Smooth mappings between foliated manifolds

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0. Introduction.

In [1], L. A. Favaro has presented the notion of stability in the tangential sense of smooth mappings between foliated manifolds. Moreover he has presented the notion of infinitesimal stability in the tangential sense in order to characterize the former. In fact, he has shown that the latter is the sufficient condition of the former.

In this paper, we shall prove that the converse of Favaro's theorem is true for foliation preserving mappings. The key point of the proof is a transversality theorem in some sense. We shall call it "the transversality theorem in the tangential sense".

In §1, we will formulate the main result. The transversality theorem in the tangential sense will be given and proved in §2. We will prove the main theorem in §3.

All manifolds, foliations and mappings considered here are differentiable of class C^∞. All manifolds should satisfy the second countability axiom.

1. Formulations.

Let $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ be foliated manifolds. Let $C^\infty(M, N)$ be the set of smooth mappings $M \to N$.

Definition 1.1. (Favaro) *$f \in C^\infty(M, N)$ is stable in the tangential sense if there is a neighbourhood $V_f \subset C^\infty(M, N)$ of f in the Whitney C^∞-topology such that for each $g \in V_f$ satisfying $g(x)$ and $f(x)$ belong to same leaf of $\mathscr{F}_N$ for each $x \in M$, there are diffeomorphisms $\phi : M \to M$, taking each leaf of $\mathscr{F}_M$ onto itself, and $\psi : N \to N$, taking each leaf of $\mathscr{F}_N$ onto itself, such that $f = \psi \circ g \circ \phi$.*

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Let $T\mathscr{F}_M$ be the subbundle of TM with fibre $T_x\mathscr{F}_M$ the tangent space of the leaf of $\mathscr{F}_M$ at x. For any $f \in C^\infty(M, N)$, we let $\theta(f)$ denote the set of vector fields along f. We let $\theta(M) = \theta(1_M)$. Let $\theta_{\mathscr{F}}(f)$ be the $C^\infty(M)$--submodule of $\theta(f)$ consisting of smooth section of $f^*T\mathscr{F}_N$. Here, $C^\infty(M)$ is the ring of smooth functions on M. We let $\theta_{\mathscr{F}}(M) = \theta_{\mathscr{F}}(1_M)$. We define mappings $tf : \theta(M) \to \theta(f)$ by $tf(\xi) = df \circ \xi$ and $\omega f : \theta(N) \to \theta(f)$ by $\omega f(\eta) = \eta \circ f$.

Definition 1.2. (Favaro) $f \in C^\infty(M, N)$ is *infinitesimally stable in the tangential sense* if $tf(\theta_{\mathscr{F}}(M)) + \omega f(\theta_{\mathscr{F}}(N)) \supset \theta_{\mathscr{F}}(f)$.

Then, Favaro has proved the following theorem.

Theorem 1.3. (Favaro [1]). *If M is compact and $f : M \to N$ is infinitesimally stable in the tangential sense, then f is stable in the tangential sense.*

Let $C^\infty_{\mathscr{F}}(M, N)$ be the set of foliation preserving smooth mappings. Our main result is the following theorem.

Theorem 1.4. *Let M be compact. For $f \in C^\infty_{\mathscr{F}}(M, N)$, the followings are equivalent*:

1) *f is stable in the tangential sense.*
2) *f is infinitesimally stable in the tangential sense.*

Remark. i) Let $f \in C^\infty_{\mathscr{F}}(M, N)$ and $\phi : M \to M$ (resp. $\psi : N \to N$) be diffeomorphisms taking each leaf of $\mathscr{F}_M$ (resp. $\mathscr{F}_N$) onto itself. Then $f(x)$ and $\psi \circ f \circ \phi(x)$ belong to the same leaf of $\mathscr{F}_N$ for each $x \in M$. We write $f \sim_{\mathscr{F}} g$ if there exist ϕ and ψ as above such that $f = \psi \circ g \circ \phi$.
ii) For $f \in C^\infty_{\mathscr{F}}(M, N)$, the notion of infinitesimal stability in the tangential sense is equivalent to

$$tf(\theta_{\mathscr{F}}(M)) + \omega f(\theta_{\mathscr{F}}(N)) = \theta_{\mathscr{F}}(f).$$

Hence, foliation preserving mappings are natural objects for which we define the notion of infinitesimal stability in the tangential sense.

2. Transversality theorem in the tangential sense.

Let M be a smooth manifold and $(N, \mathscr{F}_N)$ be a smooth foliated manifold. Let U, V be opensets in N with $U \supset \overline{V}$, where $\overline{V}$ is the closure of V in N.

Definition 2.1. *Let A be a submanifold of U, $f \in C^\infty(M, N)$. We say that f is transverse to A relative to $(\mathscr{F}_N, U)$ at x (denoted by $f \pitchfork_{(\mathscr{F}_N, U)} A$ at x) if either*

(a) *$f(x) \notin \overline{V}$, or*
(b) *$f(x) \in \overline{V}$, and*
 1) *$f(x) \notin A$, or*
 2) *$f(x) \in A$ and $df_x(T_xM) + T_{f(x)}A \supset T_{f(x)}\mathscr{F}_N$.*

Then, we have the following lemmas.

Lemma 2.2. *Let B_1, B_2 be smooth manifolds and A be a submanifold of U. Let $b_0 \in B_1$ be fixed. Let $j : B_1 \times B_2 \to C^\infty(M, N)$ be a mapping and define $\Phi : M \times B_1 \times B_2 \to N$ by $\Phi(x, b_1, b_2) = j(b_1, b_2)(x)$. Assume that*:

1) *Φ is transverse to A and $T_{(x, b_0, b_2)}\Phi^{-1}(A) \subset T_{(x, b_0, b_2)}(M \times b_0 \times B_2)$ for any $(x, b_2) \in M \times B_2$, and*
2) *$\Phi_2 \pitchfork_{(\mathscr{F}_N, U)} A$, where $\Phi_2 = \Phi \mid M \times b_0 \times B_2$.*

Then the set $\{b_2 \in B_2 \mid j(b_0, b_2) \pitchfork_{(\mathscr{F}_N, U)} A\}$ is dense in B_2.

