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Uniform stability of perturbed non-linear systems of differential equations with time delay

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I. Introduction.

The purpose of this work is to study the uniform stability of the solutions of a system of functional differential equation with time delay

$$\dot{y}(t) = f(t, y_t) + g(t, y_t)$$

perturbed from the non-linear system

$$\dot{x}(t) = f(t, x_t)$$

under the assumption that this system has at least one bounded solution.

One of the basic tools needed is the Alekseev-Shanholt's Integral Formula, which is a generalization of the Variation of Constants Formula for non-linear functional differential equations with time delay.

In Section 5, we present an application of our results to the equation

$$x(t) = -\int_{-r}^{0} a(-\theta)h(\phi(\theta))d\theta.$$

II. Preliminaries.

Suppose $r \geq 0$ is a given real number, $R = (-\infty, \infty)$, R^n is an n-dimensional real or complex linear vector space with norm $|\cdot|$, and $C([a, b], R^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into R^n with the topology of uniform convergence. If $[a, b] = [-r, 0]$, we let $C = C([-r, 0], R^n)$ and designate the norm of an element ϕ in C by $\|\phi\| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$.

If x is a continuous map of $[a - r, b]$ into R^n, then $x_t \in C$ is given, for each $a \leq t < b$, by

$$x_t(\theta) = x(t + \theta), \quad -r \leq \theta \leq 0.$$

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Let Λ be an open subset of C, $\rho \in R$, $\Gamma = (\rho, \infty) \times \Lambda$, and $f : \Gamma \to R^n$ be a given function.

A retarded functional differential equation is a relation of the form

(1) $$\dot{x}(t) = f(t, x_t),$$

where $\dot{x}(t)$ denotes the right-hand derivative of $x(u)$ at $u = t$.

For any $(t_0, \phi) \in \Gamma$, we say that x is a solution of (1) with initial function ϕ at t_0, if there exists an $A > 0$ such that:

i) $x \in C([t_0 - r, t_0 + A], R^n)$
ii) $x_{t_0} = \phi$
iii) $\dot{x}(t) = f(t, x_t)$, $t_0 \le t < t_0 + A$.

If we call $x(t_0, \phi)$ the solution x of equation (1) through (t_0, ϕ), $x_t(t_0, \phi)$ will denote the element of C given by $x_t(t_0, \phi)(\theta) = x(t + \theta, t_0, \phi)$.

Let $[t_0 - r, t^+)$, $t_0 < t^+ \le \infty$, be the maximal interval of existence of the solution $x(t_0, \phi)$. If $t^+ = \infty$ we say that $x(t_0, \phi)$ is defined in the future.

Definition 1. *We say that $x(t_0, \phi)$ is bounded if there is a constant $k > 0$ such that $|x(t, t_0, \phi)| \le k$ for any $t \in [t_0 - r, t^+)$. If $x(t_0, \phi)$ exists and is bounded in $[t_0 - r, \infty)$, we say that $x(t_0, \phi)$ is bounded in the future.*

Definition 2: Stability. *A solution $x(t_0, \phi)$ of (1), defined in the future, is stable if given $\varepsilon \in 0$, there exists $\delta = \delta(\varepsilon, t_0)$ such that $\zeta \in \Lambda$, $\| \xi - \phi \| < \delta$ implies $\| x_t(t_0, \xi) - x_t(t_0, \phi) \| < \varepsilon$ for any $t \ge t_0$. If δ does not depend on t_0, we say that $x(t_0, \phi)$ is uniformly stable.*

System (1) is said to be stable (uniformly stable) if every solution $x(t_0, \phi)$ of (1) is stable (uniformly stable).

If $f(t, \phi)$ is continuous in Γ, then for every $(t_0, \phi) \in \Gamma$ there is at least one solution of (1) through (t_0, ϕ). If, in addition, $f(t, \phi)$ is locally Lipschitzian in ϕ in each compact subset of Γ, then there is a unique solution of (1) through (t_0, ϕ), and $x_t(t_0, \phi)$ is continuous in (t, t_0, ϕ) [2-b, pp. 13-15 and 21-23].

