On mappings into $\mathbb{R}^{2\ell}$

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Abstract.

Let M be a compact orientable manifold. We know how to calculate $\chi(M)$, the Euler characteristic of M, from a stable map $f: M \to R$, with information only on $S(f)$, the singular set of f. This result was extended to stable maps into the plane by H. Levine [L-2] when M has dimension $2n$, and it is also calculated from $S(f)$. The purpose of this work is to generalize the above result for maps into $R^{2\ell}$, where $n \geq \ell$. In this case $S(f)$ is not a manifold. We use the process of resolution of singularities [L-3] to get a homomorphism having only singularities of codimension 1 and use simmilar technics as in [L-2].

1.1. Let M be a compact oriented manifold of dimension $2n$ and $F: M \to R^{2\ell}$, $n \geq \ell$, a differentiable map with $j^kF \pitchfork \Sigma^{(k)}$ and $j^1F \pitchfork \Sigma^k$, where $\Sigma^{(k)} = \Sigma^{(1,1,\cdots,1)}$ and the Σ^k are the Boardman singularities in $J^k(M, R^{2\ell})$ and $J^1(M, R^{2\ell})$ respectively. We let $S^{(k)} = (j^kF)^{-1}(\Sigma^{(k)})$ and $S^k = (j^1F)^{-1}(\Sigma^k)$.

In the case $\ell = 1$ we have the following theorem due to H. Levine L-2.

Theorem 1. *If F is as above, then the Euler characteristic of M is given by*

$$\chi(M) = \sum_C r(C)$$

where the C are the components of the singular set of F with some orientation and $r(C)$ the degree of a certain map ϕ.

For the case $\ell > 1$ we have following

Theorem 2. *Let $F: M \to R^{2\ell}$ be as above and stable; then*

$$\chi(M) = \sum_{\hat{C}} r(\hat{C})$$

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where $\widehat{C}$ are the components of the singular set of $\widehat{dF}$, the map obtained from the resolution of the manifold collection $\{S^1(F), \ldots, S^\ell(F)\}$, $\ell = max\{k/S^kF \neq \emptyset\}$ and the homorphism dF is defined as in [L-3] and r is the degree of a certain map $\widehat{\phi}$.

The proof runs parallel to [L-2] with the introduction of the new map $\widehat{\phi}$. The changes of coordinates of 2.5 are also similar. I thank Prof. H. Levine for the suggestion of the problem and kind advice.

1.2. Definition of the map $\widehat{\phi}$.

We consider the germs of stable maps $F: R^{2\ell} \to R^{2\ell}$ such that $0 \in S^1(F)$. By Martinet correspondence Theorem [M-1] between $S(2\ell\text{-}1, 1, 1)$ and $V(2\ell\text{-}1, 1)$ it is enough to study germs $f:(R, 0) \to (R, 0)$ with $df(0) = 0$ and $\text{codim}_V f \leq 2\ell$; hence $f(x) = x^k h(x)$ with $h(0) \neq 0$ and then $f \underset{V}{\sim} x^k$. The unfoldings of this maps are given by:

$$F_k: R^{2\ell-1} \times R \to R^{2\ell-1} \times R$$
$$(\bar{u}, x) \to (\bar{u}, f_k(\bar{u}, x))$$

where $f_k(\bar{u}, x) = x^{k+1} + \sum_{i=1}^{k-1} u_i x^i$.

