BOL. SOC. BRAS. MAT., VOL. 13 N.º 2 (1982), 83-93

Bounded Random Perturbations of the Liapounov Number

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We will present here a model for bounded random perturbations of the Liapounov numbers. We will prove that the Liapounov number changes in a continuous fashion (in an almost everywhere sense) with the variance of the random perturbation. Our result will be for the simplest case of a sequence of 2×2 matrix with one Liapounov number bigger than zero and the other one smaller than zero. We'll suppose here that the angles between the stable and unstable directions are going to zero subexponentially in the sense of Oseledec [1]. This will play an important role in our model. This hypothesis appears in a natural way in the Non Commutative Ergodic Theorem of Oseledec. It's the general situation of most of the points of a general system. Our model will be not the one that would correspond to Kifer's [2] concept of random perturbation. We'll explain later what we mean by bounded perturbation (see *definition* 3. page 86, and *Theorem* on page 89).

We would like to point out that, as the angles between stable and unstable directions are going to zero, we will have to consider a reescaling in our model.

It's quite reasonable to suppose in a real model that we are able to make a reescaling and that the randomness appears after the reescaling. We will make clear this point later.

Let's now state the particular case of matrix that we intend to study.

Let θ_n be a decreasing sequence of real positive numbers such that $\lim_{n\to\infty} \frac{1}{n} \log(\sin\theta_n) = 0$ and $\lim_{n\to\infty} (\sin\theta_n) = 0$. We'll define now a sequence of 2×2 matrix A_n. Let be $\lambda < 1 < \mu$. First we define A_1 as the only one matrix such that

$$A_1(1,0) = (\lambda, 0)$$

$$A_1(0,1) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} 0 \\ \mu \end{pmatrix}$$

and by induction we obtain A_n for $n > 1$ by solving

Versão definitiva em 23/3/83.

$$(A_n \circ A_{n-1} \circ A_{n-2} \cdots A_2 \circ A_1)\ (1,0) = (\lambda^n, 0)$$

$$(A_n \circ A_{n-1} \circ A_{n-2} \cdots A_2 \circ A_1)\ (0,1) = \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} 0 \\ \mu^n \end{pmatrix}.$$

Let $B_n = (A_n \circ A_{n-1} \circ \ldots \circ A_2 \circ A_1)$.

Definition 1. *We define $l(v)$ the Liapounov number in the direction $v \in \mathbb{R}^2$ as $\lim_{n\to\infty} \frac{1}{n} \log \|B_n(v)\| = l(v)$. An equivalent definition is*

$$l(v) = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} \log \frac{\| B_n(B_{n-1}(v))\|}{\| B_{n-1}(v)\|}\right).$$

As $\lim_{n\to\infty} \frac{1}{n} \log(\sin\theta_n) = 0$, it's easy to see that for any v in $\mathbb{R}^2$, that is not in the x-axis, $l(v) = \log \mu$. For v in the x-axis $l(v) = \log \lambda$. The condition $\lim_{n\to\infty} \frac{1}{n} \log(\sin\theta_n) = 0$ is the one obtained in Oseledet Theorem [1].

In our model we'll just consider bounded random perturbations of the angles of the vectors. In this direction we'll introduce the following definitions. Let be $\phi = \{(1,y) \mid y \in \mathbb{R}\}$, and for $x \in \phi$ and $n > 1$, $g_n(x)$ will be the only element in ϕ such that, if $E\ (A_n(x))$ is the one dimensional subspace generated by $A_n(x)$, then $g_n(v) \in \phi \cap E(A_n(x))$. There exists a point in ϕ such that g_n is not defined. Anyway we'll consider D the domain of g_n, the points of the form $(1,y) y > 0$. Note that for any g_n we have $g_n(D) = D$ and g_n is the homeomorphism of D. Therefore we can define the successive iterations of $(g_n \circ g_{n-1} \circ \ldots \circ g_1)\ (v)$ for $v \in D$. They will represent the successive iterations of $(A_n \circ A_{n-1} \circ \ldots \circ A_1)\ (v)$ in a more or less normalized sense. For instance $(g_n \circ g_{n-1} \circ g_1)(0,1) = (1, \sin\theta_n/\cos\theta_n)$.