Suppose that $A(\cdot)$ is a continuous from (ρ, ∞), $\rho \in R$, into the space of continuous linear operators from C into R^n. Since $\| A(t) \|$ is a continuous function of t for $t \in (\rho, \infty)$ and $| A(t)\phi | \le \| A(t) \| \cdot \| \phi \|$, $(t, \phi) \in (\rho, \infty) \times C$, it follows that for any $(t_0, \phi) \in (\rho, \infty) \times C$, the linear functional differential equation

(2) $$y(t) = A(t) y_t$$

has a unique solution $y(t_0, \phi)$ defined and continuous on $[t_0 - r, \infty)$ [2-b; pp. 80-82].

Furthermore, for any fixed t_0 and for $t \ge t_0$,

$$y(t_0, \cdot\ )\ (t) : C \to R^n$$

is a continuous linear operator. Thus, we can define a family of continuous linear operators

$$T(t, t_0) : C \to C,\ t \ge t_0 > \rho,$$

by the relation

(3) $$T(t, t_0)\phi = y_t(t_0, \phi), \qquad \phi \in C,$$

where for each $t \ge t_0$, $y_t(t_0, \phi)$ is the function in C given by $y_t(t_0, \phi)\ (\theta) = y(t_0, \phi)(t + \theta)$, $t_0 - r \le \theta \le t_0$. Then, $T(t, t_0)$ is a strongly continuous semigroup on C, for all $t \ge t_0$. For the proof of these results see [2-b].

For any piecewise continuous function

$$\psi : [-r, 0] \to R^n$$

one can define a solution of (2) with initial values ψ in t_0. Therefore, if the $n \times n$ matrix function Y_0 is defined by

(4) $$Y_0(\theta) = \begin{cases} 0 & -r \le \theta < 0 \\ \mathrm{I} & \theta = 0, \end{cases}$$

then the operator $T(t, t_0)$ can also be defined on the columns of Y_0.

If $g : \Gamma \to R^n$ is a given continuous function, then z is a solution of

(5) $$\dot{z}(t) = A(t) z_t + g(t, z_t)$$

with initial value ϕ at t_0, $(t_0, \phi) \in \Gamma$, if and only if z satisfies the integral equation

(6) $$z_t = T(t, t_0)\phi + \int_{t_0}^{t} T(t, s) Y_0 g(s, z_s) ds,\ t \ge t_0,$$

where the integral equation (6) is an integral equation in R^n and is to be interpreted as

$$z_t(\theta) = [T(t, t_0)\phi](\theta) + \int_{t_0}^{t} [T(t, s) Y_0](\theta) g(s, z_s) ds$$

for $t \ge t_0$, $-r \le \theta \le 0$. For a derivation of (6), see [2-b; pp. 80-86].

Equation (6) is the counterpart of the classical variation-of-constants formula for ordinary differential equations. A counterpart of the non--linear Alekseev formula will be developed in the next section.

III. Integral formula of Alekseev-Shanholt.

The following result, which we are going to state omitting the proof, is due to Shanholt [7, Theorem 3]. It stabilishes a relationship between the solutions of system (1) and the solutions of the perturbed system

$$\dot{y}(t) = f(t, y_t) + g(t, y_t). \tag{7}$$

It is assumed that $f(t, \phi)$ has a continuous Fréchet derivative $\frac{\partial}{\partial \phi} f(t, \phi)$ with respect to ϕ in Γ.

Corresponding to each solution $x(t_0, \phi)$ of (1), we can define a linear functional differential equation

$$\dot{u}(t) = \frac{\partial}{\partial x_t}(t, x_t(t_0, \phi))\ u_t \tag{*}$$

which is called the linear variational equation of (1) with respect to $x_t(t_0, \phi)$. In what follows, the family of linear operators associated with (*) will be denoted by $T(t, t_0 : \phi)$, $t \geq t_0$.