Proof. Let $A_\Phi = \Phi^{-1}(A)$. Let π be the restriction to A_Φ of the projection $M \times B_1 \times B_2 \to B_2$. First note that if $b_2 \notin \text{Image}(\pi)$, then $j(b_0, b_2)(M) \cap A$ is empty, so $j(b_0, b_2) \pitchfork_{(\mathscr{F}_N, U)} A$. Now if $\dim A_\Phi < \dim B_2$, then $\pi(A_\Phi)$ has measure zero in B_2 and for a dense set b_2 in B_2 — namely B_2 — Image(π), $j(b_0, b_2) \pitchfork_{(\mathscr{F}_N, N)} A$. Thus, we may assume that $\dim A_\Phi \geq \dim B_2$. We remark that if b_2 is a regular value of π, then $j(b_0, b_2) \pitchfork_{(\mathscr{F}_N, U)} A$.

To prove the remark let b_2 be a regular value for π and let x in M. If $(x, b_0, b_2) \notin A_\Phi$, then $j(b_0, b_2)(x) \notin A$ and $j(b_0, b_2) \pitchfork_{(\mathscr{F}_N, U)} A$ at x. So we may assume that $j(b_0, b_2)(x)$ is in A_Φ. Since b_2 is a regular value for π and $\dim A_\Phi \geq \dim B_2$, we have that

$$T_{(x, b_0, b_2)}M \times B_1 \times B_2 = T_{(x, b_0, b_2)}A_\Phi + T_{(x, b_0, b_2)}M \times B_1 \times b_2.$$

Since $T_{(x, b_0, b_2)}A_\Phi \subset T_{(x, b_0, b_2)}M \times b_0 \times B_2$, we have

$$T_{(x, b_0, b_2)}M \times b_0 \times B_2 = T_{(x, b_0, b_2)}A_\Phi + T_{(x, b_0, b_2)}M \times b_0 \times b_2.$$

Apply $d\Phi_{(x, b_0, b_2)}$ to both sides and obtain

$$d\Phi_{(x, b_0, b_2)}(T_{(x, b_0, b_2)}M \times b_0 \times B_2) = T_{j(b_0, b_2)(x)}A + d(j(b_0, b_2))_x(T_xM).$$

We assume that $\Phi_2 \pitchfork_{(\mathscr{F}_N, U)} A$ so

$$d\Phi_{2(x, b_0, b_2)}(T_{(x, b_0, b_2)}M \times b_0 \times B_2) + T_{j(b_0, b_2)(x)}A \supset T_{j(b_0, b_2)(x)}\mathscr{F}_N.$$

Combining two formulas we have that

$$d(j(b_0, b_2))_x(T_xM) + T_{j(b_0, b_2)(x)}A \supset T_{j(b_0, b_2)(x)}\mathscr{F}_N$$

This completes the proof.

Lemma 2.3. *Let A be a submanifold of U and $A' \subset A$ be closed in $\overline{V}$* (i.e. *closed in N*). *Then $T_{A',V} = \{f \in C^\infty(M,N) \mid f \,\overline{\pitchfork}_{(\mathscr{F}_N,U)} A$ on $A'\}$ is an open subset of $C^\infty(M,N)$ in the Whitney C^1-topology. (Here, we write that $f \,\overline{\pitchfork}_{(\mathscr{F}_N,U)} A$ on A' if for every $x \in M$ with $f(x)$ in A', $f \,\overline{\pitchfork}_{(\mathscr{F}_N,U)} A$ at x).*

Proof. Like in the usual case, it follows that the following set is open in $J^1(M,N)$:

$$W = \{\sigma \mid \beta(\sigma) \notin A'\} \cup \{\sigma \mid \beta(\sigma) \in A' \text{ and } df_{\alpha(\sigma)}(T_{\alpha(\sigma)}M) + T_{\beta(\sigma)}A \supset T_{\beta(\sigma)}\mathscr{F}_N\}.$$

Here, α is the source map, β is the target map and f is a representative of σ.

We now formulate the transversality theorem in the tangential sense.

Definition 2.4. *Let $f, g \in C^\infty(M,N)$. We say that f and g are in the same tangent (denoted by $f \sim_{S\,T} g$) if $f(x)$ and $g(x)$ belong to the same leaf of $\mathscr{F}_N$ for each $x \in M$.*

Remark. i) $\sim_{S \cdot T}$ is an equivalence relation.
ii) Let $C^\infty(M,N;h)$ be denoted the same tangent class for which $h \in C^\infty(M,N)$ is representative.
iii) We observe that $C^\infty(\overline{M,N;h})$ is a closure of $C^\infty(M,N;h)$ in the C^∞--topology.
iv) For $f \in C^\infty_{\mathscr{F}}(M,N)$, $C^\infty(M,N;f) \subset C^\infty_{\mathscr{F}}(M,N)$.
v) For $f \in C^\infty(M,N)$, the notion of stability in the tangential sense is equivalent to the following condition: there is a neighbourhood $V_f \subset C^\infty(M,N;f)$ of f in the Whitney C^∞-topology such that for each $g \in V_f$, there are diffeormosphisms $\phi : M \to M$, taking each leaf of $\mathscr{F}_M$ onto itself, and $\psi : N \to N$, taking each leaf of $\mathscr{F}_N$ onto itself, such that $f = \psi \circ g \circ \phi$.

The following proposition is obvious.

Proposition 2.5. *$C^\infty(\overline{M,N;h})$ is a Baire space.*

We define a foliated structure on $J^k(M,N)$ naturally induced by $\mathscr{F}_N$.

Let (U,η) be a coordinate system on M and (V,ψ) be a foliated coordinate system on $(N,\mathscr{F}_N)$.

Let

be the coordinate system on $J^k(M,N)$ induced by (U,η) and (V,ψ). Here, $p = \dim \mathscr{F}_N$ and $\psi : V \to \mathbb{R}^{n-p} \times \mathbb{R}^p$ is represented by

$$\psi(x) = (\psi_1(x), \psi_2(x)).$$

Let $\pi : J^k(m, n-p) \times J^k(m,p) \to J^k(m, n-p)$

be the canonical projection.

