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Foliations by closed cylindres in 3 dimensional manifolds

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1. Introduction. Let F be a differentiable (C^∞) foliation in a simply connected 3-dimensional manifold M such that all leaves are closed and diffeomorphic to some fixed surface L. We know that every leaf is proper and that the space of leaves M/F (quotient of M by the equivalence relation which identifies all points in the same leaf) is a simply connected, one dimensional manifold, not necessarily Hausdorff [Ha 1]. If $L = R^2$, then the space of leaves characterizes the foliation, i.e., given two such foliations F and F' on manifolds M and M' and a diffeomorphism $h: M/F \to M'/F'$, there exists a diffeomorphism $H: M \to M'$ such that the diagram below commutes, where p and p' are the natural projections [Pm 1]

$$\begin{array}{ccc} M & \xrightarrow{H} & M' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M/F & \xrightarrow{h} & M'/F' \end{array}$$

Such a diffeomorphism H is called a conjugacy between F and F'.

In this paper the case $L = R \times S^1$ is considered with the extra hypothesis that the manifolds are irreducible, i.e., every embedded 2-sphere bounds a ball, and it is proved that although the space of leaves no longer suffices to characterize the foliation, the space of leaves plus the set of leaves supporting vanishing cycles, plus an order induced by the foliation on each set of non separated points of the space of leaves do characterize the foliation in the above sense, i.e., up to conjugacy. Partial results can be found in [Pm 2] and [Pm 3] (the case $M = R^3$ and R^3/F diffeomorphic to R).

Let us recall the definition of a vanishing cycle, as in [Ha 2].

Let F be a codimension one foliation of a manifold M. A *vanishing cycle* is a (differentiable) map $f_0: S^1 \to M$ such that

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(a) f_0 is a map from S^1 to a leaf L_0 which is non homotopic to a constant in L_0

(b) f_0 can be extended to a differentiable map

$f : [0,1] \times S^1 \to M,\ f(t,x) = f_t(x)$ such that $\forall t > 0,\ f_t(S^1)$

is contained in a leaf L_t and is null homotopic in L_t. Without loss of generality, we may still suppose that

(c) $\forall x \in S^1$ the curve $t \to f(t,x)$ is transversal to the foliation F.

We say that the leaf L_0 *supports* the vanishing cycle f_0 and that f is its associate map.

If the leaf L_0 divides M into two connected components (such is the case for foliations with closed leaves in suply connected manifolds) then condition (c) above implies that $f((0,1) \times S^1)$ is contained in one of the two connected components of $M - L_0$. It will be shown in section 3 that this connected component is the same for all vanishing cycles supported by L_0. This component will be called the *inside* of the leaf L_0. The other component will be called the *outside* of the leaf L_0. This notion can be extended to leaves which do not support vanishing cycle, i.e., given a leaf L and a map from S^1 into L, non homotopic to zerò in L, then it is homotopic to zero in one of the connected components of M^3-L but not in the other, so the inside and the outside are well defined for any leaf. By abuse of language, given a point $x \in M^3/F$ we may call *inside* of x the projection of the inside of the leaf $p^{-1}(x)$. In the same way we may speak of the *outside* of. x.

Let M and M' be simply connected irreducible 3-manifolds.

Given two foliations by closed cylinders F and F' on M and M' respectively and a map $h : M^3/F \to M'^3/F'$, h will be said to *preserve sides* if for all $x \in M^3/F$, h takes the inside of x into the inside of $h(x)$ and the outside of x into the outside of $h(x)$. We will say that h *preserves vanishing cycle* if for all $x \in M^3/F$ such that $p^{-1}(x)$ supports a vanishing cycle, then $p'^{-1}(h(x))$ also supports a vanishing cycle.

It will be shown in section 3 that if F is a foliation of M by closed cylinders and if $\{x_i ; i \in I\} \subset M/F$ is a set of non separated points, i.e., $i, j \in I$, every neighbourhood of x_i intersects every neighbourhood of x_j, then the foliation induces an *order* in this set.

Theorem. *Given two differentiable foliations by closed cylinders F and F' on simply connected irreducible 3-manifolds M and M' respectively, and a diffeomorphism $h : M/F \to M'/F'$ which preserves sides, vanishing cycles and order, there exists a diffeomorphism $H : M \to M'$ such that the diagram below commutes, where p and p' are the natural projections.*

$$\begin{array}{ccc} M & \xrightarrow{H} & M' \\ \downarrow p & & \downarrow p' \\ M/F & \xrightarrow{h} & M'/F' \end{array}$$

The foliations F and F' are then said to be conjugate.

Corollary. *If M is a simply connected, irreducible, 3 dimensional manifold, foliated by closed cylinders, then M is diffeomorphic to R^3.*

Remark 1. Through this paper the word cylinder is applied either to surfaces diffeomorphic to $R \times S^1$ or to surfaces diffeomorphic to $[0,1] \times S^1$. In the latter case, the boundary components $(\{0\} \times S^1$ and $\{1\} \times S^1)$ will be called bases of the cylinder. In section 3.1 solid cylinders are mentioned. By that we mean $D^2 \times [0,1]$ or $S^1 \times [0,1] \times [0,1]$, and its bases are $D^2 \times \{0\}$ and $D^2 \times \{1\}$ or $S^1 \times [0,1] \times \{0\}$ and $S^1 \times [0,1] \times \{1\}$ respectively.

2. *Examples.* Let

$$A_n = \{(x,y,z) \in R^3 : x = 0;\ y \geq 0;\ z = \frac{n\pi}{2} + tan^{-1}(y)\}$$

$$B = \{(x,y,z) \in R^3 : x = 0;\ y = 0;\ z \geq 0\}$$

Example 1. Let us consider R^3 foliated by the horizontal planes $z =$ constant. Let $M = R^3 - \bigcup_{n \in Z} A_n$. It is easy to see that: M is diffeomorphic to R^3, the foliation $z =$ constant is a foliation by closed cylinders in M; the space of leaves is diffeomorphic to R; the leaves which supports vanishing cycle are given by $z = n\pi/2$, $n \in Z$ (figure 1).

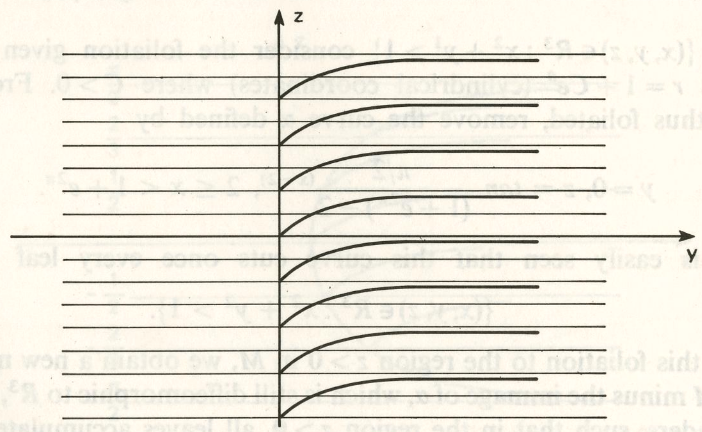

Fig. 1

Example 2. Let $M = R^3 - B - \bigcup_{n>0} A_{-n}$. Again, M is diffeomorphic to R^3, the foliation $z =$ constant is a foliation by closed cylinders, and the space of leaves is diffeomorphic to R. The leaves which support vanishing cycle are given by $z = n\pi/2$, $n = 0, -1, -2, -3, \ldots$ (figure 2).

