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Cyclical monotonicity of maximal monotone step operators

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Summary.

Maximal monotone operators $T: X \to 2^Y$ such that $\{Tx\}_{x \in X}$ is a finite family of sets are shown to be cyclically monotone.

§1. Let X and Y be two dual locally convex Hausdorff topological vector spaces paired by a bilinear form $\langle x, y \rangle$.

Definition. $T : X \to 2^y$ is said to be a step multimapping if the set family $\{Tx\}_{x \in X}$ is finite; T is said to be a local step multimapping if for each $x_0 \in X$ there is a neighborhood $U(x_0)$ such that $\{Tx\}_{x \in U(x_0)}$ is finite.

If T is a step multimapping so is T^{-1}. In fact, if $\eta_1, \eta_2, \ldots, \eta_n$ are the various distinct sets of the form Tx, then

$$T^{-1}y = \bigcup_{j \in J_y} \{x \mid Tx = \eta_j\}, \qquad J(y) = \{j \mid y \in \eta_j\},$$

which shows that there are no more than 2^n possibilities for $T^{-1}y$. No such invariance is enjoyed by local step multimappings. Indeed, on the real line the inverse of any bounded local step but not step multimapping is not locally step.

In this article we are concerned with maximal monotone operators which at the same time are local step mappings. As on the real line they are the simplest imaginable monotone operators, but in higher dimensions, unlike in one dimension, the class cannot be expected to furnish an approximation for any maximal monotone operator. The reason for this apparently odd occurrence is to be found in the coming into existence — as one leaves the real line — of maximal operators other than subdifferentials, operators which by their very nature are beyond the reach of local step ones, the latter being, as we shall see, all subdifferentials. Let us make things precise:

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A function of the type $f(x) = \sup_{i=1,2,\cdots,n} [\langle x, y_i\rangle + r_i]$ is called a *polyhedral convex function.* It is obvious that the effective domain of definition of such a function is the whole space, and that it is everywhere continuous and subdifferentiable; its epigraph is the intersection of the half spaces $\langle x, y_i\rangle + r_i - z \leq 0$, $i = 1, 2, \ldots, n$, and hence is a polyhedron. The projection on X of such a polyhedron produces a partition of the space into a finite number of closed convex polyhedra on each of which f is affine. The subdifferential of f is easily calculated:

$$(1) \qquad \partial f(x) = co\{y_j\}_{j\in J(x)}, \quad J(x) = \{j \mid \langle x, y_j\rangle + r_j = f(x)\},$$

where *co* denotes the convex hull. This shows plainly that $\partial f(x)$ cannot take more than 2^n distinct set values. Since f can be identified with a continuous convex functions $\tilde{f}$ on the finite dimensional quotient space X/N, where $N = \{x \mid \langle x, y_i\rangle = 0,\ i = 1, 2, \ldots, n\}$, and ∂f with $\partial \tilde{f}$, ∂f is maximal monotone. Thus the subdifferential of a polyhedral convex function is a maximal monotone operator[1].

A convex function is said to be *locally polyhedral* if about each point in space there is a neighborhood on which the function coincides with a polyhedral one. Like the earlier kind locally polyhedral convex functions have the whole space as effective domains, are continuous and subdifferentiable everywhere. Moreover, since convex functions coinciding on an open set have the same subgradients at points of this set the subdifferentials of locally polyhedral functions are monotone local step operators. It remains to see that they are maximal monotone. The easiest way is to realize that the locally matching polyhedral convex functions can be chosen so that their subdifferentials take all their values in the corresponding neighborhood (which can be assumed to be convex), and then to remark that the enlargement to a monotone set of the graph of the subdifferential of a locally polyhedral function by the addition of a new point $(\tilde{x}, \tilde{y})$ is incompatible with the maximal monotonicity of the matching functions around $\tilde{x}$. In conclusion, the subdifferentials of locally polyhedral convex functions are maximal monotone locally step operators.

Much less obvious and considerably more interesting is the converse of this proposition, from which the identifications of maximal monotone local step operators with subdifferentials of locally polyhedral convex functions follows.