Define a local submersion

$$P_{U,V} : J^k(U,V) \to \mathbb{R}^{n-p} \times J^k(m, n-p)$$

by

$$P_{U,V} = (\psi \times \pi) \circ \phi_{U,V}.$$

Lemma 2.6. *For every coordinate systems on M and every foliated coordinate systems on $(N,\mathscr{F}_N)$, local submersions $\{P_{U,V}\}$ defined as the above induce a foliated structure on $J^k(M,N)$.*

Proof. In this case, the structure group of $J^k(M,N)$ is $L^k(m) \times L^k(n,p)$. Here, $L^k(m)$ is the Lie group consisting of k-jets of origin preserving diffeormosphisms

$$(\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0),$$

and

$$L^k(n,p) = \{j^k_0\psi \in L^k(n) \mid \psi : (\mathbb{R}^{n-p} \times \mathbb{R}^p, 0) \to (\mathbb{R}^{n-p} \times \mathbb{R}^p, 0) \text{ and } \psi(x,y) = (\psi_1(x), \psi_2(x,y)) \text{ for } (x,y) \in (\mathbb{R}^{n-p} \times \mathbb{R}^p, 0)\}.$$

The action of $L^k(m) \times L^k(n,p)$ on $J^k(m,n-p) \times J^k(m,p)$ induces the following commutative diagram:

$$\begin{array}{ccc} J^k(m,n-p) \times J^k(m,p) & \longrightarrow & J^k(m,n-p) \times J^k(m,p) \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ J^k(m,n-p) & \longrightarrow & J^k(m,n-p). \end{array}$$

This diagram indicates that the above local submersions induce a foliated structure on $J^k(M,N)$.

We denote as $\mathscr{F}^k(M,N)$ the foliated structure on $J^k(M,N)$ induced by Lemma 2.6.

For any $(x,y) \in M \times N$, let U_x be a coordinate neighbourhood about

$U'_x \subset U_x$ (resp. V_y) be a open subneighbourhood of x (resp. y) with $\overline{U}'_x \subset U_x$ (resp. $\overline{V}'_y \subset V_y$).

Here, we may take $\psi : V_y \to \mathbb{R}^{n-p} \times \mathbb{R}^p$ as $\psi(V_y) = V_1 \times V_2$, where $V_1 \subset \mathbb{R}^{n-p}$ and $V_2 \subset \mathbb{R}^p$ are open sets. We also take $\psi(V'_y) = V'_1 \times V''_2$, where $V'_1 \subset V_1$ and $V'_2 \subset V_2$.

Then, we have $M \times N = \bigcup_{(x,y)\in M\times N} U'_x \times V'_y$.

Since $M \times N$ satisfies the second countability axiom, the above is a countable covering;

$$M \times N = \bigcup_{i=1}^{\infty} U'_i \times V'_i.$$

Then, $\{J^k(U_i, V_i)\}_{i=1}^{\infty}$ is an open covering of $J^k(M, N)$.

We say that $\mathscr{U}^k_{\mathscr{F}} = \{J^k(U_i, V_i)\}_{i=1}^{\infty}$, which is constructed as above, is *a good coordinate system of $J^k(M, N)$ with respect to $\mathscr{F}_N$*.

Definition 2.7. *Let X be a smooth manifold. Let $\mathscr{U} = \{(U_i, \psi_i)\}_{i\in I}$ be a coordinate system on N. We say that A is a local submanifolds correction with respect to $\mathscr{U}$ if the following conditions hold:*

1) *$A = \bigcup_{i\in I} A_i$, where A_i is a submanifold of U_i for any i;*
2) *if $A_i \cap A_j$ is non-empty in $U_i \cap U_j$, then*

$$\psi_j \circ \psi_i^{-1}(\psi_i(A_i \cap U_j)) = \psi_j(A_j \cap U_i)$$

by the coordinate transformation $\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$.

Let $(N, \mathscr{F}_N)$ be a foliated manifold and $\mathscr{U} = \{(U_i, \psi_i)\}_{i\in I}$ be a foliated coordinate system.

We say that A is *a local submanifold correction in $\mathscr{F}_N$ with respect to $\mathscr{U}$* if A is a local submanifold correction with respect to $\mathscr{U}$ and for any $i \in I$, there is $L \in \mathscr{F}_N$ such that $A_i \subset U_i \cap L$.

We are now ready to define the notion of transversality in the tangential sense.

Definition 2.8. *Let $\mathscr{U}^k_{\mathscr{F}}$ be a good coordinate system of $J^k(M, N)$ with respect to $\mathscr{F}_N$ and A be a local submanifolds correction with respect to $\mathscr{U}^k_{\mathscr{F}}$. Let $f : M \to N$ be a smooth mapping.*

We say that $j^k f$ is transverse to A with respect to $\mathscr{U}^k_{\mathscr{F}}$ at $x \in M$, (we write $j^k f \pitchfork_{\mathscr{F}} A$ at x), if $j^k f$ is transverse to A relative to $(\mathscr{F}^k(M, N), J^k(U_i, V_i))$ at x for any i. We also say that $j^k f$ is transverse to A with respect to $\mathscr{U}^k_{\mathscr{F}}$, (we write $j^k f \pitchfork_{\mathscr{F}} A$), if $j^k f$ is transverse to A with respect to $\mathscr{U}^k_{\mathscr{F}}$ at any $x \in M$.

Then we have the following theorem.

Theorem 2.9. (Transversality theorem in the tangential sense). *Let M be a smooth manifold and $(N, \mathscr{F}_N)$ be a smooth foliated manifold. Let $\mathscr{U}^k_{\mathscr{F}}$ be a good coordinate system of $J^k(M, N)$ with respect to $\mathscr{F}_N$ and A be a local submanifolds correction in $\mathscr{F}^k(M, N)$ with respect to $\mathscr{U}^k_{\mathscr{F}}$. Then $T_{A,\mathscr{F},h} = \{f \in C^\infty\overline{(M, N; h)} \mid j^k f \pitchfork_{\mathscr{F}} A\}$ is dense in $C^\infty\overline{(M, N; h)}$.*

Proof. We first choose a countable covering of A_i by open subsets $A_i^1, A_i^2, \dots$ such that each A_i^j satisfies:

1) the closure of A_i^j in $J^k(U_i, V_i)$ is contained in A_i.
2) $\bar{A}_i^j$ is compact.

Since $(\alpha, \beta)(A_i) \subset U_i \times V_i$, then $(\alpha, \beta)(\bar{A}_i^j) \subset U_i \times V_i$.