Let $\Lambda_p = \{\phi \in \Lambda : \phi(\theta)$ exists, is bounded, and is piecewise continuous on $[-r, 0)\}$, and for each $(t_0, \phi) \in \Gamma$ let $J = J(t_0, \phi)$ be the maximal interval of existence of $x(t_0, \phi)$.

Theorem 1. *If, for any* $(t_0, \phi) \in (\rho, \infty) \times \Lambda_p$, $J(t_0, \phi) = [t_0, \infty)$, *then*

$$y_t(t_0, \phi) = x_t(t_0, \phi) + \int_{t_0}^{t} T(t, s : y_s(t_0, \phi))\, Y_0\, g(s, y_s(t_0, \phi))ds \tag{8}$$

so long as $(t_0, \phi) \in (\rho, \infty) \times \Lambda_p$ *and* t *is in an interval in which* $y(t_0, \phi)$ *exists.*

We will reffer to relation (8) as "Alekseev-Shanholt's Integral Formula". It is indeed a generalization of the variation of constante formula.

IV. On the uniform stability of a perturbed non-linear system.

Theorem 4.1. *Let us consider the system of functional differential equations*

$$\dot{x}(t) = f(t, x_t) \tag{1}$$

where $f : \Gamma \to R^n$ *is continuous and suppose that the following conditions are satisfied:*

H_1) *There exists at least one bounded solution of system* (1).

H_2) *For every* $\alpha > 0$ *there exists a constant* $N(\alpha) > 0$ *such that if*

$$\| \psi \| \leq \alpha, \text{ then } \| T(t, t_0 : \psi) \| \leq N(\alpha) \text{ for all } t,\ t \geq t_0.$$

H_3) *The Fréchet derivative* $\frac{\partial}{\partial \phi} f(t, \phi)$ *is continuous in* Γ.

Then, for any given $c > 0$, *there exists a constant* $K = K(c)$ *such that if* $\| \phi \| \leq c$ *we have* $\| x_t(t_0, \phi) \| \leq K$ *for each* t, $t_0 \leq t < \infty$.

More than that, any solution of (1) *is uniformly stable in* $[t_0, \infty)$.

Proof. Let $\sigma(t_0, \xi)$ be a bounded solution of (1). Then there exists $k > 0$ such that $\| \sigma_t(t_0, \xi) \| \leq k$ for any $t \geq t_0$. Given $c > 0$, let $\phi \in \Lambda$ be such that $\| \phi \| \leq c$. Since $\| \xi \| \leq k$ let $\alpha = \sup\{k, c\}$.

Let $\hat{\Lambda} = \{\psi \in \Lambda : \psi = \lambda \cdot \xi + (1-\lambda)\phi, 0 \leq \lambda \leq 1\}$ be a convex subset of Λ. If $\psi \in \hat{\Lambda}$ we have $\| \psi \| \leq \alpha$.

From [7, Theorem 2],

$$\| x_t(t_0, \phi) - \sigma_t(t_0, \xi) \| \leq \sup_{\psi \in \hat{\Lambda}} \| T(t, t_0 : \psi) \|\ \| \phi - \xi \|$$

Then hypothesis H_2 implies

$$\| x_t(t_0, \phi) - \sigma_t(t_0, \xi) \| \leq N(\alpha)\ \| \phi - \xi \| \text{ for all } t \geq t_0 \text{ and hence} \tag{9}$$

$$\| x_t(t_0, \phi) \| \leq \| \sigma_t(t_0, \xi) \| + N(\alpha)\ \| \phi - \xi \| \leq \| \sigma_t(t_0, \xi) \| + N(\alpha)(\| \phi \| + \| \xi \|) \leq k + N(\alpha)(c+k) = K = K(c).$$

Then $x(t_0, \phi)$ is bounded in the future

The uniform stability of $\sigma(t_0, \xi)$ follows immediately from (9). Using the same argument as above we can see that every solution $x(t_0, \phi)$ of (1) is uniformly stable in $[t_0, \infty)$, completing the proof.