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[1] We recall that outside of reflexive Banach spaces the subdifferentials of $\ell.s.c.$ proper convex functions are not necessarily maximal monotone.

Theorem 1. *A maximal monotone operator having the whole space as domain is a local step mapping if and only if it is the subdifferential of a locally polyhedral convex function.*

The proof, somewhat long, requires various lemmas. The leading idea is to think of the operator as a flow and remark that a circulating (non-cyclical) flow in a simple connected domain must necessarily have non-vanishing vorticity somewhere.

Lemma 1. *Let M be a maximal monotone local step operator with $D(M) = X$, and K a compact set in X. Then there is only a finite number of sets of the form $M^{-1}y \cap K$, $y \in X$, and they form a closed covering of K.*

Proof. By covering K with a finite family of neighborhoods on each of which M takes a finite number of set values one concludes that the restriction of M to K is a step multimapping, and that hence so is its inverse. This amounts to say that the sets of the form $M^{-1}y \cap K$, $y \in Y$, which are closed because of maximal monotonicity, are finite in number and cover K.

Lemma 2. *Let M be a maximal monotone local step operator with domain X, and K a compact finite dimensional set in X; moreover, let $\|\ \|$ be a norm on the finite dimensional space spanned by K, and for any $x \in K$ and $\varepsilon > 0$ define $p(x, \varepsilon)$ as the distance from x to the intersection of all the sets of the form $M^{-1}y \cap K$ whose distances to x do not exceed ε, setting $p(x, \varepsilon) = +\infty$ if the intersection is empty. Then, a) there is a positive number p_K such that $\varepsilon < p_k$ implies $p(x, \varepsilon) < +\infty$, $\forall x \in K$; b) $\lim_{\varepsilon\to 0} p(x, \varepsilon) = 0$ uniformly in $x \in K$.*

Proof. Let $\eta_1, \eta_2, \ldots, \eta_n$ be the distinct sets of the form $M^{-1}y \cap K$, and for any subclasse $\eta_i, \eta_j, \ldots, \eta_k$ let $p_{i,j,\ldots,k}$ denote the radius of the smallest closed ball with center in K touching all its members. Some of these quantities vanish, indicating that $\eta_i, \eta_j, \ldots, \eta_k$ have some point in common; the others, finite in number, have a positive minimum p_K. This number has the property that if a closed ball with center in K and radius strictly smaller than p_K intersects a certain number of η_i's then these sets have a nonempty intersection. It follows that if $\varepsilon < p_K$ then $p(x, \varepsilon) < +\infty$, $\forall x \in K$ proving a). As to b) argue by contradiction and assume the existence of a sequence $\{x_n\}^\infty \dot{\subset} K$, a sequence of positive numbers converging to zero $\{\varepsilon_n\}^\infty$, and a $\delta > 0$ such that $p(x_n, \varepsilon_n) \geq \delta$, $n = 1, 2, \ldots$. Since K is compact we can assume the x_n's to be convergent to an $x \in K$, and since the intersection of the η_i's at a distance from x_n not exceeding ε_n is one of finitely many sets we may at the same time assume that the closed balls $B_{\varepsilon_n}(x_n)$ intersect a fixed group of η_i's, say $\eta_{i_1}, \eta_{i_2}, \ldots, \eta_{i_s}$. But then, the η_i's being closed, $x \in \eta_{i_1} \cap \eta_{i_2} \cap \ldots \cap \eta_{i_s}$, and $p(x_n, \varepsilon_n) \leq \|x - x_n\|$, in contradiction with the assumption $p(x_n, \varepsilon_n) \geq \delta$.

In the next lemmas we shall turn our attention to sums of the form

$$\begin{aligned} \Sigma_\pi = \sum_{i=1}^{n} \langle x_{i+1} - x_i, y_i \rangle = \langle x_2 - x_1, y_1 \rangle + \\ + \langle x_3 - x_2, y_2 \rangle + \ldots + \langle x_n - x_{n-1}, y_{n-1} \rangle + \langle x_1 - x_n, y_n \rangle \end{aligned} \tag{2}$$

associated with cycles of points $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n), (x_1, y_1) \ldots$, (where (x_{i+1}, y_{i+1}) follows (x_i, y_i), and (x_1, y_1) follows (x_n, y_n)), in the graph of maximal monotone local step operators M, aiming to prove that they are nonpositive, and in consequence that M is cyclically monotone. The letter π in the notation Σ_π referes to the closed oriented polygonal line in X obtained by joining by a straight segment each point x_i to its successor x_{i+1}. We shall indicate by (π) the set of points π consists of. Remark that neither (π) nor π surffices to determine Σ_π.

