field $\{E_1, \ldots E_{2n}\}$ on U such that E_i is tangent to U_1 for every $1 \leq i \leq k_1$ and E_i is tangent to U_2 if $1 + k_1 \leq i \leq k_1 + k_2 = 2n$. It follows that $\Omega_{ij} = 0$ if $i \leq k_1$ and $j > k_1$ or if $i > k_1$ and $j \leq k_1$.

If H_1 is a Euclidean space then by the preceding observation $\Omega_{i_{2r-1} i_{2r}} = 0$ for $1 \leq r \leq n$ and any permutation $i = (i_1, i_2 \ldots, i_{2n})$ unless both i_{2r-1} and i_{2r} are greater than k_1. This requirement cannot be satisfied simultaneously for every r with $1 \leq r \leq n$ and hence each term in the sum defining Δ is zero, contradicting our hypothesis that $\Delta \not\equiv 0$. Hence H has no Euclidean de Rham factor. Similarly if both k_1 and k_2 are odd integers then for each permutation $i = (i_1, i_2 \ldots i_{2n})$ we can find an integer r with $1 \leq r \leq n$ such that either $i_{2r-1} \leq k_1$ and $i_{2r} > k_1$ or $i_{2r-1} > k_1$ and $i_{2r} \leq k_1$. In any case $\Omega_{i_{2r-1} i_{2r}} = 0|$ by the discussion above. This shows that each term in the sum that defines Δ must be zero, again contradicting our hypothesis that $\Delta \not\equiv 0$. Therefore H has no odd dimensional non Euclidean de Rham factors.

We now begin the proof of Theorem 3. Clearly it suffices to prove Theorem 3 in the case that H admits a compact quotient manifold with nonzero Euler characteristic. As in the earlier arguments we use Proposition 4.1. of [11] and the lemma above to write H as a Riemannian product $H_1 \times B$, where H_1 is a symmetric space of noncompact type and B is a Hadamard manifold such that $I(B)$ is discrete but satisfies the duality condition. Again we consider separately the following cases: 1) H_1 is absent. 2) B is absent. 3) H_1 and B are both present.

We consider the first case where $H = B$ and $I(H)$ is a discrete group that satisfies the duality condition. To prove Theorem 3 in this case it suffices to prove that the ratio $\chi(\Gamma^*)/\mathrm{vol}(\Gamma^*)$ is a constant α that is independent of the uniform lattice Γ^* in H. We observe first that every uniform lattice Γ^* in H must have finite index in the discrete group $I(H)$. The quotient manifold H/Γ^* is compact and hence the quotient space $H/I(H)$, a manifold with singularities, must also be compact. This is possible only if Γ^* has finite index in $I(H)$ since $I(H)$ is discrete. For a more detailed argument see the proof of Proposition 2.2 of [12].

By hypothesis there exists a uniform lattice Γ in H such that $\chi(\Gamma) \neq 0$. Let Γ^* be any uniform lattice in H. By the previous paragraph both lattices Γ and Γ^* have finite index in $I(H)$ and hence $G = \Gamma \cap \Gamma^*$ has finite index k in Γ and k^* in Γ^*. It follows that $\chi(\Gamma^*)/\mathrm{vol}(\Gamma^*) = k^*\chi(\Gamma^*)/k^*\ \mathrm{vol}(\Gamma^*) = \chi(G)/\mathrm{vol}(G) = k\chi(\Gamma)/k\ \mathrm{vol}(\Gamma) = \chi(\Gamma)/\mathrm{vol}(\Gamma) = \alpha \neq 0$.

We now consider the second case where $H = H_1$ is a symmetric space of noncompact type. Let Γ be a uniform lattice in H with $\chi(\Gamma) \neq 0$, and let Γ^* be any uniform lattice in H. Theorem 3.3 of [5] states that $\chi(\Gamma^*)\ \mathrm{vol}(M') = (-1)^n \chi(M')\ \mathrm{vol}(\Gamma^*)$, where M' is the compact Riemannian symmetric space that is dual to H and $2n$ is the dimension of M' and H.

It follows that $\chi(M') \neq 0$ since $\chi(\Gamma) \neq 0$ and hence $\chi(\Gamma^*)/\mathrm{vol}(\Gamma^*) = (-1)^n \chi(M')/\mathrm{vol}(M') = \alpha \neq 0$ for any uniform lattice Γ^* in H. We note that $\alpha = -1/2\pi$ if $n=1$ by the Gauss-Bonnet theorem applied to the case $K \equiv -1$.

