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Endomorphisms of Branched one-dimensional Manifolds

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Introduction.

This paper is concerned with the description of the orbit structure of endomorphisms of a branched one-dimensional manifold K.

Endomorphisms of branched 1-manifolds have been studied by Williams [7], who showed how to unfold certain endomorphism of K to obtain diffeomorphisms with similar orbit structure. Here, we will be mainly interested in the problem of the topological classification of the endomorphisms of K.

In [4] Shub proved that the expanding endomorphisms of any compact differentiable manifold M are structurally stable. Later on Jakobson [2] considered the case $M = S^1$ and contructed in the space $C^2(S^1, S^1)$, of all C^2 endomorphisms of S^1 into S^1, an open set J consisting of structurally stable endomorphisms. He showed that J together with the expanding endomorphisms of Shub are C^1-dense in $C^2(S^1, S^1)$.

In this paper, in Part 4, we extend this result of Jakobson to the space $Im^1_B(K)$ of all C^1 immersions of K preserving the branch set B; in Part 3, we extend the result of Shub for expanding endomorphisms of K preserving B. In Part 2 we give a necessary condition for structural stability of certain endomorphisms of K. We also prove that expanding endomorphisms on a branched 1-manifold with non empty branch set are unstable. Using this fact we prove that structural stability is not generic.

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1. Preliminaries.

We recall here a few definitions in order to establish the terminology.

To define a *branched 1-manifold* K [7], one proceeds just as in the definition of a 1-manifold, except that two types of coordinate neigborhoods are allowed. These are the real line $\mathbb{R}$ and $Y = \{(x, y) \in \mathbb{R}^2;\ y = 0$ or $y = \varphi(x)\}$. Here $\varphi : \mathbb{R} \to \mathbb{R}$ is a fixed C^∞ function such that $\varphi(x) = 0$ for $x \leq 0$ and $\varphi(x) > 0$ for $x > 0$. *The branch set* B of K is the set of all points of K corresponding to $(0, 0) \in Y$. If K is compact, B is finite and $K - B$ has a finite number of components. In this case, the closures of these componentes will be called the *simplexes* of K. For a neighborhood Y of a branch point r, *the branches at* r are the two 1-submanifolds corresponding to $Y_1 = \{(x, 0) \in Y\}$ and $Y_2 = \{(x, y) \in Y;\ y = \varphi(x)\}$.

A Cr structure for a branched 1-manifold is defined as usual; note that K has a tangent bundle $T(K)$ since the two branches of a branch point r have the same tangent line at r. A differentiable map $f : K_1 \to K_2$ of branched 1-manifolds induces a map $Df : T(K_1) \to T(K_2)$ of their tangent bundles; f is an *immersion* if Df is a monomorphism on the tangent space at each point. A C^r-immersion $f : K \to K$ is called *expanding* relative to a Riemannian metric $\| \ \|$ on $T(K)$, if there are constants $c > 0$ and $\lambda > 1$ such that $\| Df^n(x) \| \geq c\lambda^n$ for all $x \in K$ and $n \in \mathbb{Z}^+$.

We denote by $End^r(K)$, $r = 1, 2, \ldots, \infty$, the space of C^r maps of a compact connected branched 1-manifold K. For $f \in End^r(K)$, a point $x \in K$ is a *periodic point* of f with period n if $f^n(x) = x$ and $f^m(x) \neq x$ for all $m < n$; x is *hiperbolic* if $|Df^n(x)| \neq 1$; x is a *source* if $|Df^n(x)| > 1$ and x is *contracting* if $|Df^n(x)| < 1$. In this case x has a local stable manifold $W^s_\varepsilon(x)$, which is either an open interval or a neighborhood Y. The *stable manifold* of x, $W^s(x)$ is defined by $W^s(x) = \bigcup_{n \geq 0} f^{-n}(W^s_\varepsilon(x))$ and the *stable manifold* of f, $\Delta(f)$ is defined by $\Delta(f) = UW^s(x)$, where the union is taken over all contracting periodic points x of f. We denote $\Sigma(f) = K - \Delta(f)$. Note that $\Delta(f)$ and $\Sigma(f)$ are invariant under f. A point x is *eventually periodic* of f if some iterate of x is a periodic point of f. A point x is called a *singularity* (turning point) of f if $Df(x) = 0$; x is non-degenerate if $D^2 f(x) \neq 0$. A point x is said to be *wandering* if x has a neighborhood U with the property that $f^n(U) \cap U = \varnothing$ for all n. Otherwise x is called *non-wandering*.