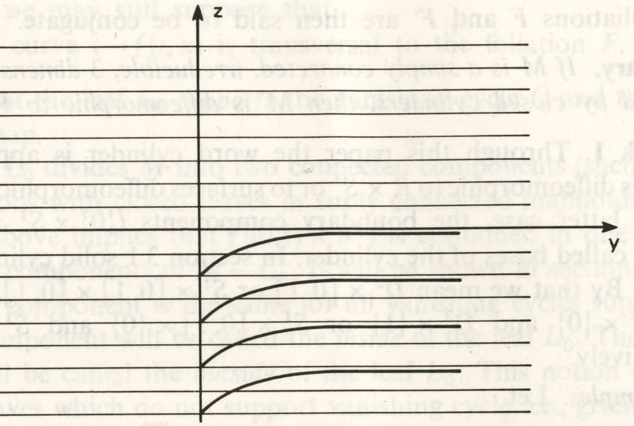

Fig. 2

Since the set of leaves which support vanishing cycle is preserved by conjugacy it is clear that the foliations in examples one and two are not conjugate.

Example 3. Foliation by cylinders in R^3 with non closed leaves.

Let us modify the foliation in example 2, in the region $z > 0$. We remark that the region $z > 0$ in M is diffeomorphic to the region $x^2 + y^2 > 1$ in R^3.

In $\{(x, y, z) \in R^3 : x^2 + y^1 > 1\}$ consider the foliation given by the surfaces $r = 1 + Ce^{\theta}$ (cylindrical coordinates) where $C > 0$. From this region thus foliated, remove the curve α defined by

$$y = 0,\ z = \tan \frac{\pi/2}{(1 + e^{2\pi}) - 2}(x-2),\quad 2 \le x < 1 + e^{2\pi}.$$

It is easily seen that this curve cuts once every leaf of

$$\{(x, y, z) \in R^3 : x^2 + y^2 > 1\}.$$

Taking this foliation to the region $z > 0$ in M, we obtain a new manifold $M_1 = M$ minus the immage of α, which is still diffeomorphic to R^3, foliated by cylinders, such that in the region $z > 0$, all leaves accumulate on the leaf $z = 0$.

This construction is equivalent to the following: Foliate the region $z > 0$ in M by helicoides with axis $x = 0$, $y = 0$, $z > 0$. Then remove a curve diffeomorphic to $[0, +\infty]$, which cuts every leaf once.

Example 4. Consider the curve $x = 0$;

$$y = \frac{1}{\frac{\pi^2}{4} - z^2};\quad -\frac{\pi}{2} < z < \frac{\pi}{2} \text{ (figure 3)}$$

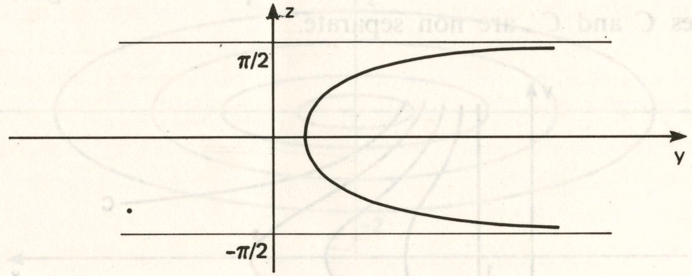

Fig. 3

Let $z_n = \frac{\pi}{2} \cdot \frac{n}{|n| + 1}$; $n \in Z$.

It is clear that if $n \to \pm\infty$ then $z_n \to \pm \frac{\pi}{2}$.

Let α_n be the curve $x = 0$; $y = \frac{1}{\pi^2/4 - z_n^2} + \tan \frac{\pi}{2} \frac{z - z_n}{z_{n+1} - z_n}$; $z_n < z < z_{n+1}$ (figure 4)

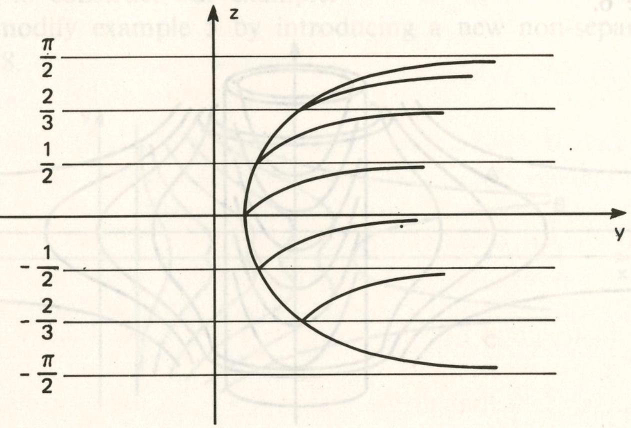

Fig. 4

Let R^3 be foliated by the planes $z=$ constant and consider $M = R^3 - \bigcup_{\substack{n\in Z\\ n\neq 0}} A_n - \bigcup_{n\in Z} \alpha_n$. As before, M is diffeomorphic to R^3, and the foliation $z=$ constant is a foliation by closed cylinders. The leaves which support vanishing cycle are given by $z = n\pi/2$ and $z = n/|n| + 1 \cdot \pi/2$. Note that the images of these leaves in the space of leaves form a set which has $-\pi/2$ and $\pi/2$ as accumulation points.

Example 5. Consider the foliation of the half-plane $x > 1$ in figure 5, where the leaves C and C' are non separate.

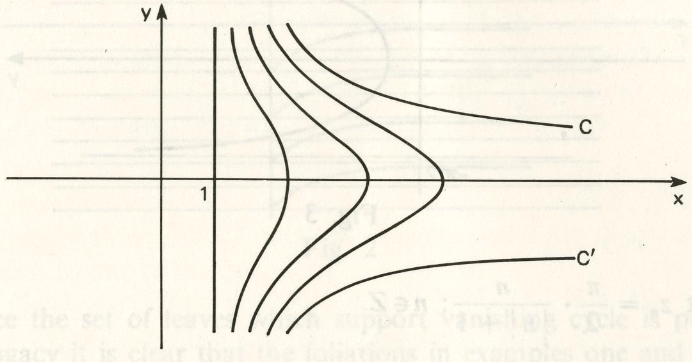

Fig. 5

Rotating the figure around the y-axis, we obtain a foliation by closed cylinders on the region $x^2 + y^2 > 1$. We complete the foliation putting in the region $x^2 + y^2 < 1$ the foliation of example 1, obtaining the foliation of figure 6.

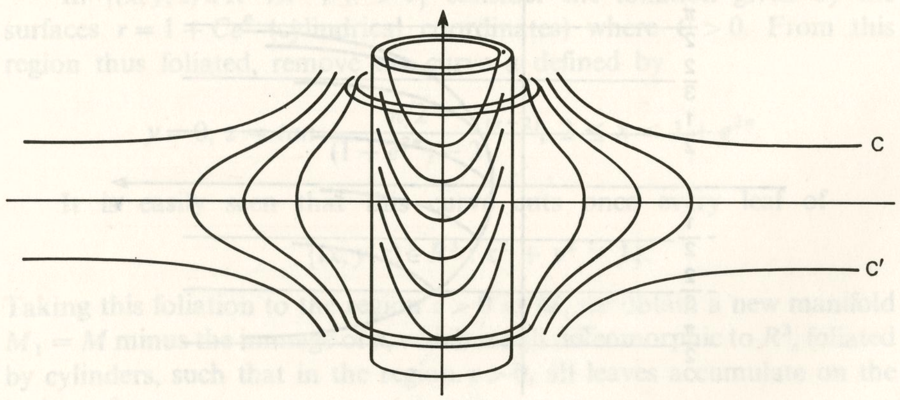

Fig. 6

Note that we have here a foliation with a non Hausdorff space of leaves. The non separate leaves do not support vanishing cycle.

Example 6. We will modify example 5 to obtain two foliations with the same space of leaves, the same set of leaves supporting vanishing cycle, but non conjugate.

Consider the foliation G of the region $x^2 + y^2 > 1$ in R^2 defined as in figure 7.