We'll identify now $\mathbb{R}$ with ϕ in such a way that $(1,0)$ corresponds to 0 and $(1,1)$ corresponds to 1. We'll think on $x \in \phi$ alternatively as a vector in $\mathbb{R}^2$ and as an element in $\mathbb{R}$, depending on the context we are.

As the Liapounov number can be defined up to a finite number of iterations, we can always suppose that θ_n is very small and therefore random perturbations of the angles of the vector on $\mathbb{R}^2$ can be represented as random perturbations in $\mathbb{R}$ (or in θ).

For $x \in \phi$ let $h_n(x) = (g_n \circ g_{n-1} \circ \ldots \circ g_1)\ (x)$. Then $h_n(x)$ represents for us the orbit of x (see picture 1). By bounded random perturbation of the orbit we mean a sequence x_n of real random variables such that x_n has mean $h_n(x)$ and a suitable variance that we'll specify later. In our model we'll need to bound the possible differences between $h_n(x)$ and x_n.

Let be $\bar{x} = (x_i)_{i\in N}$ $x_i \in D$.

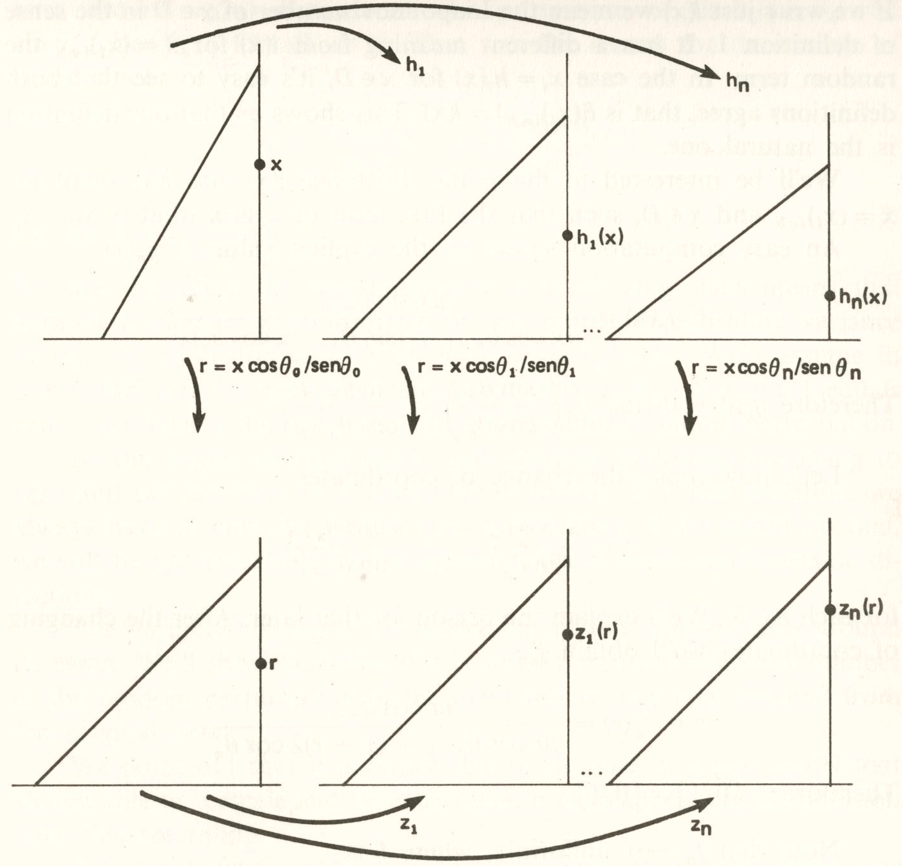

Fig. 1

Definition 2. *The Liapounov number $l(\bar{x})$, for $\bar{x} = (x_i)_{i\in N}$ the random perturbation, will be by definition the limit*