Let $T_{A_i^j} = \{f \in C^\infty(M, N) \mid j^k f \pitchfork A$ on $\bar{A}_i^j\}$, $T_{A_i^j, h} = T_{A_i^j} \cap C^\infty(M, N; h)$ and $T_{A_i^j, \bar{h}} = T_{A_i^j} \cap C^\infty\overline{(M, N; h)}$.

It is clear that $T_{A,\mathscr{F},h} = \bigcap_{j\geq 1} \bigcap_{i \geq 1} T_{A_i^j, \bar{h}}$.

Since $C^\infty\overline{(M, N; h)}$ is a Baire space, it is enough to show that $T_{A_i^j}$ is open and $T_{A_i^j, h}$ is dense in $C^\infty(M, N; h)$ for any i, j.

Define $T_{j,i} = \{g \in C^\infty(M, J^k(M, N)) \mid g \pitchfork_{(\mathscr{F}^k(M,N), J^k(M,N))} A$ on $\bar{A}_i^j\}$. By Lemma 2.3, $T_{j,i}$ is open.

Since $j^k : C^\infty(M, N) \to C^\infty(M, J^k(M, N))$ is continuous then $T_{A_i^j} = (j^k)^{-1}(T_{j,i})$ is open.

We now show that $T_{A_i^j, h}$ is dense in $C^\infty(M, N; h)$.

Since $A_i \supset A_i^j$, for coordinate systems

$$\phi : U_i \to \mathbb{R}^m \text{ and } \psi : V_i \to \mathbb{R}^{n-p} \times \mathbb{R}^p,$$

we have $(\alpha, \beta)(A_i^j) \subset \bar{U}_i \times \bar{V}_i$.

Since $\bar{A}_i^j$ is compact, there exist smooth functions $\rho : \mathbb{R}^m \to [0\ 1]$, $\rho'_1 : \mathbb{R}^{n-p} \to [0\ 1]$ and $\rho'_2 : \mathbb{R}^p \to [0\ 1]$ such that

$$\rho = \begin{cases} 1 & \text{on a neighbourhood of } \phi \circ \alpha(\bar{A}_i^j) \\ 0 & \text{off } \phi(U_i), \end{cases}$$

$$\rho'_1 = \begin{cases} 1 & \text{on a neighbourhood of } \pi_1 \circ \psi \circ \beta(\bar{A}_i^j) \\ 0 & \text{off } V_i^1, \text{ and} \end{cases}$$

$$\rho'_2 = \begin{cases} 1 & \text{on a neighbourhood of } \pi_2 \circ \psi \circ \beta(\bar{A}_i^j) \\ 0 & \text{off } V_i^2, \end{cases}$$

where $\pi_1 : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^{n-p}$ and $\pi_2 : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$ are canonical projections and $\psi(V_i) = V_i^1 \times V_i^2 \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$.

Let B'_1 (resp. B'_2) be the space of polynomial mappings of $\mathbb{R}^m \to \mathbb{R}^{n-p}$ (resp. $\mathbb{R}^m \to \mathbb{R}^p$) of degree at most k.

We will use Lemma 2.2 to show that $f \in C^\infty(M, N; h)$ can be perturbed slightly in $C^\infty(M, N; h)$ to transverse to A in the tangential sense.

For $(b_1, b_2) \in B'_1 \times B'_2$, define $g_{(b_1, b_2)}: M \to N$ by

$$g_{(b_1,b_2)}(x) = \begin{cases} f(x) & x \notin U_i \text{ or } f(x) \notin V_i \\ \psi^{-1}(\rho(\phi(x))\rho'_1(f_1(x))b_1(\phi(x)) + f_1(x), & \\ \rho(\phi(x))\rho'_2(f_2(x))b_2(\phi(x)) + f_2(x)) & \text{otherwise,} \end{cases}$$

where $\psi \circ f(x) = (f_1(x), f_2(x)) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$.

The choice of ρ, ρ'_1 and ρ'_2 guarantees that $g_{(b_1,b_2)}$ is a smooth mapping from M to N such that f and $g_{(0,b_2)}$ are in the same tangent. Hence, $g_{(0,b_2)} \in C^\infty(M, N; h)$.

Define $G: M \times B'_1 \times B'_2 \to N$ by $G(x, b_1, b_2) = g_{(b_1,b_2)}(x)$. By the definition, G is smooth. Define $\Phi(x, b_1, b_2) = j^k g_{(b_1,b_2)}(x)$. In order to apply Lemma 2.2, we need to show that Φ is transverse to A_i on $\bar{A}_i^j$,

$$T_{(x,0,b_2)}\Phi^{-1}(A_i) \subset T_{(x,0,b_2)}(M \times 0 \times B_2) \text{ and } \Phi_2 \pitchfork_{(\mathscr{F}^k(M,N),\, J^k(U_i, V_i))} A_i.$$

Since $A_i \subset L \cap J^k(U_i, V_i)$ for some $L \in \mathscr{F}^k(M, N)$, it is clear that $T_{(x,0,b_2)}\Phi^{-1}(A_i) \subset T_{(x,0,b_2)}(M \times 0 \times B_2)$.

By the same technique of the ordinary jet transversality theorem ([2], Theorem 4.9), there are a neighbourhood B_ℓ of the origin in B'_ℓ $(\ell = 1,2)$ and a neighbourhood A'^j_i of $\bar{A}_i^j$ in A_i such that $\tilde{\Phi}$ is transverse to A_i and

$$\tilde{\Phi}_2 \pitchfork_{(\mathscr{F}^k(M,N),\, J^k(M,N))} A_i \text{ on } A'^j_i.$$

Here, $\tilde{\Phi} = \Phi \mid M \times B_1 \times B_2$ and $\tilde{\Phi}_2 = \tilde{\Phi} \mid M \times 0 \times B_2$.

This completes the proof.

It is easy to generalize the above theorem to multijet transversality theorem in the tangential sense as follows.

Define $M^s = M \times \ldots \times M$ (s-times) and $M^{(s)} = \{(x_1, \ldots, x_s) \in M^s \mid x_i \neq x_j$ for $1 \leq i < j \leq s\}$. Define $\alpha^s: J^k(M, N)^s \to M^s$ in the obvious fashion. Then ${}_sJ^k(M, N) = (\alpha^s)^{-1}(M^{(s)})$ is the s-fold k-jet bundle.