The singular set of F_k is given by

$$x = 0 \text{ for } k = 1 \text{ and}$$
$$u_1 = -\left((k+1)x^k + \sum_{i=2}^{k-1} i u_i x^{i-1}\right) \text{ for } k > 1.$$

Let $k > 1$; we consider $\{\{\partial|\partial u_i\}_{i=1}^{2\ell-1}, \partial|\partial x\}$ the basis for the source and $\{\{\partial|\partial v_j\}_{j=1}^{2\ell-1}, \partial|\partial Y\}$ the basis for the target. If $p \in S^1(F)$ then $T_pS^1(F)$ is generated by:

$$\partial|\partial u_i - ix^{i-1}\partial|\partial u_1 \quad \text{for } i = 2, \ldots, k-1$$
$$\partial|\partial u_j \quad \text{for } j = k, \ldots, 2\ell - 1$$
$$\partial|\partial x - C_k\partial|\partial u_1$$

where $C_k = (k+1)kx^{k-1} + \sum_{1=2}^{k-1} i(i-1)u_i x^{i-2}$. If we let

$$A_k = S^{(k)}F - S^{(k+1)}F \text{ then } p \in A_1 \Leftrightarrow C_k \neq 0$$

The image under dF_p of this vector is

$$B_i = \partial|\partial v_i - ix^{i-1}\partial|\partial v_1 - (i-1)x^i\partial|\partial_Y \text{ for } i = 2, \ldots, k-1$$
$$\gamma_j = \partial|\partial v_j \quad \text{for} \quad j = k, \ldots, 2\ell\text{-}1$$
$$- C_k(\partial|\partial v_1 + x\partial|\partial Y)$$

Let $G_{2\ell-1}(R^{2\ell})$ be the Grassmannian manifold and $G_{2\ell-1}(TR^{2\ell})$ the Grassmannian of $TR^{2\ell}$. We let

$$\phi: A_1 \to G_{2\ell-1}(R^{2\ell})$$

be the differentiable map defined by $\phi(x) =$ the vector space generated by $\pi_2\{B_i, \gamma_j, \partial|\partial v_1 + x\partial|\partial_Y$, where $\pi_2: TR^{2\ell} \to R^{2\ell}$ is the projection onto the fiber over $\bar{o} \in R^2$. This map can be extended to a differentiable map over S^1F. If $p \in A_1$ then $\phi(p) = \pi_2 \circ dF(T_pS^1F)$ and the extension is given by $\phi(p) = \pi_2 \circ dF(T_pM)$.

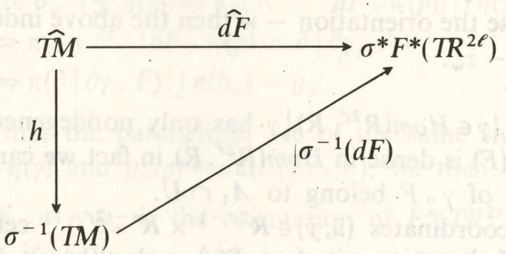
1.3. An example where S^1F is not compact: Consider $F: R^4 \to R^4$ given by

$$F(u_1, u_2, x, y) = (u_1, u_2, x^2 + u_1 y, y^2 + u_2 x).$$

In this case $S^1F = \{(u_1, u_2, x, y) \in R^4 - \bar{o} \quad 4xy - u_1u_2 = 0\}$ and $S^2F = \{\bar{o}\}$, so S^1F is not closed. Moreover it is impossible to extend to S^2F.

We state the following theorem (L-3)

Let $dF: TM \to F^*(TR^{2\ell})$ where $J^1F \pitchfork \Sigma^k$ and $\hat{\sigma}: \hat{M} \to M$ be the composition of the maps obtained by the resolution of singularities (σ is a diffeomorphism outside $\sigma^{-1}\left(\bigcup_{i=2}^{k} S^kF\right)$), then there exists a bundle $\widehat{TM}$ and homomorphisms $\widehat{dF}: TM \to \sigma^*F^*(TR^{2\ell})$ and $h: \widehat{TM} \to \sigma^*(TM)$, with h an isomorphism outside $\sigma^{-1}\left(\bigcup_{i=2}^{k} S^iF\right)$ such that the following diagram of vector bundles over $\hat{M}$ commutes



We define $\widehat{\phi}: S^1(\widehat{dF}) \to G_{2\ell-1}(R^{2\ell})$ by $\widehat{\phi}(p) = \pi_2\widehat{dF}(T_p\widehat{M})$.