Theorem 4.2. *Consider the systems of functional differential equations*

$$x(t) = f(t, x_t) \tag{1}$$

$$y(t) = f(t, y_t) + g(t, y_t) \tag{7}$$

where $f, g : \Gamma \to R^n$ *are continuous and the following hypotheses are satisfied:*

H_1) *There exists at least one bounded solution of* (1).

H_2) *For every* $\alpha > 0$ *there exists a constant* $N(\alpha) > 0$ *such that if*

$$\| \psi \| \leq \alpha, \text{ then } \| T(t, t_0 : \psi) \| \leq N(\alpha) \text{ for all } t,\ t \geq t_0.$$

H_3) *For every* $\alpha > 0$, *there exists a continuous function* $\lambda(t, \alpha) > 0$, *with* $\int_0^\infty \lambda(t, \alpha)dt < \infty$, *such that if* $\| \phi \| \leq \alpha$, *then* $\| g(t, \phi) \| \leq \lambda(t, \alpha)$.

H_4) *The Fréchet derivatives* $\frac{\partial}{\partial \phi} f(t, \phi)$ *and* $\frac{\partial}{\partial \phi} g(t, \phi)$ *are continuous in* Γ.

Then given $\varepsilon > 0$ *and* $c > 0$, *there is a number* $T = T(\varepsilon, c) > 0$ *such that if* $\phi \in \Lambda$ *with* $\|\phi\| \leq c$, *the solutions of* (7) *are bounded in the future and* $\|y_t(\tau, \phi) - x_t(\tau, \phi)\| < \varepsilon$ *for all* t, $t \geq \tau \geq T$.

Proof. By using Theorem 4.1, hypotheses H_1, H_2 and H_4 imply that for any given $c > 0$, there is a constant $K = K(c)$ such that if $\|\phi\| \leq c$, then $\|x_t(\tau, \phi)\| \leq K$ for all t, $\tau \leq t < \infty$, $\tau \geq t_0$.

From Alekseev-Shanholt's Integral Formula it follows that

$$(10) \quad y_t(\tau, \phi) = x_t(\tau, \phi) + \int_\tau^t T(t, s : y_s(\tau, \phi)) Y_0\, g(s, y_s(\tau, \phi)) ds$$

where $\|\phi\| \leq c$.

We will show that there is a number $T = T(\varepsilon, c)$ such that $y(\tau, \phi)$ is bounded in the future for every $\tau \geq T$. From (10) it follows that

$$(11) \quad \|y_t(\tau, \phi)\| \leq \|x_t(\tau, \phi)\| + \int_\tau^t \|T(t, s : y_s(\tau, \phi))\| \cdot |g(s, y_s(\tau, \phi))| ds.$$

Suppose that $y(\tau, \phi)$ is not bounded in the future. Let us take $M = K + 1$. Then there exists $\eta = \eta(\tau)$ such that $\|y_t(\tau, \phi)\| < M$ for all t, $\tau \leq t < \eta(\tau)$ and $\|y_{\eta(\tau)}(\tau, \phi)\| = M$.

It follows from hypothesis H_3 that there is a continuous function $\lambda(t, M) > 0$, with $\int_0^\infty \lambda(t, M) dt < \infty$ such that $|g(t, y_t(\tau, \phi))| \leq \lambda(t, M)$ for all t on $[\tau, \eta(\tau)]$.