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} \log \frac{\| A_i(x_{i-1})\|}{\| x_{i-1}\|}\right)$$

in case it exists. In case this limit doesn't exist, we'll denote $l(\bar{x})$ either as

$$\limsup_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} \log \frac{\| A_i(x_{i-1})\|}{\| x_{i-1}\|}\right)$$

or as

$$\liminf_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} \log \frac{\| A_n(x_{i-1})\|}{\| x_{i-1}\|}\right).$$

If we write just $l(x)$ we mean the Liapounov number of $x \in D$ in the sense of definition 1. It has a different meaning from $l(\bar{x})$ for $\bar{x} = (x_i)_{i \in N}$ the random term. In the case $x_i = h_i(x)$ for $x \in D$, it's easy to see that both definitions agree, that is $l((x_i)_{i \in N}) = l(x)$. This shows us that our definition is the natural one.

We'll be interested to make the difference $l(x)$ and $l(\bar{x})$ small for $\bar{x} = (x_i)_{i \in N}$ and $x \in D$, such that the first term of $\bar{x}$ is x (that is $x_0 = x$).

An easy computation shows us the explicit value of $g_n(x)$:

$$y = g_n(x) = \frac{\mu x \sin \theta_{n+1}}{\mu x \cos \theta_{n+1} + (\sin \theta_n - x \cos \theta_n)\lambda}.$$

Therefore $g_n(0) = 0$ and $g_n\left(\frac{\sin \theta_n}{\cos \theta_n}\right) = \frac{\sin \theta_{n+1}}{\cos \theta_{n+1}}$.

Let's now make the change of coordinates

$$r = \frac{x \cos \theta_n}{\sin \theta_n}, \quad s = \frac{y \cos \theta_{n+1}}{\sin \theta_{n+1}},$$

for each $n \in N$. We'll explain the reason for that later. After the changing of coordinates we'll obtain f_n:

$$s = f_n(r) = \frac{\mu r \cos \theta_{n+1}}{\mu r \cos \theta_{n+1} + (1 - r)\lambda \cos \theta_n}.$$

Therefore $f_n[0, 1] = [0, 1]$.

Note that $f_n \to f$ uniformly, where f is

$$f(r) = \frac{r\mu}{\mu r + (1 - r)\lambda},$$

and $f_n(0) = 0$ and $f_n(1) = 1$.

Let's define $z_n(r) = (f_n \circ f_{n-1} \circ \ldots \circ f_1)(r)$ for $r \in \mathbb{R}$ positive.
It's easy to see that $\lim z_n(r) = 1$ for any r bigger than 0. Observe that

$$z_n(r) \frac{\sin \theta_n}{\cos \theta_n} = h_n(x).$$

That is, $z_n(r)$ represents the h_n orbit of x in the new variables r.

Definition 3 — *Bounded random perturbation.* Now we are able to define a stationary stochastic process r_n such that for a fixed $r_0 \in [0, 1]$, we have that $(r_n - z_n(r_0)) = Y_n$ are independent equally distributed random variables in $\mathbb{R}$ with mean O and variance σ. This represents in our model the fact that, as $g_n(\sin \theta_n / \cos \theta_n) = \sin \theta_{n+1} / \cos \theta_{n+1}$, then the spread of the random term around $z_n(r_0)$ depends on the proportion of $\sin \theta_n / \cos \theta_n$ and $\sin \theta_{n+1} / \cos \theta_{n+1}$ for all $n \in N$. That was the reason for the change of coordinates

$$r = \frac{x \cos \theta_n}{\sin \theta_n}, \quad s = \frac{y \cos \theta_{n+1}}{\sin \theta_{n+1}}.$$

Let be $x_n = r_n \frac{\sin \theta_n}{\cos \theta_n}$.