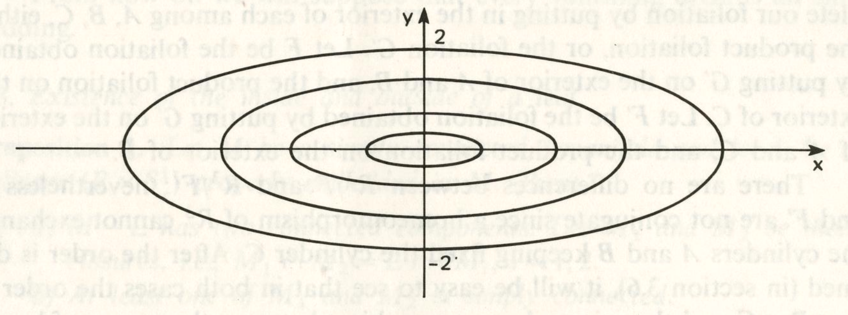

Fig. 7

The leaves in the region $|y| \geq 2$ are the lines $Y=$ constant and in the region $|y| < 2$ are the elipses $x^2/a^2 + y^2/b^2 = 1$ with a and b increasing functions of a parameter $t \in [0, 1)$ such that $a(0) = b(0) = 1$ and $\lim_{t\to 1} a(t) = \infty$ and $\lim_{t\to 1} b(t) = 2$.

Consider the product of G by R and remove the half-lines $x=0$, $y>2$, $z=0$ and $x=0$, $y\leq -2$, $z=0$. We have obtained a foliation G' in $R^3 - \{(x, y, z) : x^2 + y^2 < 1\}$. G' is a foliation by closed cylinders which will be used to construct our example.

Let us modify example 5 by introducing a new non-separate leaf as in figure 8.

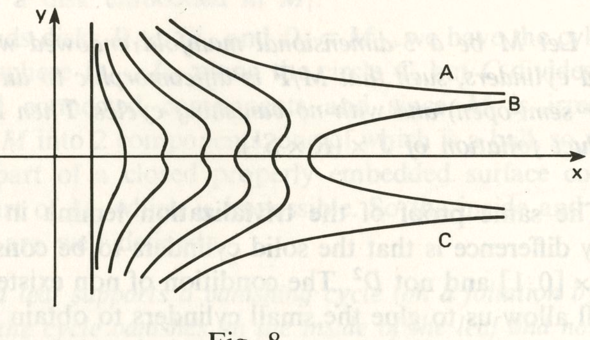

Fig. 8

There are, now, 3 non separate leaves: A, B, C. Rotating the figure around the y-axis, we obtain a foliation by closed cylinders of $R^3 - \{x, y, z) : x^2 + z^2 < 1\}$ with 3 non separate leaves which we still will call A, B, C. As in example 5, we extend the foliation to the region $x^2 + z^2 < 1$.

We will now consider the exterior of the cylinders A, B, C (i.e. the non simply connected component of their complements). The exterior of each cylinder is diffeomorphic to the region $x^2 + y^2 > 1$ and we can complete our foliation by putting in the exterior of each among A, B, C, either the product foliation, or the foliation G'. Let F be the foliation obtained by putting G' on the exterior of A and B, and the product foliation on the exterior of C. Let F' be the foliation obtained by putting G' on the exterior of A and C, and the product foliation on the exterior of B.

There are no differences between R^3/F and R^3/F'; nevertheless F and F' are not conjugate since a homeomorphism of R^3 cannot exchange the cylinders A and B keeping fixed the cylinder C. After the order is defined (in section 3.6), it will be easy to see that in both cases the order is $A < B < C$ and there is no homeomorphism between the spaces of leaves which is order preserving, since any such homeomorphism must switch A and B, therefore will not be order preserving.

3. *Proof of the Theorem.* The proof of the theorem is very similar to the proof of theorem 1 of [Pm 1]. M and M' will be decomposed into big trivialized pieces (decomposition lemma in the end of section 3.6 and trivialization lemma, section 3.1) where the conjugacy can be defined. The extension lemma (section 3.7) shows how to put together adjacent pieces. Section 3.8 wraps it up.

3.1 *Trivialization Lemma.*

Lemma. *Let M be a 3-dimensional manifold endowed with a foliation F by closed cylinders, such that M/F is diffeomorphic to an interval J (open, closed or semi-open) and with no vanishing cycles. Then F is conjugate to the product foliation of $J \times (R \times S^1)$.*

Proof. The same proof of the trivialization lemma in [Pm 1] applies. The only difference is that the solid cylinders to be considered will have base $S^1 \times [0, 1]$ and not D^2. The condition of non existence of vanishing cycle will allow us to glue the small cylinders to obtain the big cylinders containing arbitrary compact sets in their interiors.

3.2 *Vanishing Cycles and Embeddings.*

Proposition 1. *Let M be a 3-dimensional manifold foliated by closed cylinders. Let L be a leaf supporting a vanishing cycle. Then L supports a vanishing cycle which is an embedding.*

Proof. Just notice that $f : S^1 \to L$ is an embedding, if and only if it is a generator of $\pi_1(L)$ and if f is a vanishing cycle, then there exists a vanishing cycle which is a generator of $\pi_1(L)$.

From now on we will suppose that every vanishing cycle is an embedding.

3.3. *Existence of the inside and outside of a leaf*

Proposition 2. *Let M be a simply connected 3-manifold and let L be a cylinder $(R \times S^1)$ properly embedded in M. Then:*

a) *$M - L$ has two connected components. Let M_1 and M_2 be their closures, i.e., $M_1 \cap M_2 = L \cap \partial M_i$, $i = 1, 2$.*

b) *At least one of M_1 and M_2 is simply connected.*

c) *If C is a circle embedded in L, then C bounds a disk embedded in M_i, $i = 1$ or 2 and M_i simply connected.*

d) *If M is irreducible, only one of M_1 and M_2 is simply connected, and C is not null homotopic in both M_1 and M_2.*

Proof. a) If $M - L$ is connected, there is a circle cutting L only in one point. Since M is simply connected, any circle intersects L in an even number of points (intersection number is preserved by homotopy).

b) A simple application of Van Kampen's theorem shows that since $\pi_1(L) = Z$ and $\pi_1(M) = \{0\}$, we must have $\pi_1(M_1) = \{0\}$, or $\pi_1(M_2) = \{0\}$ or both.

c) We have: M_i is a 3-manifold with boundary, C is a circle embedded in ∂M_i which is null homotopy in M_i but not in ∂M_i, so by the loop theorem [He], C bounds a disk embedded in M_i.

d) If C bounds disks $D_1 \subset M_1$ and $D_2 \subset M_2$, we have the cylinder L intersecting the sphere $D_1 \cup D_2$ along the circle C, but C divides L into two umbounded connected components and since M is irreducible, $D_1 \cup D_2$ divides M into 2 components, one of which is a ball, so we have an unbounded part of a closed properly embedded surface contained in a bounded part of M, which is impossible. So the inside and outside of a cylinder L are well defined.

Corollary 1. *If a leaf supports a vanishing cycle (on a foliation by closed cylinders in M), the cycle vanishes on the inside of the leaf and not on the* *outside, i.e., if we push the cycle to nearby leaves on the outside, it stays non homotopic to zero on each leaf.*

In this situation we say that the inside is a *vanishing side* and that the outside is a *persisting side* (the cycle persists; does not vanish by homotopy on the leaf). If the leaf does not support vanishing cycle, we will say that both sides are persisting. By abuse of language, just as before we may talk of persisting and vanishing sides of points in M/F.

Let $\tilde{A} \subset M$ be the union of all leaves which support vanishing cycle and let A be the projection of $\tilde{A}$ on M/F.

Corollary 2. *A is non-empty and infinite.*

Proof. Let L be a leaf which does not support a vanishing cycle (if there is no such leaf, the corollary is true). Let C be a circle embedded in L, non homotopic to zero in L. Let D be a disk bounded by C in general position with respect to F. The well known Haefliger's argument of the disk in general position [Hal pg. 391] gives immediately the existence of a vanishing cycle on a leaf L_1. Since the inside of L_1 is simply connected we can repeat the argument and get a second vanishing cycle on a leaf L_2. Now we look at the inside of L_2 in the inside of L_1, i.e., the intersections of the insides of L_1 and L_2. This is a simply connected set (see below) and we can repeat the argument and so on. We thus obtain a countable number of leaves supporting vanishing cycles. In order to see that the intersection of the insides of L_1 and L_2 is simply connected, we apply proposition 2 to the inside of L_1.