From hypothesis H_2 there exists a constant $N(M) > 0$ such that $\|T(t, s : y_s : (\tau, \phi)\| \leq N(M)$ for all t on $[\tau, \eta(\tau)]$, $\tau \leq s \leq t$. We choose $T_1 = T_1(c)$, $T_1 < \tau$, so that $\int_{T_1}^\infty \lambda(t, M) dt < \dfrac{1}{N(M)}$. Then for all $t \in [\tau, \eta(\tau)]$ it follows from (11) that

$$\|y_t(\tau, \phi)\| \leq K + \int_\tau^t N(M)\lambda(s, M) ds \leq K + N(M) \int_{T_1}^\infty \lambda(s, M) ds < K + $$

$$+ N(M) \frac{1}{N(M)} = K + 1 = M.$$

Thus $\|y_t(\tau, \phi)\| < M$ on $[\tau, \eta(\tau)]$, a contradiction.

Then $y(\tau, \phi)$ is bounded in the future.

From (10) we have that

$$\|y_t(\tau, \phi) - x_t(\tau, \phi)\| \leq \int_\tau^t \|T(t, s : y_s(\tau, \phi))\| \cdot |g(s, y_s(\tau, \phi))| ds.$$

As $\|y_t(\tau, \phi)\| \leq M$ on $[\tau, \infty)$, it follows from hypotheses H_2 and H_3 that there exists a constant $N(M) > 0$ and a continuous function $\lambda(t, M) > 0$ satisfying $\int_0^\infty \lambda(t, M) dt < \infty$, such that $\|T(t, s : y_s(\tau, \phi))\| \leq N(M)$ and $|g(s, y_s(\tau, \phi))| \leq \lambda(s, M)$ for all $\tau \leq s \leq t$. Given any $\varepsilon > 0$, choose $T = T(\varepsilon, c) > T_1$ so that

$$\int_T^\infty \lambda(t, M) dt < \frac{\varepsilon}{N(M)}.$$

Then we have

$$\|y_t(\tau, \phi) - x_t(\tau, \phi)\| \leq \int_\tau^t N(M) \cdot \lambda(s, M) ds \leq N(M) \int_T^\infty \lambda(s, M) ds <$$

$$< N(M) \cdot \frac{\varepsilon}{N(M)} = \varepsilon.$$

Therefore

$$\|y_t(\tau, \phi) - x_t(\tau, \phi)\| < \varepsilon \text{ for all } t, \ t \geq \tau \geq T > 0.$$

The proof is complete

The next theorem is the main result of this work, in the sense that we can guarantee uniform stability of each solution bounded in the future of system (7). Onuchic, [6-b], got a similar result for linear perturbed systems.

Theorem 4.3. *Suppose systems* (1) *and* (7) *satisfy the same hypotheses as in Theorem* 4.2. *Then any solution* $y(\tau, \phi)$ *of* (7), *bounded in the future, is uniformly stable in any interval* $[\gamma, \infty)$ *contained in its maximal interval of existence.*

Proof. Let $y(\tau, \phi)$ be any solution of (7) bounded in the future, and $[\gamma, \infty)$ an interval contained in its maximal interval of existence. For some constant M and for t, $\gamma \leq t < \infty$ we have $\|y_t(\tau, \phi)\| \leq M$.

Let us prove that for any given ε, $0 < \varepsilon \leq 1$, there exists $\delta = \delta(\varepsilon)$, $0 < \delta \leq \varepsilon$, such that if $\xi \in \Lambda$ with $\|\xi - \phi\| < \delta$, imply $\|y_t(\tau, \phi) - y_t(\tau, \xi)\| < \varepsilon$ for all t, $t \geq \tau \geq \gamma$.

We have

$$(12) \quad \|y_t(\tau, \phi) - y_t(\tau, \xi)\| \leq \|y_t(\tau, \phi) - x_t(\tau, \phi)\| + \|x_t(\tau, \xi) - x_t(\tau, \phi)\| + $$
$$+ \|x_t(\tau, \xi) - y_t(\tau, \xi)\|$$

As $\|\phi\| \leq M < M + 1$, by using Theorem 4.2, with $c = M + 1$ we can find $T = T(\varepsilon/3, M + 1) > 0$ such that

(13) $$\| y_t(\tau, \phi) - x_t(\tau, \phi) \| < \frac{\varepsilon}{3} \text{ for all } t,\ t \ge \tau \ge T.$$