We are reescaling the transformation in earch step. This means that without a change of coordinates in our model the absolute variance of $(x_n - h_n(x))$ is decreasing with n as $\sigma \sin \theta_n / \cos \theta_n$. We'll assume in our model that for each $n \in N$ the probability of $(r_n - z_n(r_0)) > 1$ equals zero ... (1). That is the reason for the word bounded random perturbation.

As the angles between stable and unstable directions are going to zero and the precision of a computer is limited, in a real situation, we always have to consider in this case a reescaling. If we dont't do that, we will be not even able to distinguish between stable and unstable directions.

The variance measures the dispersion of the random term around its mean. We'll show here that if σ is small then the Liapounov number of the random perturbation of the orbit will be not very different from the original orbit.

We point out that it is reasonable to suppose in a real model that we are able to reescale and that the random term appears in the iteration after this reescaling.

Before we continue I'd like to make some comments about our definition of random perturbations.

1) Our definition assumes that the random variable is stationary, unlike [2], [3], [4] and [5].

2) Our definition is the suitable one, because in the other case ([2] and [3]) the variance would go to infinity with n, and in this situation it is not possible to control the Liapounov number, as we can see in (7) in our proof.

3) In terms of the variables x and y our model supposes that the variance is going to zero subexponentially.

Two important consequence that we have for our model are that, almost everywhere in the set of events (see Breiman [6] for definitions and theorems):

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} (r_i - z_i(r_0)) = 0 \tag{2}$$

and

(3) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} (r_i - z_i(r_0))^2 = \sigma.$$

These results are known as the Law of Large Numbers [6].

The last assumption in definition 3 means that almost everywhere in the set of events:

(4) $$|r_n - z_n(r_0)| \leq 1$$

That means that for the old variables x and y the differences $|x_n - h_n(x_0)|$ are going to zero subexponentially. This means that almost everywhere in the set of events r_i, we have that for the old variable $x_i = r_i \sin\theta_i/\cos\theta_i$ is such that $\lim_{n\to\infty} \|x_i\| = 1$. This means that

(5) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \log \frac{\|A_i(x_{i-1})\|}{\|x_{i-1}\|} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \log \|A_i(x_{i-1})\|.$$

We'll show now that the Liapounov number can change in a discontinuous fashion with the random perturbation (this will be not possible almost everywhere as we'll see later). Take a point $x_0 \in D$ that has Liapounov number $\log\mu$ and for any i in $N - \{0\}$ take $x_i = 0$. In this case the event $(x_i)_{i\in N} = \bar{x}$ is such that $l(x_0) = \log\mu$ (see definition 1) but $l(\bar{x}) = \log\lambda$ (see definition 2). Therefore, independently of the variance, there exists the possibility of a change of $|\log\mu - \log\lambda|$ in the Liapounov number of a vector and its random perturbation. We'll prove here that if the variance is small, then almost everywhere in the set of events the Liapounov number $l(\bar{x})$ is near $l(x_0)$ for $\bar{x} = (x_i)_{i\in N}$, where x_0 is the first element of $\bar{x}$.

Before we state the theorem we need the following propositions:

Proposition 1. *Suppose that* $q_i \to 0$, $p_i \to p$ *and* $p > 0$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(q_i + p_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(p_i).$$

Proof. We just have to prove that

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} \log(q_i + p_i) - \log(p_i) \right) = 0,$$

that is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log\left(\frac{q_1}{p_i} + 1\right) = 0.$$

But as

$$0 = \lim_{n\to\infty} \log\left(\frac{q_i}{p_i} + 1\right), \text{ we have}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (q_i + p_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(p_i).$$

An easy consequence of proposition 1 is

Corollary 2. *In the hypothesis of proposition 1 about* q_i, p_i *and if* $q_i' \to 0$, $p_i' \to p' > 0$, *we have:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log\left(\frac{q_i + p_i}{q_i' + p_i'}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log\left(\frac{p_i}{p_i'}\right).$$

We'll begin now the proof of the theorem.