Let $f: M \to N$ be a smooth mapping. We define ${}_sj^k f: M^{(s)} \to {}_sJ^k(M, N)$ by ${}_sj^k f(x_1, \ldots, x_s) = (j^k f(x_1), \ldots, j^k f(x_s))$.

Let $\mathscr{F}^k(M, N)^s$ be the s-fold product foliation of $\mathscr{F}^k(M, N)$. Define ${}_s\mathscr{F}^k(M, N) = \mathscr{F}^k(M, N)^s \mid {}_sJ^k(M, N)$.

For any $(x_1, \ldots, x_s, y_1, \ldots, y_s) \in M^{(s)} \times N^s$, let U_{x_i} be a coordinate neighbourhood about x_i such that $U_{x_i} \cap U_{x_j}$ is empty $(i \neq j)$ and $\bar{U}_{x_i}$ is compact. Let (V_{y_i}, ψ_i) be a foliated coordinate neighbourhood about y_i with $\psi_i(V_{y_i}) = V^1_{y_i} \times V^2_{y_i} \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$. Let $U'_{x_i} \subset U_{x_i}$ (resp. $V'_{y_i} \subset V_{y_i}$) be an open subneighbourhood of x_i (resp. y_i) with $\bar{U}'_{x_i} \subset U_{x_i}$ (resp. $\bar{V}'_{y_i} \subset V_{y_i}$) and $\psi_i(V'_{y_i}) = V'^1_{y_i} \times V'^2_{y_i} \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$.

Then, we have $M^{(s)} \times N^s = \bigcup_{(x,y) \in M^{(S)} \times N^S} U'_x \times V'_y$, where

$$U'_x = U'_{x_1} \times \ldots \times U'_{x_s},\ V'_y = V'_{y_1} \times \ldots \times V'_{y_s},\ x = (x_1, \ldots, x_s) \in M^{(s)}$$

and $y = (y_1, \ldots, y_s) \in N^s$.

Since $M^{(s)} \times N^s$ satisfies the second countability axiom, the above is a countable covering:

$$M^{(s)} \times N^s = \bigcup_{i=1}^{\infty} U'_i \times V'_i.$$

Then

$$\{{}_sJ^k(U_i, V_i) = J^k(U_i^1, V_i^1) \times \ldots \times J^k(U_i^s, V_i^s) \mid i \in \mathbb{N},\ U_i = U_i^1 \times \ldots \times U_i^s$$

and $V_i = V_i^1 \times \ldots \times V_i^s\}$ is an open covering of ${}_sJ^k(M, N)$.

We say that ${}_s\mathscr{U}^k_{\mathscr{F}} = \{{}_sJ^k(U_i, V_i)\}_{i=1}^{\infty}$, which is constructed as above, is *a good coordinate system of* ${}_sJ^k(M, N)$ *with respect to* $\mathscr{F}_N$.

Definition 2.10. *Let* ${}_s\mathscr{U}^k_{\mathscr{F}}$ *be a good coordinate system of* ${}_sJ^k(M, N)$ *with respect to* $\mathscr{F}_N$ *and* A *be a local submanifolds correction with respect to* ${}_s\mathscr{U}^k_{\mathscr{F}}$. *Let* $f: M \to N$ *be a smooth mapping.*

We say that ${}_sj^k f$ *is transverse to* A *with respect to* ${}_s\mathscr{U}^k_{\mathscr{F}}$ *at* $x \in M^{(s)}$, *(we write* ${}_sj^k f \pitchfork_{\mathscr{F}} A$ *at* x*), if* ${}_sj^k f$ *is transverse to* A_i *relative to* $({}_s\mathscr{F}^k(M, N), {}_sJ^k(U_i, V_i))$ *at* x *for any* i.

We also say that ${}_sj^k f$ *is transverse to* A *with respect to* ${}_s\mathscr{U}^k_{\mathscr{F}}$, *(we write* ${}_sj^k f \pitchfork_{\mathscr{F}} A$*), if* ${}_sj^k f$ *is transverse to* ${}_s\mathscr{U}^k_{\mathscr{F}}$ *at any* x *in* $M^{(s)}$.

The following theorem is a trivial extension of Theorem 2.9.

Theorem 2.11. (Multijet transversality theorem in the tangential sense). *Let* M *be a smooth manifold and* $(N, \mathscr{F}_N)$ *be a foliated manifold. Let* ${}_s\mathscr{U}^k_{\mathscr{F}}$ *be a good coordinate system of* ${}_sJ^k(M, N)$ *with respect to* $\mathscr{F}_N$ *and* A *be a local submanifolds correction in* ${}_s\mathscr{F}^k(M, N)$ *with respect to* ${}_s\mathscr{U}^k_{\mathscr{F}}$. *Then* $T^s_{A,\mathscr{F},h} = \{f \in C^\infty\overline{(M, N; h)} \mid {}_sj^k f \pitchfork_{\mathscr{F}} A\}$ *is dense in* $C^\infty\overline{(M, N; h)}$.

Let $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ be smooth foliated manifolds. Define $J^k_{\mathscr{F}}(M, N) = \{j^k f(x) \mid f: V \to N;$ foliation preserving, where V is some open neighbourhood of $x\}$ and

$${}_sJ^k_{\mathscr{F}}(M, N) = J^k_{\mathscr{F}}(M, N)^s \cap {}_sJ^k(M, N).$$

Proposition 2.12. $(\alpha^{(s)}, \beta^s): {}_sJ^k_{\mathscr{F}}(M, N) \to M^{(s)} \times N^s$ *is a subbundle of* $(\alpha^{(s)}, \beta^s): {}_sJ^k(M, N) \to M^{(s)} \times N^s$ *with structure group* $L^k(m, q)^s \times L^k(n, p)^s$.

Proof. For convenience, we may suppose that $s = 1$. For any $j^k f(x) \in J^k(M, N)$, taking foliated coordinate systems about x and $f(x)$ respectively, we may consider that f is a smooth mapping: $\mathbb{R}^{m-q} \times \mathbb{R}^q \to \mathbb{R}^{n-p} \times \mathbb{R}^p$ such that the following diagram commutes;

$$\begin{array}{ccc} \mathbb{R}^{m-q} \times \mathbb{R}^q & \xrightarrow{\ f\ } & \mathbb{R}^{n-p} \times \mathbb{R}^p \\ \downarrow & & \downarrow \\ \mathbb{R}^{m-q} & \xrightarrow{\ f_1\ } & \mathbb{R}^{n-p} \end{array}$$

(i.e. $f(x, y) = (f_1(x)\ f_2(x, y))$ for any $(x, y) \in \mathbb{R}^{m-q} \times \mathbb{R}^q$).