Proposition 3. *Let $x \in M/F$. Let P be a persisting side of x. Then x has a neighborhood V such that $V \cap P \cap A = \varnothing$.*

Remark. We write *a* persisting side of x because if $x \notin A$, both sides are persisting.

Corollary 3. *A is closed.*

Proof of the Corollary. If $x \notin A$ let P and P' be its sides. Then $V \cap P \cap A = \varnothing$ and $V \cap P' \cap A = \varnothing$ for some neighborhood V of x. Since $M/F = P \cup \{x\} \cup P'$, we have $V \cap A = \varnothing$, which implies that $M/F - A$ is open.

Proof of Proposition 3. Let V_1 be a neighborhood of x in M/F. We can suppose V_1 homeomorphic to $(0, 1)$. Let $x_1 \in V_1$ be such that a generator of $\pi_1(p^{-1}(x))$ lifts to a loop in $p^{-1}(x_1)$ non homotopic to zero in $p^{-1}(x_1)$ (just take $x_1 \in P$ sufficiently close to x). We claim that taking the intersection of V_1 with the connected component of $M/F - \{x_1\}$ which contains x, we get a neighborhood V that answers our question. To see this take x_2 between x and x_1. Then, the lifting of loops from $p^{-1}(x)$ to $p^{-1}(x_1)$ can be factored through $p^{-1}(x_2)$. Passing to the homomorphism induced on the fundamental groups we see that if $p^{-1}(x_2)$ supports a vanishing cycle, we will have a non trivial homomorphism from Z to Z factoring through a trivial homomorphism.

Corollary 4. *$\tilde{A}$ is countable.*

Proof. We want to associate to each $x \in A$ an open set V_x such that if $x \neq y$, then $V_x \cap V_y = \varnothing$. Since M/F has a countable base, M/F cannot support an uncountable disjoint family of open sets.

Let $B = \{x \in M/F : \exists y : x \text{ and } y \text{ are non separated}\}$. Since M/F has a countable base, B is countable [Pm 1 pg. 123]. If A is uncountable, $(M/F - B) \cap A$ is also uncountable, and there is a coordinate system $\Phi : V \to R$, where V is an open set in M/F and Φ is a homeomorphism, such that $V \cap A$ is uncountable. Since V is homeomorphic to R, using proposition 3, we associate to each $x \in A \cap V$ an open set V_x such that $V_x \cap A = \varnothing$ and such that if $x \neq y$, then $V_x \cap V_y = \varnothing$. Thus, we obtain in V an uncountable family of disjoint open sets, which is a contradition.

Corollary 5. *Let $A^0 = A$ and A^k be the set of cluster points of A^{k-1} if $k > 0$. Then there exists an integer n such that $A^{n-1} \neq \varnothing$ and $A^n = \varnothing$, i.e., A^{n-1} is a discrete set.*

Proof. If for all k, $A^k \neq \varnothing$, let $D = \bigcap_{k \geq 0} A^k$.

Then D is a perfect set, i.e., all its points are accumulation points. We know that there is a map from M/F to R which is locally a homeomorphism, so the image of D is also perfect and this implies that it is an uncountable set. So D is uncountable. But $D \subset A$ and A is countable.

3.5 *Properties of* M/F.

Preliminary remark: Let us recall that M/F is a connected, simply connected, not necessarily Hausdorff, one dimensional manifold. Such manifolds can be represented by sets of horizontal lines. A neighborhood of a point is an interval containing the point. For end points of half lines, the interval is to be continued on the nearest line. For instance, on figure 9 a neighborhood of point a is formed by the union of the intervals (f, a) and (b, e). A neighborhood of c is $(d, b) \cup (c, g)$. A neighborhood of b may be (d, e) or $(d, b) \cup (c, g)$ or $(f, a) \cup (b, e)$ but not $(f, a) \cup \{b\} \cup (c, g)$.

Given two points $a, b \in M/F$ such that there exists a chart $g: V \to R$ with $a, b \in V$ ($V \subset M/F$ is an open set), we will call open interval (a, b), the image by g^{-1} of the open interval with end points $g(a)$ and $g(b)$. In the same way we define closed and half-closed intervals in M/F. Notice that (a, b) and (b, a) are both well defined and are the same interval. In figure 9 we can consider the intervals: $(f, a) = (a, f)$, (f, e), (d, e), (c, g), (d, g) but not (f, b) nor (f, g), nor (b, g). Notice that $a \in (f, e)$ but $b \notin (f, e)$.

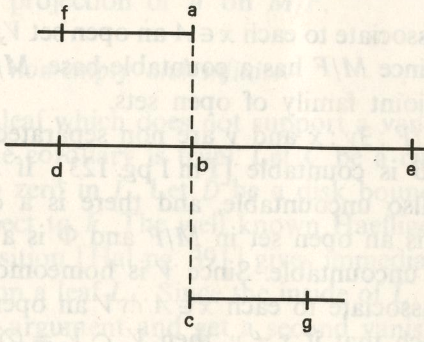

Fig. 9

Proposition 4. *Let $(a, b) \subset M/F$ be an open interval, such that $(a, b) \cap A = \varnothing$. Then (a, b) is not in the outside of both a and b. In particular if $a \in A$ and $b \in A$, (a, b) is not on the persisting side of both a and b.*

Proof. If (a, b) is in the outside of both a and b, then there is no vanishing cycle in F restricted to $p^{-1}[a, b]$ and we can apply the trivialization lemma, obtaining a conjugacy $h: p^{-1}[a, b] \to [a, b] \times (R \times S^1)$ between the foliation F in $p^{-1}[a, b]$ and the product foliation of $[a, b] \times (R \times S^1)$. Let us consider the cylinder $C = [a, b] \times \{1\} \times S^1$.

The cylinder $h^{-1}(C)$ is transversal to F and has its bases in $p^{-1}(a)$ and $p^{-1}(b)$. Just as in the proof of proposition 2 we can glue disks to the bases of $h^{-1}(C)$, obtaining a sphere which intersects $p^{-1}(a)$, transversely, along one circle which is a homotopy generator. We have arrived at a contradiction, so (a, b) cannot be in the outside of both a and b.

Proposition 5. *Let $(a, b) \subset M/F$ be such that $(a, b) \cap A = \varnothing$ (we allow $a = -\infty$ or $b = +\infty$). Then there exists $c \in \partial(a, b)$ such that (a, b) is in the outside of c, and such c is unique. See figure 10 for possible c (in this and other figures, arrows are used to indicate the inside of points).*

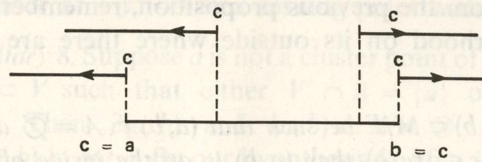

Fig. 10

Proof. Existence – Let $a' \in (a, b)$. Applying the trivialization lemma to $p^{-1}(a, b)$ we see that $\pi_1(p^{-1}(a, b)) = Z = \pi_1(clp^{-1}(a, b))$, where by clX we denote the closure of the set X. Let $\alpha: S^1 \to p^{-1}(a')$ be an embedding which is a generator of $\pi_1(p^{-1}(a'))$ so it is also a generator of $\pi_1(p^{-1}(a, b))$. By proposition 2 there exists an embedding $\beta: D^2 \to M$ which extends α. We can suppose β transversal to $\partial p^{-1}(a, b) = \bigcup_{x \in \partial(a, b)} p^{-1}(x)$. Then $\beta(D^2) \cap \partial p^{-1}(a, b)$ is a finite union of circles $C_1, \ldots, C_n$. It is easy to see that one of them must be homotopic to zero out of $p^{-1}(a, b)$.