A endomorphism $f \in End^r(K)$ is said to be structurally stable if there exists a neighborhood U of f in $End^r(K)$ such that any $g \in U$ is topologically conjugate to f; i.e. there exists a homeomorphism $h : K \to K$ satisfying $hf = gh$.

In this paper K will denote a compact connected branched 1-manifold with non empty branch set.

2. Nongenericity of structural stability.

The following result will be used to prove that structurally stable endomorphisms are not dense in $End^r(K)$.

Theorem 2.1. *Let $f \in End^r(K)$ such that $f/\Sigma(f)$ is expanding. If f is structurally stable then $B \cap \Sigma(f) = \varnothing$.*

Proof. The proof will be done by contraction; so we assume that $B \cap \Sigma(f) \neq \varnothing$ and show that there exists g, arbitrarilly close to f, which is not conjugate to f. Let $r \in \Sigma(f) \cap B$. Then either $r \in \overline{\Delta(f) \cap B}$ or $r \in \overset{\circ}{\widehat{\Sigma(f)}} \cap B$.

If $r \in \overline{\Delta(f) \cap B}$, there exists $y \in \Delta(f)$, arbitrarilly close to r. Then, in a small neigborhood Y of r, we have that $f^n(y) \notin Y$ for all $n \geq 1$. Since B is finite, we may assume by choosing Y smaller if necessary, that $f^n(X) \notin Y$ for all $n \geq 1$ and $x \in B \cap \Delta(f)$. Then, there is a small C^r-perturbation g of f such that $f = g$ outside a small neighborhood of Y and $g(r) = f(y)$. Thus $B \cap \Delta(g) \supset (B \cap \Delta(f)) \cup \{r\}$ and f is not conjugate to g, since a conjugacy preserves B and the stable manifold. If $r \in \overset{\circ}{\widehat{\Sigma(f)}} \cap B$, we have that $r \in \overline{Uf^{-n}(B)}$, since $f/\Sigma(f)$ is expanding. Thus we can find g_1 and g_2 arbitrarilly close to f such that $g^n_1(r) \in B$ and $g^n_2(r) \notin B$ for some n. Since B is finite we may assume that $g^n_1(\tilde{r}) = g^n_2(\tilde{r}) = f^n(\tilde{r})$ for $\tilde{r} \in \Sigma(f) \cap B$ with $\tilde{r} \neq r$. Then f is not conjugate to g_i for some $i = 1, 2$. This finishes the proof of the theorem.

As an inmediate Corollary we have

Corollary 2.2. *The expanding endomorphisms of K are not structurally stable.*

Now we prove that structural stability is not generic.

Theorem 2.3. *There exists an open set U in $End^r(K)$ such that no $f \in U$ is structurally stable.*

The proof of the theorem requires the following.

Lemma 2.4. *K contains a branched submanifold K_1 which is either the circle S^1 or the branched 1-manifold shown in figure 2.4.*

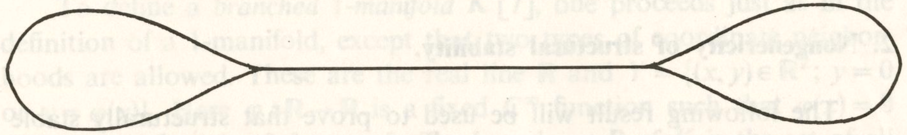

Fig. 2.4

Proof. We first show that K contains a loop L which is not contractible. In fact let $I_1 = [r_0, r_1]$ be a simplex of K. If $r_0 = r_1$ we are done. Otherwise we consider a simplex $I_2 = [r_1, r_2]$ such that the juxtaposition $I_1 \vee I_2$ is a branch at r_1. If $r_2 = r_i$ for some $i = 1, 2$, we are done. Otherwise we consider $I_3 = [r_2, r_3]$ such that $I_2 \vee I_3$ is a branch at r_2. We continue this procedure. Since B is finite, after finitely many steps we get a loop L with vertex $r \in B$ such that L is not contractible and $L - \{r\}$ is a differentiable arc. If L is differentiable at r then $L = S^1$ and the proof of lemma follows, so we assume that L is not differentiable at r. Without loss of generality we may assume that $L = I_1 \vee \dots \vee I_s$ where $I_i = [r_{i-1}, r_i]$ and $r = r_0 = r_s$. Consider the simplex $I_{s+1} = [r_s, r_{s+1}]$ such that $-I_1 \vee I_{s+1}$ and $I_s \vee I_{s+1}$ are the branches at r_0. If $r_{s+1} = r_i$ for some $i = 1, \dots, s-1$, then either