In fact $S^1(\widehat{dF}) = \sigma^{-1}(S^1F) = \overline{\sigma^{-1}(S^1F)}$.

2.1. We let M be a compact oriented manifold of dimension n with $n \geq 2\ell$. Then the maps F_k are given by

$$F_k : R^{2\ell-1} \times R \times R^{n-2\ell} \to R^{2\ell-1}$$
$$(\bar{u}, x, \bar{z}) \to (\bar{u}, f_k(\bar{u}, x) + Q(\bar{z}))$$

where Q is a nondegenerate quadratic form. We shall denote F_k by F.

We give the standard orientation to $R^{2\ell}$ and an orientation δ to $S^{2\ell-1}$ with $p \wedge \delta(p)$ being the standard orientation at p. We will follow (L-2).

We denote by $E = TM \mid S^1F$ and $F^*TR^{2\ell}$ the pull back of $TR^{2\ell}$ over S^1F, then we have exact sequence of bundles over S^1F:

$$0 \to L \to E \xrightarrow{dF} F^*TR^{2\ell} \xrightarrow{\pi} G \to 0$$

where L, the kernel of dF, is an $n-(2\ell-1)$ bundle and G a line bundle.

If $D(dF)$ denotes the quadratic differential then we also have the sequence

$$0 \to TS^1F \to E \xrightarrow{D(dF)} L^* \otimes G \to 0$$

If we restrict to Lp, then we have

$$0 \to R_p \to L_p \xrightarrow{D(dF)} p(L^* \otimes G)_p \to K_p \to 0$$

and if $p \in A_1(F)$ then $D(dF)_p$ is an isomorphism

The following construction will be done restricted to a chart (u, φ) of M around $p \in A_1$.

Let w be an orientation for S^1F; since $p \in A_1$ then kernel $(dF_p) + T_pS^1F = T_pM$ and then $\wedge^{2\ell-1} dF_p \circ w_p$ is nowhere zero. Let $h_w \in F^*T_pR^{2\ell}$ with $\wedge^{2\ell-1} dF_p \circ w_p \wedge h_w(p)$ be the orientation of $F^*T_pR^{2\ell}$ and $g_w = \pi(h_w)$. Since G is a trivial line bundle the orientation g_w gives a trivialization of G and we have an isomorphism

$$D(dF)_p : L_p \otimes L_p \to R$$

We denote by τ_w the index of this matrix.

If we choose the orientation $-w$ then the above index would change to $n-(2\ell-1)-\tau_w$.

2.2 Let $\psi(F) = \{\gamma \in Hom(R^{2\ell}, R) \mid \gamma$ has only nondegenerate singularities on $U\}$. Then $\psi(F)$ is dense in $Hom(R^{2\ell}, R)$, in fact we can obtain that the singular points of $\gamma \circ F$ belong to $A_1 \cap U$.

Consider coordinates $(\bar{u}, \bar{y}) \in R^{2\ell-1} \times R^{n-(2\ell-1)}$ centered at $p \in A_1$ and $(V, Y) \in R^{2\ell-1} \times R$ centered at $F(p)$ such that on U we have

$$\gamma \circ F = a_{2\ell} Q(\bar{y}) + \sum_{i=1}^{2\ell-1} a_i u_i + \sum_{k=1}^{2\ell-1} C_k U_k^2 + \sum_{i<j} b_{ij} u_i u_j + \ldots$$

and $\partial \mid \partial \bar{u} = a \cdot w\pi(\partial \mid \partial Y F) = b \circ g_w$ with $a > 0 > b > 0$.

Since $\bar{o}$ is a nondegenerate singularity we get $a_{2\ell} \neq 0$, $a_i = 0$ for $1 \leq i \leq 2\ell-1$ and A_1 is given by $\bar{y} = 0$, then

$$\gamma \circ F|_{A_1 \cap U} = \sum_{k=1}^{2\ell-1} C_k U_k^2 + \sum_{i<j} b_{ij} u_i u_j +$$

if s is the index of the above quadratic form, then the index of $\gamma \circ F$ on $\bar{o}$ is just the index of $Q(\bar{y}) + s = \tau_w + s$.