Hence, we have the following correspondences:

$$\begin{array}{c} J^k(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^m \times \mathbb{R}^n \times J^k(m, n-p) \times J^k(m, p) \\ \bigcup \qquad\qquad\qquad\qquad \bigcup \\ J^k(\mathbb{R}^m, \mathbb{R}^n) \cap J^k_{\mathscr{F}}(M, N) \cong \mathbb{R}^m \times \mathbb{R}^n \times J^k(m-q, n-p) \times J^k(m, p). \end{array}$$

This completes the proof.

Let ${}_s\mathscr{U}^k_{\mathscr{F}}$ be a good coordinate system. We say that

$${}_s\mathscr{U}^k_{\mathscr{F}, P} = \{{}_sJ^k(U, V) \cap {}_sJ^k_{\mathscr{F}}(M, N) \mid {}_sJ^k(U, V) \in {}_s\mathscr{U}^k_{\mathscr{F}}\}$$

is a *good coordinate system of* ${}_sJ^k_{\mathscr{F}}(M, N)$ *with respect to* $\mathscr{F}_M \times \mathscr{F}_N$ if each U^j is a foliated coordinate system of M, where $U = U^1 \times \ldots \times U^s$.

In this situation, we can canonically induce the foliated structure ${}_s\mathscr{F}^k(M, N)$ on ${}_sJ^k_{\mathscr{F}}(M, N)$.

Then, we have the following proposition as a corollary to Theorem 2.11.

Proposition 2.13. *Let* $(M, \mathscr{F}_M)$ *and* $(N, \mathscr{F}_N)$ *be smooth foliated manifolds. Let* ${}_s\mathscr{U}^k_{\mathscr{F}, P}$ *be a good coordinate system of* ${}_sJ^k_{\mathscr{F}}(M, N)$ *with respect to* $\mathscr{F}_M \times \mathscr{F}_N$ *and A be a local submanifolds correction in* ${}_s\mathscr{F}^k(M, N)$ *with respect to* ${}_s\mathscr{U}^k_{\mathscr{F}, P}$. *If* $h: M \to N$ *is foliation preserving, then*

$$T^s_{A, \mathscr{F}, h} = \{f \in C^\infty\overline{(M, N; h)} \mid {}_sj^k f \pitchfork_{\mathscr{F}, P} A\}$$

is dense in $C^\infty\overline{(M, N; h)}$. *Here, we denote* ${}_sj^k f \pitchfork_{\mathscr{F}, P} A$ *if* ${}_sj^k f$ *is transverse to* A_i *relative to* $({}_s\mathscr{F}^k(M, N), {}_sJ^k(U_i, V_i \cap {}_sJ^k_{\mathscr{F}}(M, N))$ *for any* i.

3. Stability theorem in the tangential sense.

In this section, we will prove Theorem 1.4.

Let $L^k_q(m)$ denote the group of invertible jet $j^k\phi(0)$ such that the following diagram commutes;

$$\begin{array}{ccc} (\mathbb{R}^{m-q} \times \mathbb{R}^q, 0) & \xrightarrow{\ \phi\ } & (\mathbb{R}^{m-q} \times \mathbb{R}^q, 0) \\ & \searrow \quad \swarrow & \\ & (\mathbb{R}^{m-q}, 0). & \end{array}$$

There is a natural action of $L^k_q(m) \times L^k_p(n)$ on $J^k(m, n)^s$ defined as follows.

If $((j^k\phi_1(0), \ldots, j^k\phi_s(0), j^k\psi(0)) \in L^k_q(m) \times L^k_p(n)$ and

$$z = (j^k f_1(0), \ldots, j^k f_s(0)) \in J^k(m, n)^s,$$

then we set

$$((j^k\phi_1(0), \ldots, j^k\phi_s(0), j^k\psi(0)) \cdot z = (j^k\psi \circ f_1 \circ \phi_1^{-1}(0), \ldots, j^k\psi \circ f_s \circ \phi_s^{-1}(0)).$$

This action canonically induces the action on $J^k(m-q, n-p)^s \times J^k(m, p)^s$.

Now let ${}_s\mathscr{U}^k_{\mathscr{F}, P}$ be a good coordinate system of ${}_sJ^k_{\mathscr{F}}(M, N)$ with respect to $\mathscr{F}_M \times \mathscr{F}_N$.

For any ${}_sJ^k(U_i, V_i) \cap {}_sJ^k_{\mathscr{F}}(M, N) \in {}_s\mathscr{U}^k_{\mathscr{F}, P}$, we have

$${}_sJ^k(U_i, V_i) \cap {}_sJ^k_{\mathscr{F}}(M, N) \cong U_i \times V_i \times J^k(m-q, n-p)^s \times J^k(m, p)^s.$$

In this situation, let $\psi^j_i: V^j_i \to \mathbb{R}^{n-p} \times \mathbb{R}^p$ be a foliated coordinate function, where $V_i = V_i^1 \times \ldots \times V_i^s$. For any

$$(c_1, \ldots, c_s) \in (\mathbb{R}^{n-p})^s \text{ and } z \in J^k(m-q, n-p)^s \times J^k(m, p)^s,$$

we define

$$O_i(c_1, \ldots, c_s, z) = U_i \times (\psi^1_i)^{-1}(c_1 \times \mathbb{R}^p) \times \ldots \times (\psi^s_i)^{-1}(c_s \times \mathbb{R}^p) \times \times (L^k_q(m)^s \times L^k_p(n))(z).$$

Here, $(L^k_q(m)^s \times L^k_p(n))(z)$ denotes the orbit of z by the above action in $J^k(m-q, n-p)^s \times J^k(m, p)^s$.

Definition 3.1. *We say that* $O \subset {}_sJ^k_{\mathscr{F}}(M, N)$ *is a pseudo orbit with respect to* ${}_s\mathscr{U}^k_{\mathscr{F}, P}$ *if* $O = \bigcup_{i=1}^{\infty} O_i(c^i_1, \ldots, c^i_s, z^i)$ *for some* $(c^i_1, \ldots, c^i_s) \in (\mathbb{R}^{n-p})^s$ *and* $z^i \in J^k(m-q)^s \times J^k(m, p)^s$.