Uniqueness: Let $c_1, c_2 \in \partial(a, b)$ be such that (a, b) is on the outside of both c_1 and c_2. Let C_i be a circle embedded in $p^{-1}(c_i)$, non homotopic to a constant in $p^{-1}(c_i)$, $i = 1, 2$. Since (a, b) is on the outside of c_i, when we push C_i to a nearby leaf $p^{-1}(c'_1)$ with $c'_i \in (a, b)$, we obtain a circle C'_i non null homotopic on the leaf. Using the product structure of $p^{-1}(a, b)$ we can join the circles C'_1 and C'_2 by a cylinder obtaining thus a cylinder with bases C_1 and C_2. Let us recall that on the other side of $p^{-1}(c_i)$ there is a disk bounded by C_i. If we join these disks and the cylinder with bases C_1 and C_2 just obtained, we get a sphere intersecting the leaf $p^{-1}(c_i)$ transversely along C_i and as before this is a contradition. We have thus proved that there is only one point $c \in \partial(a, b)$ such that (a, b) is on the outside of c.

Corollary 6. *Let $(a, b) \subset M/F$ and let $\{a_i, i \in I\}$ be a (countable) set of points non separated from a and such that $\forall i \in I$, $\exists (a_i, b)$. We consider that $a \in \{a_i, i \in I\}$. Then either: $\exists i_0 \in I$ such that (a, b) is on the outside of a_{i_0} and this implies that $\exists b' \in (a, b)$ such that $(a, b') \cap A = \varnothing$ and $\forall i \neq i_0$, (a, b) is on the inside of a_i; or: $\forall i \in I$, (a, b) is on the inside of a_i* (figure 11).

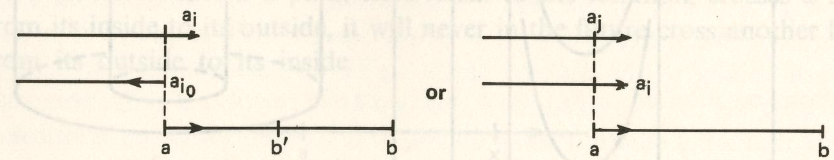

Fig. 11

Proof. Immediate from the previous proposition, remembering that every leaf has a neighborhood on its outside where there are no vanishing cycles.

Corollary 7. *Let $(a, b) \subset M/F$ be such that $(a, b) \cap A = \varnothing$ and (a, b) is on the outside of a. If $c \in \partial(a, b)$ then (a, b) is on the inside of c.*

Given $a \in M/F$ let us denote by in a and out a respectively the inside and the outside of a.

Corollary 8. *Let $a \in M/F$ be such that there exists a neighborhood V of a such that $\forall x \in V \cap$ in a, $a \in$ in x. Then, either a is an accumulation point of A or there exists a unique c non separated from a such that $V \cap$ in $a \subset$ out c and $\forall x \in V \cap$ in a, $\exists(x, c)$.*

Such a point a will be called a *turning point*. The set of all turning points not in A will be denoted by T. The set of points c obtained this way will be denoted by T'.

The first case can be seen in example 4 (section 2). The point $\pi/2$ (identifying R^3/F with the z-axis) is a turning point which is a cluster point of A. For the second case rotate the foliation of figure 12 around the z-axis and complete as in example 5.

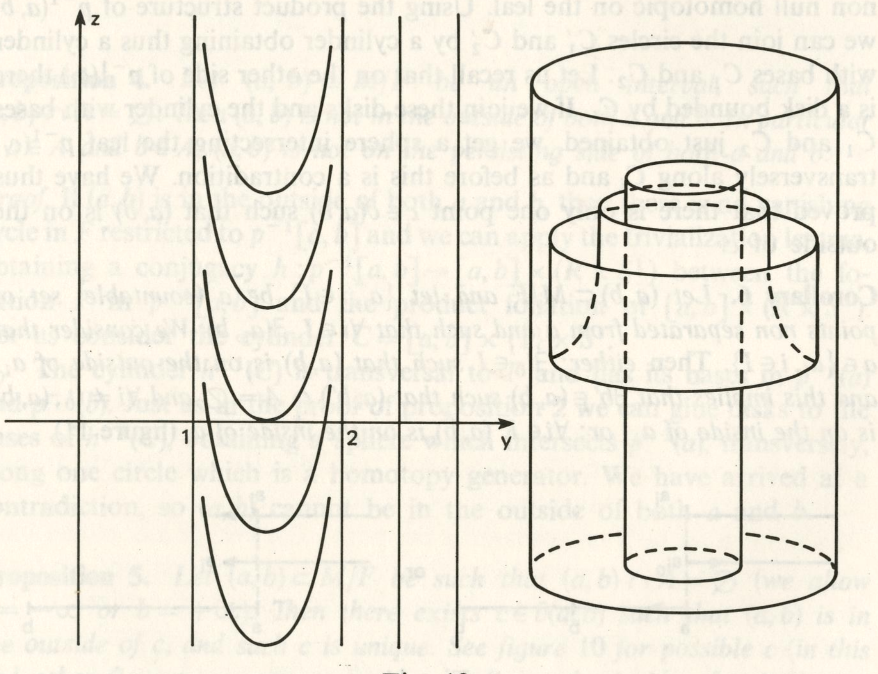

Fig. 12

The cylinder $x^2 + y^2 = 4$ is a turning point with no vanishing cycle.

Proof of Corollary 8. Suppose a is not a cluster point of A. Then a has neighborhood $V' \subset V$ such that either $V' \cap A = \{a\}$ or $V' \cap A = \varnothing$. Let $b \in$ in $a \cap V'$. Then $\exists(a, b)$ and $(a, b) \cap A = \varnothing$. By proposition 5, $\exists c \in \partial(a, b) : (a, b) \subset$ out c. By our hypothesis $(a, b) \subset$ in a and $(a, b) \subset$ in b, so $a \neq c \neq b$ (figure 13).

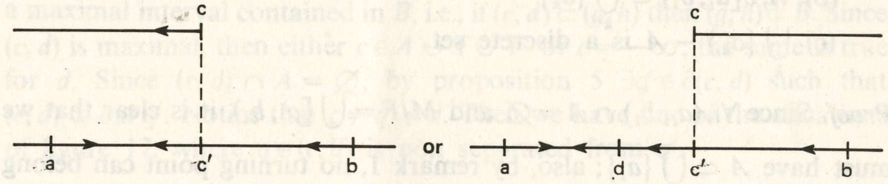

Fig. 13

The first case ($\exists(c, b)$) cannot happen by corollary 7 applied to (c', b), i.e. (c', b) cannot be on the outside of both c and c'. The second case cannot happen by proposition 4 applied to (c, d), i.e., since $(c, d) \cap A = \varnothing$, (c, d) cannot be on the outside of both c and d. If $c' = b$, since $c' \in \partial(a, b)$, then $\exists(a, c)$ and the same reasoning, as in the second case above, applies. So the only other possibility is $c' = a$ and since $c \in \partial(a, b)$, then $\exists(c, b)$ (figure 14). Corollary 6 implies that c is unique.

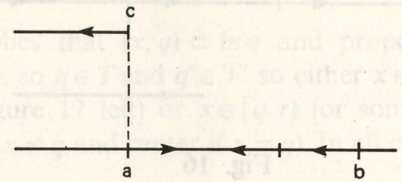

Fig. 14

Remark 1. We could have defined also another kind of turning point: let $a \in M/F$ have a neighborhood V such that $\forall x \in V \cap$ *out a*, then $a \in$ *out x*. It is clear that by proposition 4 such turning points don't exist (figure 15). As a matter of fact if a path, transversal to the foliation, crosses a leaf from its inside to its outside, it will never in the future cross another leaf from its outside to its inside.

