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On truncations of entire transcendental functions

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Abstract.

In this paper it is shown that outside every circle, every finite value is assumed by some truncation of the Taylor series of an entire transcendental function (in fact, by every truncation of sufficiently high degree). This result differs from both Casorati-Weierstrass and Picard theorems inasmuch as the truncations actually assume the preassigned values and that there are no exceptional values.

Theorem. *Let c be a complex number and f an entire transcendental function given by*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Then outside every circle, for some natural number h, *it is the case that*:

$$c - \sum_{n=0}^{k} a_n z^n$$

has a zero for every natural number k *with* $k > h$.

Proof. Let R be a nonnegative real number. In order to prove the Theorem it is enough to show that there exists a natural number h such that $c - \sum_{n=0}^{k} a_n z^n$ has a zero in the region $|z| > R$ for every $k > h$.

Let R_2 be a real number such that $R_2 > R$. Since $c - f(z)$ is an entire function, we see that $c - f(z)$ has finitely many, say, m zeros (counting the multiplicities of zeros and including the case $m = 0$) in the compact region $|z| \leq R_2$. Consequently, there exists a real number R_1 such that $R < R_1 < R_2$ and such that $c - f(z)$ has no zeros on $|z| = R_1$, i.e.,

(1) $$c - f(z) \neq 0 \ \textit{with} \ |z| = R_1 \ \textit{and} \ R < R_1 < R_2 .$$

Clearly, $|c-f(z)|$ is a continuous function on $|z|=R_1$ and therefore in view of (1) it has a positive minimum p on $|z|=R_1$, i.e.,

(2) $$|c-f(z)| \geq p > 0 \quad \textit{with} \quad |z| = R_1.$$

Since $\sum_{n=0}^{\infty} a_n z^n$ has uniform convergence on $|z|=R_1$ and since $p>0$, we see that for some natural number $h>m$ we have:

(3) $$|(f(z) - \sum_{n=0}^{k} a_n z^n| < p \textit{ with } k>h>m \textit{ and } a_h \neq 0$$

and consequently, from (2) and (3) we obtain:

(4) $$|c-f(z)| > |f(z) - \sum_{n=0}^{k} a_n z^n| \textit{ with } k>h>m \textit{ and } |z| = R_1.$$

But then from (2) and (4), by Rouché's theorem [1, p. 157], it follows that in the region $|z|<R_1$ it is the case that:

$$(c-f(z)) + (f(z) - \sum_{n=0}^{k} a_n z^n) = c - \sum_{n=0}^{k} a_n z^n$$

has as many zeros as $c-f(z)$ has. Since $R_1 < R_2$ we see that $c-f(z)$ has at most m zeros in the region $|z|<R_1$. Since $a_h \neq 0$ and $k>h>m$ we see that the polynomial $c - \sum_{n=0}^{k} a_n z^n$ must have at least one zero in the region $|z| \geq R_1$. But then since $R<R_1$ it follows that indeed $c - \sum_{n=0}^{k} a_n z^n$ has a zero in the region $|z|>R$ for every $k>h$. Thus, the Theorem is proved.

Remark. The statement of the Theorem is not valid in the setup of real-valued functions of a real variable. For example, no truncation of the Taylor expansion, in the powers of x, of e^{x^2} has a zero.

References

[1] Saks, S. and Zygmund, A.; *Analytic functions*, Warsaw, 1971.

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