Lemma 3. *For given* $\Sigma_\pi = \sum_{j=1}^{n} \langle x_{j+1} - x_j, y_j \rangle$, *and a finite number of points* $\{z_i\}_1^k \subset (\pi)$ *there is a sum* $\tilde{\Sigma}_{\tilde{\pi}} = \sum \langle \tilde{x}_{i+1}, -\tilde{x}_i, \tilde{y}_i \rangle$ *such that*

$$(\tilde{\pi}) = (\pi) \tag{3}$$

$$z_\ell \textit{ is a vertex of } \tilde{\pi}, \; \ell = 1, 2, \ldots, k \tag{4}$$

$$\tilde{y}_i \in M\tilde{x}_{i+1} \cap M\tilde{x}_i, \; \forall i \tag{5}$$

$$\Sigma_\pi \leq \tilde{\Sigma}_{\tilde{\pi}}. \tag{6}$$

Proof. Proceeding in cyclical order divide the segment $[x_j, x_{j+1}]$ into a finite number of segments individually contained in an $M^{-1}y$, including as division points the z_ℓ's left over from previous similar operations. Lemma 1 guarantees the possibility of such a division. Call the dividing points $\tilde{x}_{j_1} = x_j, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_q} = x_{j+1}$, and picking $\tilde{y}_{j_h} \in M\tilde{x}_{j_h} \cap M\tilde{x}_{j_{h+1}}$ write

$$\begin{aligned} \sum_j \langle x_{j+1} - x_j, y_j \rangle = \sum_j [\langle \tilde{x}_{j_{q-1}} - \tilde{x}_{j_q}, \tilde{y}_{j_{q-1}} \rangle + \\ + \langle \tilde{x}_{j_{q-2}} - \tilde{x}_{j_{q-q}}, \tilde{y}_{j_{q-2}} \rangle + \ldots + \langle \tilde{x}_{j_1} - \tilde{x}_{j_2}, \tilde{y}_{j_1} \rangle + \langle \tilde{x}_{j_q} - \tilde{x}_{j_1}, y_j \rangle] \\ + \sum_j [\langle \tilde{x}_{j_2} - \tilde{x}_{j_1}, \tilde{y}_{j_1} \rangle + \langle \tilde{x}_{j_3} - \tilde{x}_{j_2}, \tilde{y}_{j_2} \rangle + \ldots + \langle \tilde{x}_{j_q} - \tilde{x}_{j_{q-1}}, \tilde{y}_{j_{q-1}} \rangle]. \end{aligned} \tag{7}$$

The first on the right is a sum of sums over one-dimensional cycles and as such is non-positive simply because in one-dimensional spaces all monotone operators are cyclically monotone; the second is a sum $\tilde{\Sigma}_{\tilde{\pi}}$ of the sought type.

Lemma 4. *For a closed polygon* π *of vertices* $\{x_i\}_1^n$ *and any two of its points* u, v $(u \in [x_p, x_{p+1}], v \in [x_q, x_{q+1}], p \leq q)$ *let* π_1 *and* π_2 *denote the polygons with vertices at* $u, x_{p+1}, x_{p+2}, \ldots, x_q, v$ *and at* $v, x_{q+1}, x_{q+2}, \ldots, x_n, x_1, \ldots, x_p, u$ *respectively. Then, for any given sum* Σ_π *there are sums* $\tilde{\Sigma}_{\tilde{\pi}_1}$ *and* $\tilde{\Sigma}_{\tilde{\pi}_2}$ *such that*

$$\Sigma_\pi \leq \tilde{\Sigma}_{\tilde{\pi}_1} + \tilde{\Sigma}_{\tilde{\pi}_2}, \quad (\tilde{\pi}_1) = (\pi_1), \quad (\tilde{\pi}_2) = (\pi_2). \tag{8}$$