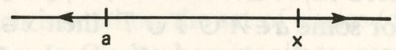

Fig. 15

Remark 2. Since T is contained in the set of non Hausdorff points of M/F and this is countable, T is countable. So T' is also countable. Actually it is easy to see that T is a discrete set, then so is T'.

Decomposition lemma. There is a decomposition of M/F in a countable, disjoint union of intervals $[a_i, b_i)$ such that:

(a) $\forall i$, $(a_i, b_i) \cap A = \varnothing$ and $(a_i, b_i) \subset out\ a_i$

(b) $\forall i$, $\partial[a_i, b_i) \subset \bigcup_k \{a_k\}$

(c) $\bigcup_k \{a_k\} - A$ is a discrete set.

Proof. Since $\forall i, (a_i, b_i) \cap A = \varnothing$ and $M/F = \bigcup_i [a_i, b_i)$, it is clear that we must have $A \subset \bigcup_i \{a_i\}$; also, by remark 1, no turning point can belong to an interval (a, b) with $(a, b) \subset out\ a$, so $\forall a \in A \cup T$ let (a, b) be a maximal interval such that $(a, b) \cap A = \varnothing$ and $(a, b) \subset out\ a$. We may have $b = \infty$. Notice that if $x \in \partial(a, b)$ and $x \neq a$, then $(a, b) \subset in\, x$ and if $x \notin A$ then we will have $x \in T$ or $x \notin T$ according to whether $\nexists(x, a)$ or $\exists(x, a)$ (figure 16).

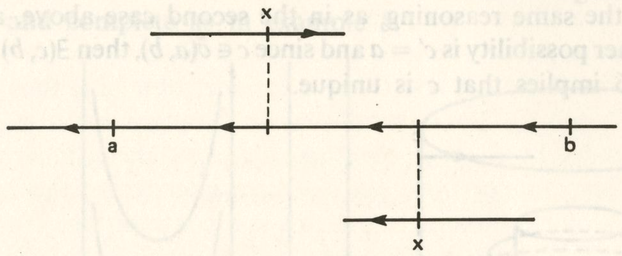

Fig. 16

Now $\forall x \in \partial[a, b)$, $x \notin A$, we consider the maximal interval $[x, y)$ such that $(x, y) \cap A = \varnothing$ and $(x, y) \subset out\ x$; and so on, i.e., $\forall c \in \partial(x, y)$, $c \notin A$, we consider the maximal interval (c, d) such that ...

We have thus obtained a disjoint countable family of intervals. It is countable because the end point belonging to the interval is chosen either in $A \cup T$ or in the set of non Hausdorff points and both sets are countable. We make the same construction for any points of T' which do not belong to the disjoint union so far obtained and we will prove now that we have covered M/F. Let $x \in M/F$, $x \notin A \cup T \cup T'$. It is easy to see that if $x \in out\ a$ for some $a \in A \cup T \cup T'$ then $x \in \cup \{(a, b) : a \in A \cup \cup T \cup T'\}$. We can suppose then that $x \in B = \cap\, in\, a$. It suffices to show this intersection is empty. This is done in two steps:

first step. B is open: given $x \in B$ let V be an interval containing x. Let $y \in V$, $y \notin B$. Then for some $a \in A \cup T \cup T'$, either $y = a$ or $y \in out\ a$, and in this latter case, since $x \in in\, a$, we will have $a \in V$. So if there is no V such that $V \subset B$ then x is a cluster point of $A \cup T \cup T'$ and since T and T' are discrete and A is closed, this implies that $x \in A$, so $x \notin B$, which is a contradiction, so B is open.

second step. Let (c, d) be an interval containing x and such that (c, d) is a maximal interval contained in B, i.e., if $(c, d) \subset (g, h)$ then $(g, h) \not\subset B$. Since (c, d) is maximal, then either $c \in A \cup T \cup T'$ or $c = -\infty$; the same is true for d. Since $(c, d) \cap A = \varnothing$, by proposition 5 $\exists q' \in \partial(c, d)$ such that $(c, d) \subset out\ q'$. Notice that $c \neq q' \neq d$. Then we have one of the situations of figure 17, where $q \in (c, d)$ is non separated from q'.

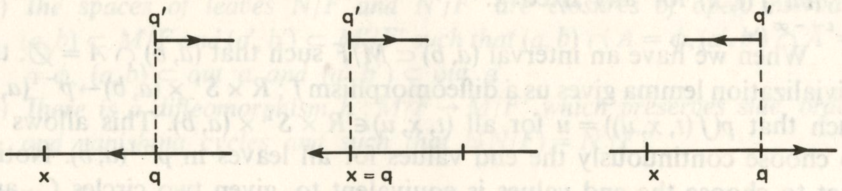

Fig. 17

Corollary 6 implies that $(x, q) \subset in\, q$ and proposition 4 implies that $\forall y \in (x, q)$, $q \in in\, y$, so $q \in T$ and $q' \in T'$ so either $x \in [q', r)$ for some r with $(q', r) \subset out\ q'$ (figure 17 left) or $x \in [q, r)$ for some r with $(q, r) \subset out\ q$ (figure 17 right if $x \neq q$ and center if $x = q$). In all cases we arrive at $x \notin B$, so $B = \varnothing$.

3.6. *Order on Non-Separated Points.*

The existence of an order on a set of non-separated points (a set of points a_i such that $\forall i \neq j$, every neighborhood of a_i intersects every neighborhood of a_j) follows from the fact that we can define an order on disjoint circles embedded in a cylinder L, in the following way: let us call $-\infty$ and $+\infty$ the ends of the cylinder L. If C_1 and C_2 are disjoint embedded circles, we will say that $C_1 < C_2$ if every continuous path joining C_1 to $+\infty$ intersects C_2 (figure 18). It is easy to see that this is a well defined partial order which is total on the set of circles (disjoint, embedded) non null homotopic on L. Circles homotopic to zero are comparable only if one is contained in the disk bounded by the other.

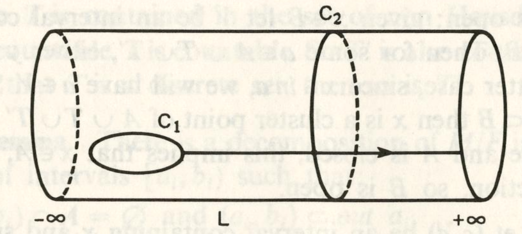

Fig. 18

The choice between the two end values is made by giving a diffeomorphism $f_L : R \times S^1 \to L$ and taking $+\infty = \lim_{t \to +\infty} f(t, x)$ and $-\infty = \lim_{t \to -\infty} f(t, x)$ for any fixed x.

When we have an interval $(a, b) \subset M/F$ such that $(a, b) \cap A = \varnothing$, the trivialization lemma gives us a diffeomorphism $f : R \times S^1 \times (a, b) \to p^{-1}(a, b)$ such that $p(f(t, x, u)) = u$ for all $(t, x, u) \in R \times S^1 \times (a, b)$. This allows us to choose continuously the end values for all leaves in $p^{-1}(a, b)$. Notice that to choose the end values is equivalent to, given two circles C_1 and C_2 non homotopic to zero, state whether $C_1 < C_2$ or $C_2 < C_1$. The circles are supposed to be disjoint.

On a leaf L which supports vanishing cycles there is a canonical way to do this: we take the cylinder bounded by C_1 and C_2 and push it to a nearby leaf L' on the vanishing side of L.

We have now an annulus on L', null homotopic on L'. Let C_1' and C_2' be the images of C_1 and C_2 respectively. We will say that $C_i < C_j$ if C_i' is contained in the disk bounded by C_j'. Notice that for whatever choice of end values in L' we will always have $C_i' < C_j'$.