Proposition 3.2. *Any pseudo orbit O with respect to ${}_s\mathscr{U}^k_{\mathscr{F},P}$ is a local submanifolds correction in $\mathscr{F}^k(M,N)$ with respect to ${}_s\mathscr{U}^k_{\mathscr{F},P}$.*

Proof. For convenience, we may suppose $s=1$. It is clear that each $O_i(c^i,z^i)$ is a submanifold of $J^k(U_i,V_i)\cap J^k(M,N)$. It is enough to check the condition 2). Let $O_i(c^i,z^i)\cap O_j(c^j,z^j)$ be non-empty. Since the structure group dependent on the good coordinate system $\mathscr{U}^k_{\mathscr{F},P}$ is a subgroup of $L^k(m,q)\times L^k(n,p)$, it is enough to show that any $L^k_q(m)\times L^k_p(n)$-orbit in $J^k(m-q,n-p)\times J^k(m,p)$ is mapped on a $L^k_q(m)\times L^k_p(n)$-orbit by the action of $L^k(m,q)\times L^k(n,p)$.

Let

$$\begin{array}{ccc}
(\mathbb{R}^{m-q}\times\mathbb{R}^q,0) & \xrightarrow{\ \phi\ } & (\mathbb{R}^{m-q}\times\mathbb{R}^q,0)\\
\downarrow & & \downarrow\\
(\mathbb{R}^{m-q},0) & \xrightarrow{\ \phi_1\ } & (\mathbb{R}^{m-q},0),
\end{array}$$

$$\begin{array}{ccc}
(\mathbb{R}^{n-p}\times\mathbb{R}^p,0) & \xrightarrow{\ \psi\ } & (\mathbb{R}^{n-p}\times\mathbb{R}^p,0)\\
\downarrow & & \downarrow\\
(\mathbb{R}^{n-p},0) & \xrightarrow{\ \psi_1\ } & (\mathbb{R}^{n-p},0),
\end{array}\qquad \text{(commutative)}$$

$$\begin{array}{ccc}
(\mathbb{R}^{m-q}\times\mathbb{R}^q,0) & \xrightarrow{\ h\ } & (\mathbb{R}^{m-q}\times\mathbb{R}^q,0)\\
& \searrow\ \ \swarrow & \\
& (\mathbb{R}^{m-q},0) &
\end{array}$$

and

$$\begin{array}{ccc}
(\mathbb{R}^{n-p}\times\mathbb{R}^p,0) & \xrightarrow{\ k\ } & (\mathbb{R}^{n-p}\times\mathbb{R}^p,0)\\
& \searrow\ \ \swarrow & \\
& (\mathbb{R}^{n-p},0) &
\end{array}$$

be local diffeomorphisms germs.

Let

$$\begin{array}{ccc}
(\mathbb{R}^{m-q}\times\mathbb{R}^q,0) & \xrightarrow{\ f\ } & (\mathbb{R}^{n-p}\times\mathbb{R}^p,0)\\
\downarrow & & \downarrow\\
(\mathbb{R}^{m-q},0) & \xrightarrow{\ f_1\ } & (\mathbb{R}^{n-p},0)
\end{array}$$

be a smooth map germ.

If we observe the relation between $\psi\circ f\circ\phi^{-1}$ and $\psi\circ k\circ f\circ h^{-1}\circ\phi^{-1}$, we have

$$\psi\circ k\circ f\circ h^{-1}\circ\phi^{-1}=(\psi\circ k\circ\psi^{-1})\circ\psi\circ f\circ\phi^{-1}\circ(\phi\circ h^{-1}\circ\phi^{-1}).$$

Since $j^k(\phi\circ h^{-1}\circ\phi^{-1})(0)\in L^k_q(m)$ and $j^k(\psi\circ k\circ\psi^{-1})(0)\in L^k_p(n)$, we have the following diffeomorphism:

$$Q: L^k_q(m)\times L^k_p(n)\,(j^kf(0))\to L^k_q(m)\times L^k_p(n)\,(j^k(\psi\circ f\circ\phi^{-1})(0))$$

by

$$Q(j^k(k\circ f\circ h^{-1}))=j^k(\psi\circ k\circ f\circ h^{-1}\circ\phi^{-1})(0).$$

By the definition, $O_i(c^i,z^i)\subset J^k(U_i,V_i)\cap J^k_{\mathscr{F}}(M,N)\cap L$ for some $L\in\mathscr{F}^k(M,N)$.

Proposition 3.3. *Let O be a pseudo orbit with respect to ${}_s\mathscr{U}^k_{\mathscr{F},P}$ and O' be another pseudo orbit with respect to ${}_s\mathscr{U}'^k_{\mathscr{F},P}$.*

If $z\in O\cap O'$, then we have $T_zO=T_zO'$.

The proof is the same as Proposition 3.2.

We now give a formula for the tangent space of a pseudo orbit. Let $S=\{x_1,\dots,x_s\}$, where $x_1,\dots,x_s$ are s distinct points in M. For any $z=(j^kf_1(x_1)\dots,j^kf_s(x_s))\in{}_sJ^k(M,N)$, let $f:(M,S)\to(N,f(S))$ be a smooth map germ with $f\,|\,(M,x_i)=f_i$.

There is a natural identification of $\mathbb{R}$-vector spaces;

$$T({}_sJ^k(M,N)_x)_z=\theta(f)_S/\mathfrak{M}^{k+1}_S\,\theta(f)_S,$$

where ${}_sJ^k(M,N)_x$ denotes the fibre of ${}_sJ^k(M,N)$ over $x=(x_1,\dots,x_s)\in M^{(s)}$ (See Mather [3], §2). Here, $\theta(f)_S$ is defined as the set of germs at S of smooth vector fields along f and ${}_S$ is the ideal of zero valued function at S in the ring $C^\infty_S(M)$ of smooth function germs at S.