Theorem. – *Let* $(\bar{r}) = (r_i)_{i\in N}$ *satisfying def. 3. Then for any* $\varepsilon > 0$ *there exists* $\bar{\sigma} > 0$ *such that, if* $r_0 \in (0, 1)$ *and* s_i *is a sequence such that* $s_i = z_i(r_0)$, *then for almost everywhere* $\bar{r} = (r_i)_{i\in N}$, *such that* $s_0 = r_0$, $\sigma < \bar{\sigma}$, σ *as in def.* 3, *we have* $l(\bar{s}) - l(\bar{r}) < \varepsilon$.

Proof. Suppose

$$x_i = r_i \frac{\sin\theta_i}{\cos\theta_i}, \quad y_i = s_i \frac{\sin\theta_i}{\cos\theta_i}.$$

Let $c_i = \|A_i(y_{i-1})\|$, $d_i = \|A_i(x_{i-1})\|$.

From the way we define z_i (see (5) and (1)), all we have to prove is that:

$$|l(\bar{s}) - l(\bar{r})| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log\left(\frac{c_i}{d^i}\right)$$

is small if σ is small.

As $\lim_{n\to\infty} z_i(r_0) = 1$, we have that for $\delta > 0$ and i very large, r_i and s_i are much bigger than $-\delta$ and smaller than 2.

As the map $t_n : [-\delta, \infty] \to \mathbb{R}$ such that

$$t_n = \left\{ \left[\mu r \frac{\sin\theta_{n-1}}{\cos\theta_{n-1}} \frac{\sin\theta_{n+1}}{\sin\theta_n} \right]^2 + \right.$$

$$\left. + \frac{1}{\sin^2\theta_n} \left[\lambda \sin\theta_n - r \frac{\sin\theta_{n-1}\cos\theta_n}{\cos\theta_{n-1}} + \mu r \frac{\sin\theta_{n-1}\cos\theta_{n+1}}{\cos\theta_{n-1}} \right]^2 \right\}^{1/2}$$

is continuous and $(\sin\theta_{n+1}/\sin\theta_n)<1$, it's easy to see that there exist M and M' bigger than zero such that if n is big enough then $0<M<c_n<M'$ and $0<M<d_n<M'$, where c_n and d_n are in the r and s coordinates:

$$c_n = \| A_n(x_{n-1}) \| = \left\{ \left[\mu x_{n-1} \frac{\sin\theta_{n+1}}{\sin\theta_n} \right]^2 + \right.$$

$$\left. + \left[\frac{\lambda(\sin\theta_n - x_{n-1}\cos\theta_n) + \mu x_{n-1}\cos\theta_{n+1}}{\sin\theta_n} \right]^2 \right\}^{1/2} =$$

$$= \left\{ \left[\mu r_{n-1} \frac{\sin\theta_{n-1}}{\cos\theta_{n-1}} \frac{\sin\theta_{n+1}}{\sin\theta_n} \right]^2 + \right.$$

$$\left. + \frac{1}{\sin^2\theta_n} \left[\lambda\left(\sin\theta_n - r_{n-1} \frac{\sin\theta_{n-1}\cos\theta_n}{\cos\theta_{n-1}} \right) + \mu r_{n-1} \frac{\sin\theta_{n-1}}{\cos\theta_{n-1}} \cos\theta_{n+1} \right]^2 \right\}^{1/2}$$

and

$$d_n = \| A_n(y_{n-1}) \| = \left\{ \left[\mu s_{n-1} \frac{\sin\theta_{n-1}}{\cos\theta_{n-1}} \frac{\sin\theta_{n+1}}{\sin\theta_n} \right]^2 + \right.$$

$$\left. + \frac{1}{\sin^2\theta_n} \left[\lambda\left(\sin\theta_n - s_{n-1} \frac{\sin\theta_{n-1}\cos\theta_n}{\cos\theta_{n-1}} \right) + \mu s_{n-1} \frac{\sin\theta_{n-1}}{\cos\theta_{n-1}} \cos\theta_{n+1} \right]^2 \right\}^{1/2}$$