Since $\bar{o}$ is a singular point of $\gamma \circ F$ then $dF_p(T_pS^1F) \subset$ kernel of γ and then $\pi(\partial \mid \partial\gamma)$ is non zero; hence we have the following two cases:

(*) I) $\pi\left(\dfrac{\partial}{\partial\gamma} \circ F\right)$ is a positive multiple of g_w then

index $\gamma \circ F = \tau_w + s$.

II) $\pi(\partial \mid \partial\gamma \circ F)$ is a negative multiple of g_w then

index $\gamma \circ F = n-(2\ell-1) - \tau_w + (2\ell-1) - s = n - \tau_w - s$.

Let $\gamma \in \psi(F)$ and denote by $\eta(\gamma) = \eta_1(\gamma) \wedge \ldots \wedge \eta_{2\ell-1}(\gamma)$ where $\{\eta_i(\gamma)\}_{1 \leq i \leq 2\ell-1}$ generates the kernel of γ and $\eta_1(\gamma) \wedge \ldots \wedge \eta_{2\ell-1}(\gamma) \wedge \partial \mid \partial_\gamma$ is the orientation of $R^{2\ell}$.

Define a new map $\theta_w : U \cap A_1 \to \wedge^{2\ell-1}(R^{2\ell}) \approx R^{2\ell-1}$ by

$$\theta_w(p) = \frac{\pi_2(\wedge^{2\ell-1} dF \circ w(p))}{\| \pi_2 \wedge^{2\ell-1} dF \circ w(p) \|}$$

where $\pi_2 : \wedge^{2\ell-1} TR^{2\ell} \to \wedge^{2\ell-1} R^{2\ell}$ is the projection onto the zero fiber, then we can consider

$$\theta_w : U \cap A_1 \to S^{2\ell-1}$$

The singular points of $\gamma \circ F$ on U are just $\theta_w^{-1}(\pm\eta(\gamma))$; since $\eta(\gamma) \wedge \partial \mid \partial\gamma \neq 0$ we translate lemma 2,3 in [L-2].

$$p \in \theta_w^{-1}(\pm\eta(\gamma)) \Leftrightarrow \pi_2(\wedge^{2\ell-1} dF \circ w(p)) \,||\, \eta(\gamma)$$
$$\Leftrightarrow \pi_2 \wedge^{2\ell-1} dF \circ w(p) \wedge \partial \mid \partial\gamma_p \neq 0$$
$$\Leftrightarrow \pi(\partial \mid \partial\gamma \circ F) \,||\, \pi(h_w) = g_w$$

We claim that the parallelities are of the same sign.

Put $\theta_w = \epsilon\eta(\gamma)$ and $g_w(p) = \mu\pi(\partial \mid \partial\gamma \circ F)$; we then have

$\wedge^{2\ell-1} dF \circ w \wedge g_w$ is the orientation of $F^*(TR^{2\ell})$

$\Leftrightarrow \mu\pi_2(\wedge^{2\ell-1} dF \circ w) \wedge \partial \mid \partial\gamma|_0$ is the orientation of $R^{2\ell}$

$\Leftrightarrow \mu\theta_w(p) \wedge \partial \mid \partial\gamma|_0$ is the orientation of $R^{2\ell}$

$\Leftrightarrow \mu \in \eta(\gamma) \wedge \partial \mid \partial\gamma|_0$ is the orientation of $R^{2\ell}$

Hence $\mu \in > 0$ since $\eta(\gamma)$ was chosen in such a way that $\eta(\gamma) \wedge \partial \,|\, \partial\gamma\,|_0$ is the orientation of $R^{2\ell}$.