$$I_{s+1} \vee I_{i+1} \vee I_{i+2} \vee \dots \vee I_s \quad \text{or} \quad -I_{s+1} \vee I_1 \vee \dots \vee I_i$$

are circles and in this case the lemma follows, so we assume that $r_{s+1} \neq r_i$ for all $i = 1, \dots, s$. Consider $I_{s+2} = [r_{s+1}, r_{s+2}]$ such that $I_{s+1} \vee I_{s+2}$ is a branch at r_{s+1}. If $r_{s+2} = r_{s+1}$ then either I_{s+2} is a circle S^1 or $I_1 \vee \dots \vee I_s \vee I_{s+1} \vee I_{s+2}$ is a branched submanifold as shown in figure 2.4. If $r_{s+2} = r_i$ for some $i = 2, \dots, s-1$ then following the same procedure above we get a circle S^1. In both cases the lemma follows. We continue this procedure since B is finite, after finitely many steps we get either a circle S^1 or a branched submanifold as shown in figure 2.4.

Proof of Theorem 2.3. Let K_1 be as in lemma 2.4. It is clear that K_1 allows an expanding endomorphism $\tilde{f}: K_1 \to K_1$. Extend $\tilde{f}$ to an endomorphism f on K. Since K_1 is compact and contains no singularities of f, there are neighborhoods U of f in $End^r(K)$ and V of K_1 satisfying the following conditions

1) No simplex I of $K - K_1$ is contained in V
2) $g(K_1) \subset V$ and V contains no singularities of g, for any $g \in U$.

We claim that no $g \in U$ is structurally stable. First we show that K_1 is

g-invariant. In fact, for any simplex I of K_1, it follows easily from property 2) that $g(I)$ is a juxtaposition of simplexes of K, $g(I) = I_1 \vee \dots \vee I_n$. Then from 1) it follows that $g(I) \subset K_1$, so K_1 is g-invariant. We may assume by choosing U smaller if necessary that $g/K_1: K_1 \to K_1$ is expanding. Then as in the proof of theorem 2.1, it follows that g is not structurally stable and the proof of theorem is finished.

Remark 2.5. Call J the set of $f \in End^2(K)$ satisfying the following conditions:

J_1) f has a finite (non zero) number of contracting periodic points and all critical points of f lie in $\Delta(f)$.

J_2) All critical points of f are nondegenerate and no critical point is eventually periodic.

J_3) $|Df^n(x)| > \alpha c^n$, $\alpha = \alpha(f) > 0$, $c = c(f) > 1$, for $x \in \Sigma(f)$.

J_4) Iterates $f^k(g)$ and $f^{\ell}(z)$ of distinct critical points y and z do not coincide for any k and ℓ.

J_5) The branch set is contained in $\Delta(f)$, no branch point is eventually periodic point of f and iterates $f^k(y)$ and $f^{\ell}(z)$ of distinct branch points y and z do not coincide for any k and ℓ.

J_6) No critical point of f is eventually a branch point and no branch point is eventually a critical point of f.

Using the same techniques as in [2] it can be shown that J consists of structurally stable endomorphisms. Also it can be shown, using arguments similar to those used in the proof of Theorem 2.1 that in presence of the condition J_3), the other conditions are necessary for structural stability, so it is reasonable to expect that J consists of the all C^2 structurally stable endomorphisms of K. This however is unknown even for $B = \emptyset$ (Jacobson [2]).

3. Expanding Endomorphisms preserving the Branch set.

From this chapter on, we study the classification of endomorphisms of K preserving B by the relation of topological equivalence.

We denote by $End^r_B(K)$ the set of endomorphisms of K preserving B. Clearly $End^r_B(K)$ is a closed subspace of $End^r(K)$ which contains all the diffeomorphisms of K. We remark that no $f \in End^r_B(K)$ is structurally stable. In fact, by a small perturbation of f we get $g \in End^r(K)$ such that B is not g-invariant. Then f and g are not topologically conjugate.

This section is devoted to the proof of the following

Theorem 3.1. *Let $f,\ g \in End^r_B(K)$ be expanding endomorphisms homotopic relative to B. Then f and g are topologically conjugate.*

Before proving this theorem we establish several preliminary results. The following two lemmas are reformulation of Lemmas 1a. and 1b, page 164 of [2]. They can be proved in a similar way as in [2].