Proof. By the previous lemma Σ_π admits a majoration by a sum $\tilde{\Sigma}_{\tilde{\pi}}$ on a polygon such that $(\tilde{\pi}) = (\pi)$ having u and v as vertices. It is clear then that it suffices to prove the lemma for $\tilde{\Sigma}_{\tilde{\pi}}$, and in consequence that we may assume from the start that u and v are vertices of the given polygon. By cyclical re-numbering of the vertices u and v can be taken to coincide with x_1 and x_q respectively. Now by means of points $\tilde{x}_1 = x_1, \tilde{x}_2, \ldots, \tilde{x}_m = x_q$ make a partition of $[x_1, x_q]$ into a finite number of smaller segments individually contained in sets of the form $M^{-1}y$, $y \in Y$, and letting $\tilde{y}_i \in M\tilde{x}_{i+1} \cap M\tilde{x}_i$, $i = 1, 2, \ldots, m$, write

$$\begin{aligned} \Sigma_\pi = \sum_{j=1}^{n} \langle x_{j+1} - x_j, y_j \rangle = [\langle x_2 - x_1, y_1 \rangle + \langle x_3 - x_2, y_2 \rangle + \ldots + \\ + \langle x_q - x_{q-1}, y_{q-1} \rangle + \langle \tilde{x}_{m-1} - \tilde{x}_m, \tilde{y}_{m-1} \rangle + \langle \tilde{x}_{m-2} - \tilde{x}_{m-1}, \tilde{y}_{m-2} \rangle + \ldots + \\ + \langle \tilde{x}_1 - \tilde{x}_2, \tilde{y}_1 \rangle] + [\langle \tilde{x}_2 - \tilde{x}_1, \tilde{y}_1 \rangle + \langle \tilde{x}_3 - \tilde{x}_2, \tilde{y}_2 \rangle + \ldots + \\ + \langle \tilde{x}_m - \tilde{x}_{m-1}, \tilde{y}_{m-1} \rangle + \langle x_{q+1} - x_q, y_q \rangle + \langle x_{q+2} - x_{q+1}, y_{q+1} \rangle + \ldots + \\ + \langle x_n - x_{n-1}, y_{n-1} \rangle + \langle x_1 - x_n, y_n \rangle]. \end{aligned}$$

A simple inspection shows that the sums in brackets on the right are sums $\tilde{\Sigma}_{\tilde{\pi}_1}$, $\tilde{\Sigma}_{\tilde{\pi}_2}$ of the desired type.

Lemma 5. *Any maximal monotone locally step operator* M *with* $D(M) = X$ *is cyclically monotone.*

Proof. The point of the proof is to show that all sums Σ_π of type (2) associated with M are nonpositive. Let K be the closed convex hull of (π); K is compact and finite dimensional. By repeated appeals to Lemma 4 Σ_π may be majorated by a finite sum of similar sums over cycles on the boundaries of triangles contained in K. Since in turn these triangles can be divided ad libitum into smaller triangles, the problem is reduced to see that any Σ_π over a sufficiently small triangle contained in K is nonpositive.

Let us recall that by Lemma 1 there is only a finite number of sets $M^{-1}y \cap K$, and that they form a covergin $\{\eta_i\}_1^m$ of K. If (π) is a triangle and $\Sigma_\pi = \sum_{i=1}^{n} \langle x_{i+1} - x_i, y_i \rangle$, then Lemma 3 allows us to assume that for each i the segment $[x_i, x_{i+1}]$ is contained in a η_{j_i}. By Lemma 2 these sets have a point x in common as soon as the diameter δ of (π) is smaller than $p_K/2$, implying that $Mx \cap Mx_i \neq \emptyset$, $i = 1, 2, \ldots, n$. By the same lemma, if δ is sufficiently small, x can be so chosen that its distance to (π) is smaller than $p_K/2$. Now, for each i choose $\tilde{y}_i$ in $Mx \cap Mx_i$, and write down the identity

$$\Sigma_\pi = \sum_{i=1}^{n} \langle x_{i+1} - x_i, y_i \rangle = \sum_{i=1}^{n} [\langle x_{i+1} - x_i, y_i \rangle + \langle x - x_{i+1}, \tilde{y}_{i+1} \rangle + \langle x_i - x, \tilde{y}_i \rangle].$$