Therefore there exists $\bar K>0$ such that $-\bar K<\log(c_n/d_n)<\bar K$ for n big enough. We will need this estimate later.

Let now $V_n=\{i\in\{0,1,\dots,n-1,n\}\mid 0<(\sin\theta_i/\sin\theta_{i-1})<\lambda\}$.

Note that as $\sin\theta_i$ is decreasing and as

$$0=\lim_{n\to\infty}\frac{1}{n}\log(\sin\theta_n)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log(\sin\theta_i/\sin\theta_{i-1}),$$

we have that $\lim\limits_{n\to\infty}\frac{1}{n}\#V_n=0$. The union of all V_n is what is usually called a set of 0 density on N.

Using now the estimates $-\bar K<\log(c_i/d_i)<\bar K$ and the fact that the union of all V_n has zero density we have that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{i=1\\ i\notin V_n}}^{n}\log\frac{c_i}{d_i}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log\frac{c_i}{d_i}.$$

The meaning of the above equality is that in order to compute the Liapounov number we can avoid a set of zero density (that determines a subsequence that is bounded). This means that we just have to prove that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{i=1\\ i\notin V_n}}^{n}\log\frac{c_i}{d_i}$$

is near $l(\bar x)$ if σ is small. By the considerations that we had done before we have the extra condition that

$$(6)\qquad 1>\frac{\sin\theta_i}{\sin\theta_{i-1}}>\lambda.$$

Let now q_n, p_n, q'_n and p'_n be such that

$$p_n=\frac{1}{\sin^2\theta_n}\left[\lambda\left(\sin\theta_n-r_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_n\right)+\mu r_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_{n+1}\right]^2$$

$$q_n=\left[\mu r_{n-1}\frac{\sin\theta_{n+1}}{\cos\theta_{n-1}}\frac{\sin\theta_{n-1}}{\sin\theta_n}\right]^2$$

$$p'_n=\frac{1}{\sin^2\theta_n}\left[\lambda\left(\sin\theta_n-s_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_n\right)+\mu s_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_{n+1}\right]^2$$

$$q'_n=\left[\mu s_{n-1}\frac{\sin\theta_{n+1}}{\cos\theta_{n-1}}\frac{\sin\theta_{n-1}}{\sin\theta_n}\right]^2.$$

Therefore we have

$$\frac{c_n^2}{d_n^2}=\frac{q_n+p_n}{q'_n+p'_n}.$$

As $(\sin\theta_{n+1}/\sin\theta_n)<1$ and r_n and s_n are almost everywhere bounded (4), we have that $q_i\to 0$ and $q'_i\to 0$. Therefore by corollary 2

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log\frac{c_i}{d_i}=\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{p_i}{p'_i}\right).$$

We can express $\log\frac{p_n}{p'_n}$ as

$$\log\frac{p_n}{p'_n}=2\left\{\log\left[\lambda\left(\sin\theta_n-r_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_n\right)+\right.\right.$$

$$\left.+\mu r_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_{n+1}\right]-$$

$$\left.-\log\left[\lambda\left(\sin\theta_n-s_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_n\right)+\mu s_{n-1}\frac{\sin\theta_{n-1}}{\cos\theta_{n-1}}\cos\theta_{n+1}\right]\right\}=$$