Lemma 3.2. a) *Let $f \in End^2(K)$ and $I_0, I_1, \ldots, I_n, \ldots$ a sequence of intervals contained in K and such that*

i) $I_i = f(I_{i-1})$ *for all* $i \in \mathbb{N}$

ii) $\sum_{i=0}^{\infty} \mu(I_i) < \infty$, *where μ is the usual Borel measure;*

iii) *There is a constant $c > 0$ such that $|Df(x)| > c$ for all $x \in \bigcup_{i=0} I_i$.*

Then $\sum_{n=0}^{\infty} |Df^n(x_0)| < \infty$ *for all* $x_0 \in I_0$.

Lemma 3.2. b) *Let $f \in End^2_B(K)$ and $x_0 \in K$ such that*

i) $\sum_{n=0} |Df^n(x_0)| < \infty$,

ii) $|Df(f^n(x_0))| > c > 0$ *for all* $n \in \mathbb{N}$.

Then, there exists a neighborhood I of x_0 such that

$$\sum_{n=0}^{\infty} |Df^n(x)| < \infty \quad \text{for all } x \in I_0.$$

Also we need the following lemmas

Lemma 3.3. *Let $f \in End^r_B(K)$ be an immersion such that all its periodic points are sources. Then*

a) *The set $P = \bigcup f^{-n}(B)$ is a countable dense set of K.*

b) $\Omega(f) = \overline{Per(f)}$. *Here $\Omega(f)$ denotes the set of nonwandering points of f and $Per(f)$ denotes the set of all periodic points of f.*

Proof. a) First we show that P is dense. Let I be a connected component of $K - P$. We must show that I is a point. Suppose this is false. Then $f^n(I)$ is an interval for all $n \in \mathbb{N}$, since $K - P$ is f-invariant. We claim that $f^n(I) \cap f^m(I) \neq \varnothing$ for some $n \neq m$. Otherwise $\sum_{n=1}^{\infty} \mu(f^n(I)) < \infty$ and by lemma 3.2.a) we have that $\sum_{n=1}^{\infty} |Df^n(x)| < \infty$ for all $x \in I$. In particular

$\sum_{n=1}^{\infty} |Df^n(a)| < \infty$ for any endpoint a of I. It follows from lemma 3.2.b) that there exists a neigborhood U of a such that $\sum_{n=1}^{\infty} |Df^n(x)| < \infty$ for al $x \in U$. But $U \cap P \neq \varnothing$ and for all $x \in P$, $\sum_{n=1}^{\infty} |Df^n(x)|$ is divergent because x is eventually a source. This contradiction proves that $f^n(I) \cap f^m(I) \neq \varnothing$ for some $n \neq m$. Let $\tilde{I}$ be the component of $K - P$ which contains $f^n(I) \cap f^m(I)$. Then $f^{|n-m|}(\tilde{I}) \subset \tilde{I}$, so I contains a periodic point of f which is either contracting or neutral. But by hypothesis all periodic points of f are sources. This contradiction proves that I is a point. Hence P is dense.

Now we prove that P is countable. Let $n \in \mathbb{N}$. Since f^n is an immersion we have that for each $x \in K$ there exists a neigborhood U of x such that f^n is at most 2-to-1 in U. It follows from this that $f^{-n}(B)$ is finite because B is finite and K compact. Hence P is countable.

b) Let $x \in \Omega(f)$. Since B is finite, $K - B$ is dense. So it is enough to take $x \in K - B$. Let I be any interval around of x and such that $I \cap B = \varnothing$. By the part a) we may assume that the endpoints of I belong to P. Then there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, the endpoints of $f^n(I)$ belong to B, so $f^n(I)$ is a juxtaposition of simplexes of K because by hypothesis f has no singularities. Take $n > n_0$ such that $f^n(I) \cap I \neq \varnothing$. Then I is contained in some simplex of $f^n(I)$ because $I \cap B = \varnothing$. Hence I contains a periodic point of f. Then $\Omega(f) = \overline{Per(f)}$.