Each term in square brackets is a sum of type (2) over the vertices of a triangle in K, each of whose sides is shorter than $p_K/2$ and contained in a $M^{-1}y$. Thus the lemma will be proved as soon as it is shown that all such sums are nonpositive. So let x_1, x_2, x_3 be three points in K with $\|x_i - x_j\| < p_K/2$, $i, j = 1, 2, 3$, such that any two belong to the same η_i, and y_1, y_2, y_3 any three points in Mx_1, Mx_2, Mx_3 respectively. From the definition of p_K (Lemma 2) it follows the existence of an x_0 belonging simultaneously to the η_i's containing the sides. This means that each of the triangles (x_1, x_2, x_0), (x_1, x_0, x_3), (x_0, x_2, x_3) is contained in a single η_i. Let $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$ be such that $(x_1, x_2, x_0) \subset M^{-1}\tilde{y}_1$, $(x_0, x_2, x_3) \subset M^{-1}\tilde{y}_2$, $(x_1, x_0, x_3) \subset M^{-1}\tilde{y}_3$, and remark that

$$\langle x_2 - x_1, y_1 \rangle + \langle x_3 - x_2, y_2 \rangle + \langle x_1 - x_3, y_3 \rangle = \langle x_2 - x_1, y_1 - \tilde{y}_1 \rangle + \langle x_3 - x_2, y_2 - \tilde{y}_2 \rangle + \langle x_1 - x_3, y_3 - \tilde{y}_3 \rangle + \langle x_1 - x_0, \tilde{y}_3 - \tilde{y}_1 \rangle + \langle x_2 - x_0, \tilde{y}_1 - \tilde{y}_2 \rangle + \langle x_3 - x_0, \tilde{y}_2 - \tilde{y}_3 \rangle.$$

By the way the $\tilde{y}_i$'s have been chosen and the monotonicity of M all terms on the right are nonpositive, and the lemma is proved.

Proof of Theorem 1. Any maximal monotone, cyclically monotone operator M is a subdifferential. Indeed, if z is any fixed point in $D(M)$,

$$(9) \qquad f(x) = \sup \left[\langle x - x_n, y_n \rangle + \sum_{1}^{n-1} \langle x_{i+1} - x_i, y_i \rangle \right], \quad (x_i, y_i) \in G(T),\ i = 1, 2, \ldots, n, \quad x_1 = z, \quad \forall n$$

is a proper $\ell.s.c.$ convex function whose subdifferential extends and hence coincides with M. The proof, an immediate consequence of cyclical monotonicity and the definition of f, is left to the reader. With this remark and with the assistance of Lemma 5 the proof of the theorem is reduced to showing that an everywhere defined $\ell.s.c.$ convex function having a local step subdifferential is locally polyhedral. To this end let us consider any closed, convex set U with non-empty interior on which ∂f is a step

multimapping. There is then a finite number of distinct sets of the form $(\partial f)^{-1}y \cap U$, say $\eta_i = (\partial f)^{-1}y_i \cap U$, $i = 1, 2, \ldots, p$; they are closed and convex, and cover U. For each i pick a point x_i in η_i. Now, y_i being a subgradient of f at x_i, $f(x) \geq f(x_i) + \langle x - x_i, y_i \rangle$, $\forall x \in X$, $i = 1, 2, \ldots, p$, and therefore $f(x) \geq \sup_i [f(x_i) + \langle x - x_i, y_i \rangle]$, $\forall x \in X$. On the other hand if $x \in \eta_j$, y_j is a subgradient of f at x, and hence $f(x) \leq f(x_j) + \langle x - x_j, y_j \rangle \leq \sup_i [f(x_i) + \langle x - x_i, y_i \rangle]$. But then, since any $x \in U$ belongs to some η_j, $f(x) = \sup_i [f(x_i) + \langle x - x_i, y_i \rangle]$, $\forall x \in U$, and f is locally polyhedral.

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