$$=2\log\left[1-\left\{\left\{\lambda\frac{\sin\theta_{n-1}\cos\theta_n}{\cos\theta_{n-1}}(r_{n-1}-s_{n-1})+\right.\right.\right.$$

$$\left.+\mu\frac{\sin\theta_{n-1}\cos\theta_{n+1}}{\cos\theta_{n-1}}(s_{n-1}-r_{n-1})\right\}\Big/$$

$$\left.\left.\left\{\sin\theta_{n-1}\left[\lambda\left(\frac{\sin\theta_n}{\sin\theta_{n-1}}-s_{n-1}\frac{\cos\theta_n}{\cos\theta_{n+1}}\right)+\mu s_{n-1}\frac{\cos\theta_{n+1}}{\cos\theta_{n-1}}\right]\right\}\right\}\right]$$

If we define a_n as

$$a_n = \frac{\left(\lambda \frac{\cos\theta_n}{\cos\theta_{n-1}} - \mu \frac{\cos\theta_{n+1}}{\cos\theta_{n-1}}\right)(r_{n-1} - s_{n-1})}{\lambda\left(\frac{\sin\theta_n}{\sin\theta_{n-1}} - s_{n-1}\frac{\cos\theta_n}{\cos\theta_{n-1}}\right) + \mu s_{n-1}\frac{\cos\theta_{n+1}}{\cos\theta_{n-1}}},$$

then $\log \frac{p_n}{p_n'} = 2 \log |1 + a_n|$.

We are going to show now that

$$\lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} \log(1 + a_i) < \varepsilon$$

if σ is small.

Let v_n be

$$v_n = \frac{\log(1 + a_n) - a_n}{a_n^2}.$$

As $\frac{\sin\theta_{i+1}}{\sin\theta_i} > \lambda$ by (6), $|r_i - s_i| < 1$ by (4), and

$$\lim_{n\to\infty} s_{n-1} \frac{\cos\theta_n}{\cos\theta_{n+1}} = 1$$

we have that

$$a_n > \frac{\lambda - \mu}{(-\lambda) + \mu + \lambda^2} = c > -1.$$

Therefore v_n is well defined.

As $\lim_{x\to 0} \frac{\log(1 + x) - x}{x^2} = -\frac{1}{2}$, we have that there exists K such that

$$-K < \frac{\log(1 + x) - x}{x^2} < K, \; x \in [c, -c].$$

Note that $\lim_{x\to -1} \frac{\log(1 + x) - x}{x^2} = -\infty$.

As $\log(1 + a_n) = a_n + v_n a_n^2$, we have that

$$\lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} \log \frac{c_i}{d_i} = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} \log(1 + a_i) = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} (a_i + v_i a_i^2) =$$

$$= \lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} a_i + \lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} v_i a_i^2.$$

As the a_n are bounded and using again the fact that we are avoiding a sequence with zero density we have that

$$\lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} a_i + \lim_{n\to\infty}{}' \frac{1}{n} \sum_{i=1}^{n} v_i a_i^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} v_i a_i^2$$

where the $'$ in the summation means that we avoid elements of the sequence that are in the V_n sets.

Using now (2) we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{c_i}{d_i} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} v_i a_i^2.$$

As $-|r_i - s_i|^2 \, K \, c^2 \leqq v_i a_i^2 \leqq K \, c^2 |r_i - s_i|^2$ and by (3), we have that

$$-K c^2 \sigma \leqq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} v_i a_i^2 \leqq K c^2 \sigma. \tag{7}$$

Therefore

$$|l(\bar{s}) - l(\bar{r})| = \left| \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \frac{\|A_i(y_{i-1})\|}{\|A_i(x_{i-1})\|} \right| = \left| \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \frac{c_i}{d_i} \right| =$$

$$= \left| \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} v_i a_i^2 \right| \leqq K c^2 \sigma.$$

The conclusion is that $l(\bar{s}) - l(\bar{r})$ is small if σ is small enough.

This is the end of our proof.

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