Lemma 3.4. *Let $f,\ g \in End^r_B(K)$ be expanding endomorphisms homotopic relative to B. Then for each simplex I of K, $f(I) = g(I)$ and $f(I)$ is a juxtaposition of simplexes of K, $f(I) = g(I) = J_1 \vee \ldots \vee J_n$.*

Proof. Let $I = [a, b]$ be a simplex of K and let $a = t_0 < t_1 < \ldots < t_n = b$ be the partition in I given by $f^{-1}(B)$. Since f and g are homotopic relative to B we have that $f(I)v - g(I)$ is contractible and since f and g are expanding, $f(I) = J_1 \vee \ldots \vee J_n$ and $g(I) = \tilde{J}_1 \vee \ldots \vee \tilde{J}_m$ with $J_i \neq J_{i+1}$ and $\tilde{J}_j \neq \tilde{J}_{j+1}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We claim that $J_n = \tilde{J}_m$. Otherwise $f(I)v - g(I)$ contains a closed curve which is not contractible. Then $f(I)v - g(I)$ is not contractible. This contradiction proves that $J_n = \tilde{J}_m$. By exactly the same argument it follows that $J_{n-1} = \tilde{J}_{m-1}, \ldots, J_{n-r} = \tilde{J}_{m-r}$ for all $r \leq n$. Then $n = m$ and $f(I) = g(I) = J_1 \vee \ldots \vee J_n$.

Proof of Theorem 3.1. Call $P_1 = f^{-1}(B)$ and $P'_1 = g^{-1}(B)$; we will define a homeomorphism $h_1 : P_1 \to P'_1$ satisfying $h_1 f = g h_1$. Let $I = [r_1, r_2]$ be a simplex of K. By lemma 3.4 there exist two partitions $x_1 = r_1 < x_2 < \ldots < x_n = r_2$ and $x'_1 = r_1 < x'_2 < \ldots < x'_n = r_2$ in I given by P_1 and P'_1 respectively and such that $f[x_s, x_{s+1}] = g[x'_s, x'_{s+1}] = J_s$ where J_s is a simplex of K and $s = 1, \ldots, n-1$. We define h_1 on $P_1 \cap I$ by $h_1(x_i) = x'_i$ for all $i = 1, \ldots, n$. It is clear that h_1 is a homeomorphism on P_1 satisfying $h_1 f = g h_1$. Now we extend h_1 to a conjugacy h_2 on $P_2 = f^{-2}(B)$. Let us write $I_s = [x_s, x_{s+1}]$ and $f_s = f/I_s$ for $s = 1, \ldots, n-1$; $I'_s = [x'_s, x'_{s+1}]$ and $g_s = g_s/I'_s$. It's clear that f_s and g_s are homeomorphisms onto J_s. We define h_2 on $P_2 \cap I$ by $h_2 = h_1$ on $P_1 \cap I$ and $h_2(x) = g_s^{-1} h_1 f_s(x)$ for $x \in (P_2 - P_1) \cap I_s$. Since f_s and g_s are both increasing homeomorphism or both decreasing, h_2 is a increasing homeomorphism. It's clear that $h_2 f = g h_2$. Inductivily we obtain a sequence of increasing homeomorphisms $h_n : P_n \to P'_n$ with $h_{n+1} = h_n$ on P_n and $h_{n+1} f = g h_{n+1}$. Then we can define a conjugacy h on P between f and g by $h(x) = h_n(x)$, where $x \in P_n$. By lemma 3.3, $P \cap I$ and $P' \cap I$ are denses in I, for all simplexes I of K. This together with the fact that h is increasing in I, imply that h can be extended to a conjugacy on K. Hence f and g are topologically conjugate.

4. Structural stability in $End^r_B(K)$.

An endomorphism $f \in End^r_B(K)$ is said to be B-structurally stable if there exists a neighborhood U of f in $End^r_B(K)$ such that if $g \in U$ then f and g are topologically conjugate. It follows from theorem 3.1 that the expanding endomorphisms of K preserving B are B-structurally stable.

Call J_B the set of $f \in End^2_B(K)$ satisfying the conditions J_1 to J_4 of Remark 2.5. In this section we prove the following theorems.

Theorem 4.1. *J_B is an open set of $End^2_B(K)$ consisting of B-structurally stable endomorphisms.*

Theorem 4.2. *Let $Im^1_B(K)$ be the space of immersions of K preserving B with the C^1-topology. Then the set of C^1 B-structurally stable immersions is dense in $Im^1_B(K)$.*

Before proving the theorems, we have to establish some preliminary lemmas.

Let $f \in End^r_B(K)$. Denote by $K^1(f)$ the union of the simplexes I of K such that $I \subset \Sigma(f)$. Call $K^2(f) = K - K^1(f)$; $\Sigma^1(f) = \bigcup_{n=0}^{\infty} f^{-n}(K^1(f))$ and $\Sigma^2(f) = \{x \in \Sigma(f) : f^n(x) \in \overline{K^2(f)} \text{ for all } n\}$. Clearly $\Sigma^1(f)$ and $\Sigma^2(f)$ are compact and $\Sigma(f) = \Sigma^1 \cup \Sigma^2$.