It is clear that $\pm\eta(\gamma)$ are regular values of θ_w, since this is equivalent for o being a regular value of $\gamma \circ \pi_2 \circ (\wedge^{2\ell-1} dF \circ w)$ which is equivalent to $\gamma \circ F$ has nondegenerate points;

Then (*) can be rewritten as

I) $\theta_w(p) = \eta(\gamma)$ then

index $(\gamma \circ F) = \tau_w + s$ — s even if θ_w preserves orientation; s odd if θ_w doesn't

II) $\theta_w(p) = -\eta(\gamma)$ then

index $(\gamma \circ F) = N - \tau_w - s$ — s even if θ_w preserves orientation; s odd if θ_w doesn't

Let $N_u(\sigma)$ be the number of points in U, where $\gamma \circ F$ is singular with its singularities having index σ and $\#(\eta(\gamma), \theta_w)$ be the number of θ_w preimages of $\eta(\gamma)$ counting + when θ_w preserves orientation and -1 when it doesn't. We then have:

$$\#(\eta(\gamma), \theta_w) = \sum_{i=0}^{\ell-1} N_u(\tau_w + 2i) - N_u(\tau_w + 2i + i)$$

$$\#(-\eta(\gamma), \theta_w) = \sum_{i=0}^{\ell-1} N_u(n - \tau_w - 2i) - N_u(n - \tau_w - (2i+1))$$

Let $\alpha: S^{2\ell-1} \to p^{2\ell-1}$ be the canonical orientation preserving map from the sphere to the projective space.

Then $\alpha \circ \theta_w = \phi \,|\, U$ and we have (**)

(**) $$\#([\eta(\gamma)], \phi \,|\, U) = \#(\eta(\gamma), \theta_w) - \#(-\eta(\gamma) \cdot \theta_w).$$

Now, if C is a component of A_1, then $C = \bigcup_{i=1}^{n} U_i$ where U_i are locally trivial and (**) can be restated as

$$\#([\eta(\gamma)], \phi\,|_C) = \#(\eta(\gamma), \theta_w\,|_C) + \#(-\eta(\gamma), \theta_w\,|_C).$$

We state a lemma and delay proof until the next section.

Lemma. *Suppose $F: M \to R^{2\ell}$ satisfies the conditions of theorem 2, then there exists an unique orientation w for $S^1(F)$, such that if c is a component of A_1, the w-index of c is even.*

Proof of theorem 2.

Observe that $\widehat{M}$ is a compact manifold and so are the components $\hat{c}$ of $S^1(dF)$. We can have $\gamma: R^{2\ell} \to R$ such that the singularities of $\widehat{\phi}$ are in $\sigma^{-1}(A_1)$ and then as in [L-2]

$$\chi(M) = \sum_{\hat{c}} \#([\eta(\gamma)], \gamma \circ \hat{\phi} \,|\, (\hat{c}) = \sum_{\hat{c}} r(\hat{c}), \ \hat{c} \text{ component of } S^1F.$$

If $S^2 = \emptyset$ then

$$\chi(M) = \sum_c r(c), \ c \text{ component of } S^1(dF).$$

2.3. Suppose $\hat{c}$ is a component of $S^1(dF)$ and $\hat{c} \subset \hat{A}_1$; since M is even dimensional we can give, in a unique way, an orientation to $\hat{c}$, such that the index is even. This is equivalent to the statement that if B is a matrix of size $(2\ell-1) \times (2\ell-1)$ then either B or $-B$ has even index.

We consider c with $c \cap A_k \neq \emptyset$, $k > 1$ and let $p \in c \cap A_k$ then under change of coordinates

$$F_k(\bar{u}, x, \bar{z}) = (\bar{u}, C_k(u, x) + Q(z)) \text{ where}$$

$$C_k(u, x) = x^{k+1} + \sum_{i=2}^{k-1} u_i x^i + a u_1 x, \text{ with } |a| = 1.$$

The singular set of F_k, S^1F_k is defined by

$$u_1 = -\frac{1}{a}\left((k+1)x^k + \sum_{i=2}^{k-1} i u_i x^{i-1}\right) \bar{z} = 0$$

and A_k is defined by $u_2 = \ldots = u_{k-1} = x = 0$.