By continuity, this choice of end values can be extended to all leaves in $\bigcup_{a \in A} p^{-1}[a, b)$ where (a, b) is a maximal interval such that $(a, b) \cap A = \varnothing$ and $(a, b) \subset out\ a$. As in the decomposition lemma, the choice remains to be made only in $\bigcap_{a \in A} in\ a$, and this is an open set which can easily be seen to be connected (follows from the fact that M/F is one dimensional and simply connected). Choosing the end values for one leaf on $p^{-1}(\bigcap_{a \in A} in\ a)$, we have the order defined on every leaf.

It remains to be seen how this order on circles on a leaf induces an order on every set of non-separated points in M/F. So let $\{a_i, i \in I\}$ be a set of non-separated points. Let x be such that $\forall i \in I, \exists (a_i, x)$ and

$(a_i, x) \cap A = \varnothing$ $\forall i \in I$, let C_i be a circle non homotopic to zero on its leaf, embedded in $p^{-1}(a_i)$. If x is close enough to a_i for all i, we can push C_i to the leaf $p^{-1}(x)$ in such a way that the obtained circles C_i' are all disjoint. We will say that $a_i < a_j$ if and only if $C_i' < C_j'$. It is easy to see that this order is well defined (independent of the choice of C_i for all i).

Given two foliations by closed cylinders F and F', on simply connected irreducible 3-manifolds M and M', a diffeomorphism $h : M/F \to M'/F'$ is said to preserve order if $a < b \Rightarrow h(a) < h(b)$.

3.7. *Extension Lemma.*

Extension lemma. *Let F and F' be foliations by closed cylinders in M and M'. Let N and N' be 3 dimensional submanifolds of M and M' respectively, connected, with boundary, saturated respectively by F and F' and such that:*

(a) *The spaces of leaves N/F and N'/F' are closures of open intervals $(a, b) \subset M/F$ and $(a', b') \subset M'/F'$ such that $(a, b) \cap A = \phi$, $(a', b') \cap A' = \phi$, $(a, b) \subset out\ a$ and $(a', b') \subset out\ a'$.*

(b) *There is a diffeomorphism $h : M/F \to M/F'$ which preserves side, order and vanishing cycles and such that $h(N/F) = N'/F'$.*

Then:

(A) *There exists a diffeomorphism $H : N \to N'$ which induces h by passing to the quotient.*

(B) *If a priori is given a conjugacy H_1 defined on a neighborhood of ∂N in M-int N, then H can be constructed such that H extends H_1 provided H_1 preserves the orientation of the leaves, induces h by passing to the quotient and can be extended differentiably to $-\infty$ on all leaves of ∂N.*

Before proving the lemma it is convenient to make some preliminary remarks.

Let us recall that $h : M/F \to M'F'$ is said to preserve sides if for all x in M/F, h takes the outside of x into the outside of $h(x)$ and the inside of x into the inside of $h(x)$. A choice of orientation of M and the transversal orientations of F and F' induce orientations on the leaves. This is the orientation considered in (B) above.

The proof of the lemma is analogous to the proof of the Main Lemma [Pm 1, pg. 118].

Proof of the extension lemma. Parallel to every construction involving F and N, the corresponding construction is carried out for N' and F'.

Since N and N' are saturated, it is clear that the connected components of ∂N and $\partial N'$ are leaves; and so, diffeomorphic to $R \times S^1$.

If in some connected component L of ∂N the conjugacy is not yet defined, we choose an orientation preserving diffeomorphism between L and $p'^{-1} \circ h \circ p(L)$. Now the conjugacy is defined in ∂N and we want to extend it to the interior of N. We are going to obtain neighborhoods (which will be called fundamental neighborhoods) of the connected components of ∂N and $\partial N'$ and we are going to extend the conjugacy to these neighborhoods.

Since N/F is the closure of (a, b) and $(a, b) \cap A = \phi$, we may use the trivialization lemma to define a conjugacy $\tilde{H} : int\ N \to int\ N'$. Using the parameter version of a diffeomorphism extension lemma from Palais [P1], [Pm 1, th. 2, p. 130], we will glue $\tilde{H}$ and the conjugacy defined on the union of the fundamental neighborhoods.

Construction of the fundamental neighborhoods:

Let $\{a_i, i \in I\}$ be the boundary of N/F, where I is a set of natural numbers. We can suppose $a_0 = a$. Let $L_i = p^{-1}(a_i)$.

Since L_i and $\bigcup_{j \neq i} L_j$ are closed disjoint sets, let $V_i \subset N$ be a neighborhood of L_i such that $\bar{V}_i \cap \bigcup_{j \neq i} \bar{V}_j = \varnothing$ and $\bigcup_j \bar{V}_j$ is closed. The procedure of construction of the fundamental neighborhood of a leaf L_k will depend on whether or not L_k supports a vanishing cycle with N on its vanishing side. Let us first suppose not. Let $\gamma_k : [0, 1) \to N$ be an arc transversal to F such that $\gamma_k(0) \in L_k$ and $\gamma_k([0, 1)) \subset V_k$. Let U_k be the saturated by F of $\gamma_k([0, 1))$. Let $g_k : D_0 \times [0, 1) \to U_k$ be the conjugacy given by the trivialization lemma ($D_0 = D^2 - \{(0, 0)\}$). We are going to construct a surface G_k in $D_0 \times [0, 1)$, separating L_k from $\bigcup_{j \neq k} L_j$. The image by g_k of the region between D_0 and G_k will be the fundamental neighborhood of L_k. We will denote it by B_k.

Construction of G_k:

Let $\delta_k : [0, 1] \to [0, 1]$ be a differentiable function such that $\delta_k(0) = 0$, $\delta_k(1) = 0$, $\delta_k'(t) > 0$ for $t < 1/2$, $\delta_k'(t) < 0$ for $t > 1/2$, $\delta_k'(1/2) = 0$. Let G_k be the graph of the map $D_0 \to (0, 1)$. We must take δ_k such that $g_k(G_k) \subset V_k$. Notice that $x \to \delta_k(|x|^2)$. Each leaf cuts B_k along a bounded cylinder and it is easy to extend H_1 from L_k to B_k. (Figure 19).

Let us consider now the case in which L_k supports a vanishing cycle and N is on its vanishing side. Let us consider a countable family of vanishing cycles on $L_k : \{f_0^n; n$ is a non negative integer$\}$ such that:

(a) If $n \neq m$, $f_0^n(S^1) \cap f_0^m(S^1) = \varnothing$.

(b) If $n < m$, $f_0^n(S^1) < f_0^m(S^1)$ (in the L_k order).

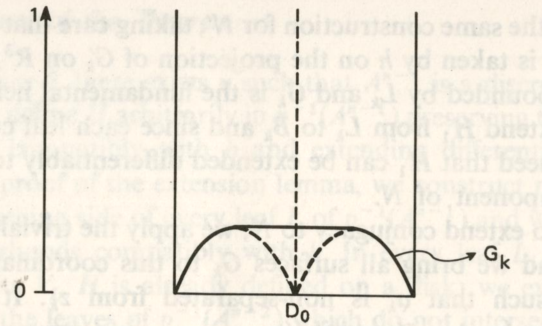

Fig. 19

(c) If $Q_n \subset L_k$ is the cylinder with bases $f_0^n(S^1)$ and $f_0^{n+1}(S^1)$, then $\bigcup_{n=0}^{\infty} Qn =$ one of the connected components of $L_k - f_0^0(S^1)$.

(d) For all n, we take the associated map to f_0^n to be defined not in $[0, 1) \times S^1$ but in $[0, E_n) \times S^1$ for some E_n, such that $f^n(S^1 \times \{t\})$ and $f^m(S^1 \times \{t\})$ are contained in the same leaf for all t, n, m.

Let us notice that as $n \to \infty$, we will probably have $E_n \to 0$.