Lemma 4.3. *Let $f \in End^2_B(K)$ with all singularities of f in $\Delta(f)$. Then $K^1(f)$ is f-invariant. Moreover if $f/K^1(f)$ is expanding, there exists a neigborhood U of f in $End^2_B(K)$ such that for any $g \in U$, $K^1(g)$ is g-invariant and $g/K^1(g)$ is expanding.*

Proof. Let I be a simplex of $K^1(f)$. Since I contains no singularities of f, $f(I)$ is a juxtaposition of simplexes of K, $f(I) = J_1 \vee \ldots \vee J_n$. Then $J_i \subset K^1(f)$ for all $i = 1, \ldots, n$, because $f(I) \subset \Sigma(f)$. Thus $K^1(f)$ is f-invariant. Now if $f/K^1(f)$ is expanding, there exists a neighborhood U of f such that for any $g \in U$, $K^1(f)$ contains no singularities of g. Then $K^1(f)$ is g-invariant. By choosing U smaller if necessary we may assume that $g/K^1(f)$ is expanding. Then to prove the second part of the lemma it is enough to show that $K^1(f) = K^1(g)$. It is clear that $K^1(f) \subset K^1(g)$ because $g/K^1(f)$ is expanding. By choosing U smaller if necessary we may assume that $K^2(f) \subset K^2(g)$. Then $K^1(g) \subset K^1(f)$, so $K^1(f) = K^1(g)$ and the proof of the lemma is finished.

Using the same arguments as lemma 3.3, one can easily prove the following

Lemma 4.4. *Let f be as lemma 4.3. Suppose f contains no neutral periodic points. Then $\Sigma^2(f)$ is totally disconnected.*

Now, let α be a contracting periodic point of $f \in End^r_B(K)$. The local stable manifold of α, $W^s_{loc}(\alpha)$, is the connected component of $W^s(\alpha)$ which contains α.

The following lemma describes the structure of $W^s_{loc}(\alpha)$.

Lemma 4.5. *Let α be a contracting periodic point of $f \in End^r_B(K)$:*

a) *If $\alpha \in K - B$, then $W^s_{loc}(\alpha)$ is an interval of $K - B$.*

b) *If $\alpha \in B$, then either $W^s_{loc}(\alpha)$ is a loop L as in the figure 4.5 with $L \cap B = \{\alpha\}$ or $W^s_{loc}(\alpha)$ is a coordinate neighborhood Y of α* (See Part 1).

c) *There exists $n_0 \in N$ such that for $n > n_0$, the connected components of $f^{-n}(W^s_{loc}(\alpha)) - f^{n_0}(W^s_{loc}(\alpha))$ are intervals of $K - B$.*

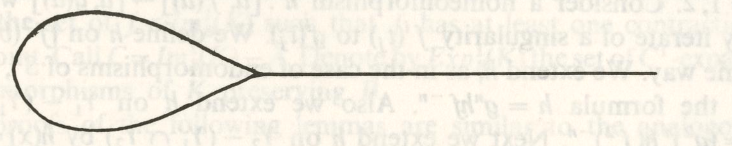

Fig. 4.5

Proof. By looking at a power of f if necessary we may assume, without loss of generality, that α is fixed.

a) follows immediately from the fact that B is f-invariant.

b) It is enough to show $W^s_{loc}(\alpha) \cap B = \{\alpha\}$. Let $\gamma \in W^s_{loc}(\alpha) \cap B$. Then $\lim_{n\to\infty} f^n(\gamma) = \alpha$, and so $f^n(\gamma) = \alpha$ for some n, because B is finite and f-invariant. Hence $\gamma = \alpha$ by definition of $W^s_{loc}(\alpha)$.

Assertion c) follows from the fact that B is finite.

Lemma 4.6. *Let $f \in J_B$ and let α be a contracting periodic point of f. There are neighborhoods V of f in $End^2_B(K)$ and U of α in K such that for any $g \in V$, there exists a unique contracting periodic point $\tilde{\alpha}$ of g in U and $g/W^s_{loc}(\tilde{\alpha})$ is topologically conjugate to $f/W^s_{loc}(\alpha)$.*

Proof. By looking at a power of f, if necessary we may assume, without loss of generality, that α is fixed. If $\alpha \in K - B$, by lemma 4.5 $W^s_{loc}(\alpha)$ is an open interval of $K - B$. In this case the proof of lemma follows as in the case of endomorphisms of S^1, so we assume that $\alpha \in B$.