Let $p \in A_k$ then $T_pS^1F_k = \langle \partial|\partial u_2), \ldots, \partial|\partial u_{2\ell-1}, \partial|\partial x\rangle$ and $L_p = \text{kernel}$ $DF_p = \langle \partial\,|\,\partial x, \partial\,|\,\partial z_1, \ldots, \partial/\partial z_{(2n-\ell)}\rangle$.

Given λ_p orientation of L_p, we let $V_p \in \wedge^{2\ell-1} T_p{:}\,M$ such that $V_p \wedge \lambda_p$ is the orientation of M at p. Choose

$$y \in F^*(TR^{2\ell}) \text{ with } \wedge^{2\ell-1} dF(V_p) \wedge y$$

be the orientation of $TR^{2\ell}$ (or a positive multiple of it) and let $g_v(p) = \pi(y)$. Since $V_p \notin TpS^1F = \text{kernel } D(dF)$ then $\wedge^{2\ell-1} D(dF)_p(v_p) = a_p(w_p^* \otimes g_v(p))$ modulo $D(dF)_p(L_p)$.

If we would choose $-\lambda_p$ then we still get a_p since M is even dimensional, hence we have a map α_w, dependent only on w, defined by

$$\alpha_w(p) = \frac{a_p}{|a_p|}.$$

Note: α_w is just the coefficient of $u_1 x$.

We state a simmilar proposition as in [L-2].

Proposition. *If the dimension of M is even and $F: M \to R^{2\ell}$ is as before, the we can give an orientation to C in such a way that:*

$$\alpha_w(p) = (-1)^{\tau_w(p)+1}$$

The proof for $\ell = 1$ is in [L-1] and the proof of this proposition is completely similar since L_p continues to be an odd dimensional vector space.

Proof of lemma:

We choose coordinate systems $(\bar{u}, x, \bar{z})$ and (V, y) such that:

1) $V \circ F(\bar{u}, x, \bar{z}) = \bar{u}$

2) $Y \circ F(\bar{u}, x, \bar{z}) = Q(z) + x^{k+1} + \sum_{i=2}^{k-1} u_i x^i + a u_1 x$

where $a = (-1)^{\tau+1}$, τ the index of Q

3) $\partial | \partial x \wedge \partial | \partial u_2 \wedge \ldots \wedge \partial | \partial u_{2\ell-1}$ is a positive multiple of w

$\partial | \partial v \wedge \partial | \partial Y$ is the standard orientation in $R^{2\ell}$

Let $Q = (u', s, 0)\ A_j$. Then Q is of the form

$$Q = \left(-\frac{1}{a}((k+1)s^k + \sum_{i=2}^{k-1} i u_i' s^{i-1}), u_2', \ldots, u_{2\ell-1, s, o}'\right) \text{ and}$$

$$F(Q) = \left(-\frac{1}{a}((k+1)s^k + \sum_{i=2}^{k-1} i u_i' s^{i-1}, u_2', \ldots, u_{2\ell-1}', -(ks^{k+1} + \sum_{i=2}^{k-1}(i-1)u_i' s^i\right) \text{ with}$$

$$k(k+1)s^{k-1} + \sum_{i=2}^{k-1}(i-1) i u_i' s^{i-2} \neq 0.$$

Consider the following changes of coordinates

$$\bar{u}_1 = u_1 + \frac{1}{a}((k+1)s^k + \sum_{i=2}^{k-1} i u_i s^{i-1}$$

$$\bar{u}_j = u_j - u_j'$$

$$\bar{x} = x - s$$

$$\bar{z} = z$$

and in the target

$$\bar{v}_1 = v_1 + \frac{1}{a}\left((k+1)s^k + \sum_{i=2}^{k-1} i u_i s^{i-1}\right)$$