Let D_t^n be the disk bounded by $f^n(S^1 \times \{t\})$ on its leaf. The union $\bigcup_{n,t} D_t^n$ is a neighborhood of L_k in N. Just as in section 2 of [Pm 1] we can glue the cylinders $f^n(S^1 \times [0, E_n])$ and $f^{n+1}(S^1 \times [0, E_{n+1}])$ to obtain a surface G_k diffeomorphic to R^2, which cuts every leaf transversely a circle (except one leaf where it is tangent) and which separates L_k from $\bigcup_{i \neq k} L_i$ (we can suppose that all our constructions were made in V_k) (figure 20).

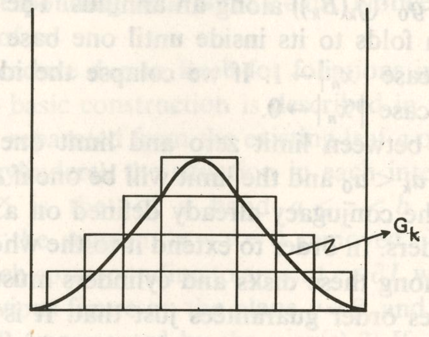

Fig. 20

We make the same construction for N', taking care that the projection of G_k on R^3/F is taken by h on the projection of G'_k on R^3/F'. As before, the region B_k bounded by L_k and G_k is the fundamental neighborhood of L_k. Now we extend H_1 from L_k to B_k and since each leaf cuts B_k along a disk, here we need that H_1 can be extended differentiably to $-\infty$ in each boundary component of N.

In order to extend conjugacy to N, we apply the trivialization lemma to $p^{-1}[a, b)$ and we bring all surfaces G_k to this coordinate system. Let $z_k \in [a, b)$ be such that a_k is non-separated from z_k. It is clear that $g_0^{-1}(G_k) \cap (D_0 \times \{z_k\}) = \emptyset$ and if $\{(a_n, y_n)\}_{n \in Z_+}$ is a sequence in $g_0(G_k) \subset \subset D_0 \times [a, b)$ such that $y_n \to z_k$ then either $|x_n| \to 0$ or $|x_n| \to 1$. Since G_k cuts every leaf along a disk or along a bounded cylinder, we have one of the situations of figure 21.

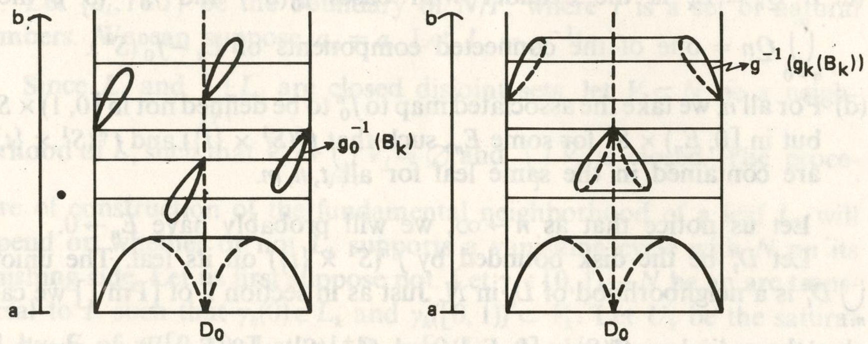

Fig. 21

Figure 21 left corresponds to the vanishing cycle case. Each leaf cuts B_k along a disk. Figure 21 right corresponds to the non vanishing cycle case. Each leaf cuts B_k along a bounded cylinder, so each leaf on $D_0 \times [a, b)$ will cut $g_0^{-1}(g_k(B_k))$ along an annulus. The surface $g_0^{-1}(g_k(G_k))$ is a cylinder which folds to its inside until one base coincides with the other. This is the case $|x_n| \to 1$. If we colapse the identified bases to a point, we get the case $|x_n| \to 0$.

What decides between limit zero and limit one is the order. The limit will be zero if $a_k < a_0$ and the limit will be one if $a_0 < a_k$. So on each leaf we will have the conjugacy already defined on a collection of disks and bounded cylinders. In order to extend it to the whole leaf, the relative position (order) among these disks and cylinders must be respected. The fact that h preserves order guarantees just that. It is sufficient to apply the one parameter version of Palais' diffeomorfhism extension theorem [P1] or [Pm 1, appendix] and proceed as in the main lemma of [Pm 1].

3.8. *Actual proof of the Theorem.*

By corollary 5, there exists n such that A^{n-1} is a discrete non empty set ($n \geq 1$). We define H arbitrarily in $p^{-1}(A^{n-1})$ preserving the orientation of the leaves, compatibly with h and extending differentiably to $-\infty$. Just as in the proof of the extension lemma, we construct neighborhoods B_L on the vanishing side of every leaf L of $p^{-1}(A^{n-1})$ and we extend H to these neighborhoods compatibly with h. In every leaf L_i in $|p^{-1}(A^{n-2})$ which intersects V_L, H is already defined on a disk; we extend it to the whole L_i. On the leaves of $p^{-1}(A^{n-2})$ which do not intersect $\bigcup_{p(L) \in A^{n-1}} B_L$ we define H arbitrarily with the same restrictions as before. We repeat the procedure and extend H to $p^{-1}(A^{n-3})$ and so on. We finally have H defined on A. We decompose M/F as in the decomposition lemma and carry the decomposition to M'/F' via h. We define H on the leaves $p^{-1}(a_k)$ of the decomposition lemma (remember that $\{a_k\} - A$ is discrete) and apply the extension lemma to the closure of each $p^{-1}[a_k, b_k)$.

4. *Proof of the Corollary.*

Let M be a 3-dimensional, simply connected, irreducible manifold with a foliation by closed cylinders F. Let A be the projection on M/F of the set of leaves supporting vanishing cycles. Let J be a maximal interval in M/F, i.e., $J = (-\infty, +\infty)$.

It is easy to construct a foliation by closed cylinders in R^3, with space of leaves R and a diffeomorphism $h: R \to J$ which preserves sides and vanishing cycles.(there is no question of order here). Just use the technique of examples 1, 2, 4.

In view of the decomposition lemma, it is sufficient to show how to add a branch corresponding to a point $c \in \partial J$. Notice that J will always be on the inside of c.

We use a procedure due to Reeb for foliations in R^2, which can be found in [R]. The basic construction is described in figure 22 where we added a leaf a non separated from the existing leaf a on a foliation in R^2.

In our case (cylinders), the situation in each interval $(a, b) \subset J$ such that $(a, b) \cap A = \emptyset$ is that of a band $a \leq z < b$ in R^3 foliated by $z =$ constant, with the segment $x = y = 0$ removed.

To add a branch corresponding to a point $c \in \partial J$, we perform the construction of the previous figure on the plane $y = 0$, and 1) if $c \in A$, we take the product with R (represented by the x-axis) 2) If $c \notin A$, we rotate the figure around the z-axis.

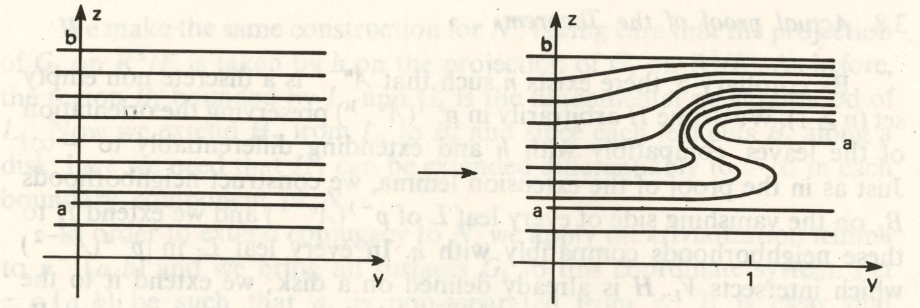

Fig. 22

As in Reeb's original construction, at the n^{th} step, the construction is made on the region $y \geq n-1$, which assures that it can be done a countable number of times, if needed, without disturbing, at each step, any of the previous ones.

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