Let g be close enough to f in $End^2_B(K)$. Then α is a contracting fixed point of g. Moreover it follows from lemma 4.5 that $W^s_{loc\,f}(\alpha)$ and $W^s_{loc\,g}(\alpha)$ are both loops as in the figure 4.5 or both coordinate neighborhoods of α. By hypothesis f has a finite number of singularities $t_1, \ldots, t_n$, so g has a finite number of singularities $\tilde{t}_1, \ldots, \tilde{t}_n$ with $\tilde{t}_i$ close enough to t_i for all $i = 1, \ldots, n$ and such that iterates $g^k(\tilde{t}_i)$ and $g^\ell(\tilde{t}_j)$ do not coincide for any k and ℓ and $i \neq j$. Take a coordinate neighborhood Y of α in $W^s_{loc\,f}(\alpha)$ such that $f(Y) \subset Y$, $0 < |df| < 1$ on Y. Let Y_1, Y_2 be the branches of Y. For definiteness assume $df(\alpha) > 0$, $f(Y_1) \subset Y_2$ and $f(Y_2) \subset Y_1$; the other cases are similar. Since g can be taken close enough to f, there exists a coordinate neighborhood $\tilde{Y}$ of α with branches $\tilde{Y}_1$ and $\tilde{Y}_2$ and with the same behavior of Y. Moreover $g^2/\tilde{Y}_1 : \tilde{Y}_1 \to \tilde{Y}_1$ is topologically conjugate to f^2/Y_1 by a conjugacy $h_1 : Y_1 \to \tilde{Y}_1$ close to the identity. Now we will define a conjugacy h on $W^s_{loc\,f}(\alpha)$. Since $df(\alpha) > 0$, $f(Y_1 \cap Y_2) \subset Y_1 \cap Y_2$. Let $a_1 \in \overset{\frown}{Y_1 \cap Y_2}$ and $a_2 \in Y_1 - (Y_1 \cap Y_2)$ such that $a_i \neq f^\ell(t_j)$ for any ℓ and $i = 1, 2$, $j = 1, \ldots, n$. Also we take $\tilde{a}_1 \in \tilde{Y}_1 \cap \tilde{Y}_2$ and $\tilde{a}_2 \in \tilde{Y}_1 - (\tilde{Y}_1 \cap \tilde{Y}_2)$ with the same behavior and such that $\tilde{a}_i = h_1(a_i)$, $i = 1, 2$. Consider a homeomorphism $h : [a, f(a)] \to [\tilde{a}, g(\tilde{a})]$ which maps any iterate of a singularity $f^i(t_j)$ to $g^i(\tilde{t}_j)$. We define h on $[f^2(b), b]$ in the same way. We extend h, as in the case of endomorphisms of S^1, to $Y_1 \cap Y_2$ by the formula $h = g^n h f^{-n}$. Also we extend h on $Y_1 - (Y_1 \cap Y_2)$ by $h = (g^2)^n h (f^2)^{-n}$. Next we extend h on $Y_2 - (Y_1 \cap Y_2)$ by $h(x) = g^{-1} h f(x)$ where g^{-1} maps $\tilde{Y}_2 - (\tilde{Y}_1 \cap \tilde{Y}_2)$ to $\tilde{Y}_1 - (\tilde{Y}_1 \cap \tilde{Y}_2)$. It is clear that $hf = gh$ on $(Y_1 \cap Y_2) \cup (Y_2 - (Y_1 \cap Y_2))$. If $x \in Y_1 - (Y_1 \cap Y_2)$ then $hf(x) = g^{-1}hf^2(x) = g^{-1}g^2(x) = gh(x)$. Hence $hf = gh$ on Y. Since $W^s_{loc\,f}(\alpha)$ and $W^s_{loc\,g}(\alpha)$ are both loops or both coordinate neighborhoods, the correspondence of singularities of f and g gives us a correspondence of intervals on which f and g are $1-1$, so we can extend h, as in the case of endomorphisms of S^1, to $W^s_{loc\,f}(\alpha)$ using the formula $h = g^{-1}hf$. Hence the lemma is proved.