$$\bar{v}_j = v_j - u_j'$$

$$Y = Y - s^{k+1} - \sum_{i=2}^{k-1} s^i v_i - a s v_1$$

Then $(\bar{v}, y) \circ F \circ (\bar{u}, \bar{x}, \bar{z}) = (\bar{u}_1, \ldots, \bar{u}_{2\ell-1}, Q(\bar{z}) + h(\bar{u}, \bar{x}))$

where $h(\bar{u}, \bar{x}) = \sum_{i=1}^{k+1} \binom{k+1}{i} s^i \bar{x}^{k+1-i} + \sum_{i=2}^{k-1} (\bar{u}_i + u_i')(\bar{x} + s)^i + a\bar{x}\,\bar{u}_1 -$

$$-(k+1)\bar{x}s^k - \bar{x}\sum_{i=2}^{k-1} i u_i' s^{i-1} - \sum_{i=2}^{k-1} s^i(\bar{u}_i + (\bar{u}_i'))$$

hence $S^1 F$ is given by

$$\bar{u}_1 = -\frac{1}{a}\left(\sum_{i=1}^{k-1}\binom{k+1}{i}(k+1-i)\, s^i x^{-k-i} + \sum_{i=2}^{k-1} i(\bar{u}_i + u_i')(\bar{x} + s)^{i-1} - \right.$$

$$\left. - \sum_{i=2}^{k-1} i u_i' s^{i-1}\right), \ \bar{z} = 0.$$

we erase the suprascripts and calculate

$$\wedge^{2\ell-1} dF \circ \left[\left(-\frac{2s}{a}\partial | \partial u_1 + \partial | \partial u_2\right) \wedge \left(-\frac{6s^2}{a}\partial | \partial u_1 + \partial | \partial u_3\right) \wedge \ldots \wedge \right.$$

$$\wedge \ldots \wedge \left(-\frac{(k-1)(k-2)}{a}\partial | \partial u_1 + \partial | \partial u_{k-2}\right) \wedge \ldots \wedge \partial | \partial u_{k-1} \wedge \ldots \wedge$$

$$\left. \wedge \ldots \wedge \partial | \partial u_{2\ell-1} \wedge B\right] \text{ where}$$

$$B = -\frac{1}{a}\left[2\binom{k+1}{2}s^{k-1} + \sum_{i=2}^{k-1} i(i-1) u_i' s^{i-2}\right]\partial | \partial u_1 + \partial | \partial x$$

This is equal to

$$\left[-\frac{1}{a}\left((k+1)ks^{k+1} + \sum_{i=2}^{k-1} i(i-1)\right) u_1' s^{i-2}\right] \partial | \partial u_1 \wedge \ldots \wedge \partial | \partial u_{2\ell-1}$$

which from (*) is not zero, hence we choose

$$h_w = \left[-\frac{1}{a}(k\,|\,k+1)s^{k-1} + \sum_{i=2}^{k-1} i(i-1) u_i s^{i-2}\right]\partial | \partial \gamma$$

and then we have to calculate the index of

$$(**)\qquad -\frac{1}{a}C'_k[Q(z)+C'_{k/2}x^2]=(-1)^{\tau}C'_k[Q(z)+C'_{k/2}x^2]$$

which h is the same index as the one of

$$(-1)^{\tau}[c'_kQ(z)+x^2] \text{ where } C'_k=\left[(k+1)ks^{k-1}+\sum_{i=2}^{k+1} i(i-1)u'_is^{i-2}\right];$$

we have two cases

(I) If τ is even then

$$\text{index }[c'_kQ(z)+x^2]=\text{index }c'_kQ(z)=\begin{cases}\tau \text{ if } c'_k>0\\ 2(n-\ell)-\tau \text{ if } c'_k<0\end{cases}$$

(II) If τ is odd then

$$\text{index}[-(c'_kQ(z)+x^2)]=1+\text{index}[-c_kQ(z)]=\begin{cases}2(n-\ell)-\tau+1 \text{ if } c'_k>0\\ \tau+1 \text{ if } c'_k<0\end{cases}$$

hence the index is always even.

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