In terms of this identification, we have

$$T_z({}_s\mathscr{F}^k(M,N)_x)=\theta_{\mathscr{F}}(f)_S/\mathfrak{M}^{k+1}_S\theta_{\mathscr{F}}(f)_S.$$

By the same argument of Mather's paper [3], §2, we have

$$T_z(O_x)=\frac{tf(\mathfrak{M}_S\theta_{\mathscr{F}}(M)_S)+\omega f(\theta_{\mathscr{F}}(N)_{f(S)})+\mathfrak{M}^{k+1}_S\theta_{\mathscr{F}}(f)_S}{\mathfrak{M}^{k+1}_S\theta_{\mathscr{F}}(f)_S}.$$

Here, we may take $f:(M,S)\to(N,f(S))$ as foliation preserving map germ.

Let $f: M\to N$ be a foliation preserving mapping. Then ${}_sj^kf$ is a section of ${}_sJ^k_{\mathscr{F}}(M,N)$. By Proposition 3.3, the notion of which ${}_sj^kf$ is transverse to a pseudo orbit with respect to the good coordinate system is not dependent on the choice of good coordinate systems. Hence, we shall say that ${}_sj^kf$ is *transverse to a pseudo orbit in the tangential sense* if ${}_sj^kf$ is transverse to a pseudo orbit with respect to the good coordinate system.

Proposition 3.3. *Let $f : M \to N$ be a foliation preserving smooth mapping and O be a pseudo orbit. Suppose $_sj^kf(x) \in O$. Then, $_sj^kf$ is transverse to O in the tangential sense at x if and only if*

$$tf(\theta_{\mathscr{F}}(M)_S) + \omega f(\theta_{\mathscr{F}}(N)_{f(S)}) + \mathfrak{M}_S^{k+1}\theta_{\mathscr{F}}(f)_S = \theta_{\mathscr{F}}(f)_S.$$

The proof is the same as the usual case (cf [3], §2).

Let $\phi : M \to M$ (resp. $\psi : N \to N$) be a diffeomorphism, taking each leaf of $\mathscr{F}_M$ (resp. $\mathscr{F}_N$) onto itself. Let $_s\mathscr{U}^k_{\mathscr{F},P}$ be a good coordinate system of $_sJ^k_{\mathscr{F}}(M, N)$ with respect to $\mathscr{F}_M \times \mathscr{F}_N$. For any

$$_sJ^k(U_i, V_i) \cap {_sJ^k_{\mathscr{F}}}(M, N) \in {_s\mathscr{U}^k_{\mathscr{F},P}},$$

we define a set

$$\psi \circ {_sJ^k}(U_i, V_i) \circ \phi = \{j^k(\psi \circ f \circ \phi)(\phi^{-1}(x)) \mid j^kf(x) \in {_sJ^k}(U_i, V_i)\}.$$

Then,

$$\psi \circ {_s\mathscr{U}^k_{\mathscr{F},P}} \circ \phi = \{\psi \circ {_sJ^k}(U_i, V_i) \circ \phi \cap {_sJ^k_{\mathscr{F}}}(M, N) \mid {_sJ^k}(U_i, V_i) \cap {_sJ^k_{\mathscr{F}}}(M, N) \in {_s\mathscr{U}^k_{\mathscr{F},P}}\}$$

is a good coordinate system of $_sJ^k(M, N)$ with respect to $\mathscr{F}_M \times \mathscr{F}_N$.

We now prove the following theorem instead of Theorem 1.4.

Theorem 3.4. *Let $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ be smooth foliated manifolds. Suppose M is compact, $s \geq p + 1$ and $k \geq p$, where $p = \dim \mathscr{F}_N$. Let f be a foliation preserving smooth mapping.*

Then the following are equivalent:

(a) *f is stable in the tangential sense;*

(b) *f is infinitesimally stable in the tangential sense;*

(c) *For any pseudo orbit O in $_sJ^k_{\mathscr{F}}(M, N)$, there exists a foliation preserving smooth mapping g, with $_sj^kg$ transverse to O in the tangential sense, such that $f \sim_{\mathscr{F}} g$.*

Proof. If f is infinitesimally stable in the tangential sense, then f is stable in the tangential sense by [1].

Suppose f is stable in the tangential sense. For s, k, the set of $g \in C^\infty(M, N; f)$ such that $_sj^kg$ is transverse to a pseudo orbit O in the tangential sense is dense in $C^\infty(M, N; f)$, by Proposition 2.13. Since f is stable in the tangential sense, there are diffeomorphisms $\phi : M \to M$, taking each leaf of $\mathscr{F}_M$ onto itself, and $\psi : N \to N$, taking each leaf of $\mathscr{F}_N$ onto itself, such that $f = \psi \circ g \circ \phi$.

For any $O_i \subset {_sJ^k}(U_i, V_i) \cap {_sJ^k_{\mathscr{F}}}(M, N) \in {_s\mathscr{U}^k_{\mathscr{F},P}}$, we denote

$$\psi \circ O_i \circ \phi = \{j^k(\psi \circ h \circ \phi)(\phi^{-1}(x)) \mid j^kh(x) \in O_i\}.$$

By the proof of Proposition 3.2, $\psi \circ O \circ \phi = \bigcup_i \psi \circ O_i \circ \phi$ is a pseudo orbit with respect to $\psi \circ {_s\mathscr{U}^k_{\mathscr{F},P}} \circ \phi$.

Since $_sj^kg \pitchfork_{\mathscr{F}} O$, we have $_sj^kf \pitchfork_{\mathscr{F}} \psi \circ O \circ \phi$. Hence, for an element $x = (x_1, \ldots, x_s) \in M^{(S)}$ with $_sj^kf(x) \in \psi \circ O_i \circ \phi$, we have

$$tf(\theta_{\mathscr{F}}(M)_S) + \omega f(\theta_{\mathscr{F}}(N)_{f(S)}) + \mathfrak{M}_S^{k+1}\theta_{\mathscr{F}}(f)_S = \theta_{\mathscr{F}}(f)_S,$$

by Proposition 3.3.

In while, by the proof of Theorem 1 in [1], the infinitesimal stability in the tangential sense is equivalent to the condition:

(*) for any $y \in N$ and each finite subset S of $f^{-1}(y)$ with no more than $p + 1$ points, it follows that

$$tf(\theta_{\mathscr{F}}(M)_S) + \omega f(\theta_{\mathscr{F}}(N)_y) + \mathfrak{M}_S^{k+1}\theta_{\mathscr{F}}(f)_S = \theta_{\mathscr{F}}(f)_S.$$

This completes the proof.

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