Proof of Theorem 4.1. It is clear that properties J_2 and J_4 are open. We show the openness of J_1 and J_3. We follow arguments of Jakobson in [2]. Let $f \in J_B$ and let g be close enough to f in $End^2_B(K)$ satisfying J_2 and J_4. By lemma 4.3. $K^1(g) = K^1(f)$ and $g/K^1(g)$ is expanding. Then there are constants $c_1 > 0$ and $\lambda_1 > 1$ such that $|Dg^n(x)| > c_1\lambda_1^n$ for all $x \in \Sigma^1(g)$. Now by lemma 4.4 $\Sigma^2(f)$ is totally disconnected. From this and from the proof of a theorem of Jakobson [2, Theorem 4, page 177], we conclude that $K^2(g)$ contains a finite number of contracting periodic points and that there are constants $c_2 > 0$ and $\lambda_2 > 1$ such that $|Dg^n(x)| > c_2\lambda_2^n$ for all $x \in \Sigma^2(g)$. Take $\lambda = \min\{\lambda_1, \lambda_2\}$ and $c = \min\{c_1, c_2\}$. Then $|Dg^n(x)| > c\lambda^n$ for $x \in \Sigma(g)$. Hence g satisfies the conditions J_1 and J_3.

Now we show that f is topologically conjugate to g. Since $K^1(f) = K^1(g)$ and $f/K^1(f)$ and $g/K^1(g)$ are expanding endomorphisms, there is a homeomorphism $\phi : K^1(f) \to K^1(g)$ such that $\phi f = g\phi$. Moreover ϕ is increasing on the simplexes of $K^1(f)$. By lemma 4.5 we can extend ϕ on $K^1(f) \cup \left(\bigcup_{i=1}^{n} W^s_{loc}(\alpha_i)\right)$, where $\alpha_1, \ldots, \alpha_n$ are the contracting periodic points of f. Now we extend ϕ, as in the case of endomorphisms of S^1, to $\Sigma^1(f) \cup \Delta(f)$. Since $\Sigma^1(f) \cup \Delta(f)$ and $\Sigma^1(g) \cup \Delta(g)$ are denses in K, we can extend ϕ on K.

Remark 4.7. *B*-structural stability is nongeneric in $End^r_B(K)$; $r \geq 2$. In fact using arguments similar to those used in the proof of theorem 2.3 we can define an open subset V of $End^r_B(K)$ such that for any $f \in V$, $K - B$ contains a singularity α of f with $\bigcup_n f^n(\alpha) \cap B \neq \emptyset$. It follows from this that no $f \in V$ is *B*-structurally stable.

The remainder of this section is devoted to the proof of theorem 4.2. Let A be the set of $f \in Im^1_B(K)$ such that f has at least one contracting periodic point. Call $C = Im^1_B(K) - \overline{A}$. Denote by $Exp^1_B(K)$ the set of C^1-expanding endomorphisms of K preserving B.

The proofs of the following lemmas are similar to the analogous results of [2].

Lemma 4.8. *$Exp_B^1(K)$ is C^1-dense in C.*

Proof. This follows from lemma 3.3 and from the proof of lemma 5 of Jakobson [2, page 179].

Lemma 4.9. *Let $g \in Im_B^1(K)$ satisfying the conditions*

a) *All periodic points of f are hiperbolic and $\Delta(f)$ is nonempty.*

b) *$g/K^1(g)$ is expanding.*

Then there exists $g_1 \in Im_B^1(K)$ close enough of g such that g_1 satisfies the condition a); *$g_1/\Sigma(g_1)$ is expanding and g_1 has only a finite number of contracting periodic points.*

Proof. This follows from lemma 4.4 and from the proof of lemma of Slenk [2, page 174].

Lemma 4.10. *$J_B \cap Im_B^1(K)$ is C^1-dense in A.*

Proof. Let $f \in A$. We apply Shub's generalization of the Kupka-Smale theorem [4] to the case of endomorphisms and approximate f by $f_1 \in Im_B^1(K)$ without neutral periodic points. Then by lemma 4.8 we can approximate $f_1/K^1(f_1)$ by a expanding endomorphism $f_2: K^1(f_1) \to K^1(f_1)$. Now we extend f_2 to a immersion $f_3 \subset Im_B^1(K)$ close enough to f_1 and such that $K^1(f_3) = K^1(f_1)$. By the Kupka-Smale theorem f_3 is approximated by $f_4 \in Im_B^1(K)$ without neutral periodic points and with the same behavior of f_3. Then by lemma 4.9 we can approximate f_4 by $f_5 \in J_B \cap \cap Im_B^1(K)$. Hence the proof of lemma is finished.

Proof of Theorem 4.2. Follows from lemmas 4.8 and 4.10.

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