We conclude the proof of Theorem 3 by considering the third case, where H is a nontrivial Riemannian product $H_1 \times B$ such that H_1 is a symmetric space of noncompact type and B is a Hadamard manifold whose isometry group $I(B)$ is discrete and satisfies the duality condition. Every isometry of H preserves the factors of this decomposition and hence $I(H) = I(H_1) \times I(B)$. Let $p_1 : I(H) \to I(H_1)$ and $p_2 : I(H) \to I(B)$ denote the projection homomorphisms.

Let Γ^* be any uniform lattice in H. The group $p_2(\Gamma^*) \subseteq I(B)$ is discrete and hence by Theorem 4.1 and Proposition 2.2 of [12] it follows that $\Gamma_1^* \times \Gamma_2^*$ has finite index in Γ^*, where $\Gamma_1^* = \mathrm{kernel}(p_2) = \Gamma \cap I(H_1)$ and $\Gamma_2^* = \mathrm{kernel}(p_1) = \Gamma \cap I(B)$. (When applying Theorem 4.1 of [12] we use the fact that Γ^* contains no Clifford translations by Theorem 1 of [30] and the fact that H has no Euclidean de Rham factor). Therefore the compact manifold H/Γ^* admits a finite Riemannian covering by the Riemannian product manifold $(H_1/\Gamma_1^*) \times (B/\Gamma_2^*)$. If k denotes the multiplicity of this covering then $k\chi(\Gamma^*) = \chi(\Gamma_1^* \times \Gamma_2^*) = \chi(\Gamma_1^*) \cdot \chi(\Gamma_2^*)$ and $k\,\mathrm{vol}(\Gamma^*) = \mathrm{vol}(\Gamma_1^* \times \Gamma_2^*) = \mathrm{vol}(\Gamma_1^*) \cdot \mathrm{vol}(\Gamma_2^*)$. We obtain

$$\chi(\Gamma^*)/\mathrm{vol}(\Gamma^*) = [\chi(\Gamma_1^*)/\mathrm{vol}(\Gamma_1^*)] \cdot [\chi(\Gamma_2^*)/\mathrm{vol}(\Gamma_2^*)]$$

for every uniform lattice Γ^* in H.

Finally let Γ be a uniform lattice in H with $\chi(\Gamma) \neq 0$. Let $\Gamma_1 \times \Gamma_2$ be a subgroup of Γ of finite index in Γ such that $\Gamma_1 \subseteq I(H_1)$ and $\Gamma_2 \subseteq I(B)$. By the equation above and the way in which Γ_1 and Γ_2 were constructed we see that Γ_1 and Γ_2 are uniform lattices with nonzero Euler characteristic in H_1 and B respectively. By applying the previous two cases to the lattices Γ_1, Γ_1^* and Γ_2, Γ_2^* we conclude that $\chi(\Gamma_1^*)/\mathrm{vol}(\Gamma_1^*) = \chi(\Gamma_1)/\mathrm{vol}(\Gamma_1)$ and $\chi(\Gamma_2^*)/\mathrm{vol}(\Gamma_2^*) = \chi(\Gamma_2)/\mathrm{vol}(\Gamma_2)$ for any uniform lattice Γ^* in H. It now follows from the equation of the previous paragraph that $\chi(\Gamma^*)/\mathrm{vol}(\Gamma^*) = \chi(\Gamma)/\mathrm{vol}(\Gamma) = \alpha \neq 0$ for every uniform lattice Γ^* in H. This concludes the proof of Theorem 3.

We now prove the two corollaries of Theorem 3.

Proof of Corollary 1. Assertions 1), 2), 3) and the first part of assertion 4) are immediate consequences of Theorem 3. The second part of assertion 4) follows from the first part when one recalls that H is diffeomorphic to Euclidean space and hence any isomorphism between the fundamental groups of M_1, M_2 is induced by a homotopy equivalence between the compact quotient spaces M_1 and M_2.

Proof of Corollary 2. The proofs of both assertions 1) and 2) follow immediately from the lemma of this section, assertion 4) of Corollary 1 and Theorem 1 of this paper.

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