We have also $\| \xi - \phi \| < \delta \le 1$. This implies $\| \xi \| < \| \phi \| + 1 \le M + 1$. By using Theorem 4.2 we have

(14) $$\| y_t(\tau, \xi) - x_t(\tau, \xi) \| < \frac{\varepsilon}{3} \text{ for all } t,\ t \ge \tau \ge T.$$

From (13) it follows that $\| x_t(\tau, \phi) \| < \| y_t(\tau, \phi) \| + \varepsilon/3 < M + 1$ for all t, $t \ge \tau \ge T$.

By using Theorem 4.1, with $c = M + 1$ and $k = M$, we can find a number $\delta = \delta(\varepsilon/3, M, M+1) > 0$, such that $\| \xi - \phi \| < \delta$ implies

(15) $$\| x_t(\tau, \xi) - x_t(\tau, \phi) \| < \frac{\varepsilon}{3} \text{ for all } t,\ t \ge \tau \ge T.$$

From (12), (13), (14) and (15) it follows that

$$\| y_t(\tau, \phi) - y_t(\tau, \xi) \| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ for all } t,\ t \ge \tau \ge T.$$

Let us take τ, $\gamma \le \tau \le T$. The solution $y_t(t_0, \phi)$ is continuous uniformly in t, with respect to the initial value ϕ for any compact subset of Γ[2-b]. Then we can find a $\delta_1 > 0$, $\delta_1 = \delta_1(\delta) = \delta_1(\varepsilon) < \delta$ such that $\| \xi - \phi \| < \delta_1$ implies

$$\| y_t(\tau, \phi) - y_t(\tau, \xi) \| < \delta \le \varepsilon \text{ for all } t,\ \gamma \le \tau \le t \le T.$$

Therefore $\| \xi - \phi \| < \delta_1$ implies

$$\| y_t(\tau, \phi) - y_t(\tau, \xi) \| < \varepsilon \text{ for all } t,\ \gamma \le \tau \le t < \infty.$$

The proof is complete.

Remark. If we take $\Lambda = \Lambda_p$ we can delete hypothesis of continuity of the Fréchet derivative, $\frac{\partial}{\partial \phi} g(t, \phi)$, in Γ. Thus, we can also delete the hypothesis of unicity of the solutions of the perturbed system (7), if we restrict ourselves to initial data ϕ in Λ_p.

V. Application.

Suppose

$$f(\phi) = - \int_{-r}^{0} a(-\theta) h(\phi(\theta)) d\theta$$

where

(i) $h(x)$ is a scalar function of class C^1, $-\infty < x < \infty$, satisfying, $h(0) > 0$, $x \,.\, h(x) > 0$ for $x \ne 0$, and

$$H(x) = \int_0^x h(s) ds \to \infty \text{ as } | x | \to \infty$$

(ii) For $0 \le t \le r$, $a(r) = 0$, $a(t) \ge 0$, $\dot{a}(t) \le 0$, $\ddot{a}(t) \ge 0$ are continuous. We consider the especial case of (1) given by

(16) $$x(t) = f(\phi) = - \int_{-r}^{0} a(-\theta) h(x(t+\theta)) d\theta = = - \int_{t-r}^{t} a(t-u) h(x(u)) du.$$

Any solution of (16) satisfies

(17) $$\ddot{x}(t) + a(0) h(x(t)) = - \int_{t-r}^{t} \dot{a}(t-u) h(x(u)) du$$

or

(18) $$\ddot{x}(t) + a(0) h(x(t)) = -a(r) \int_{t-r}^{t} h(x(t+\theta)) d\theta + + \int_{-r}^{0} \ddot{a}(-\theta) \left(\int_{\theta}^{0} h(x(t+u)) du \right) d\theta.$$

Equation (17) is the model of a special type of circulating fuel nuclear reactor, as well as can also serve as a one-dimensional model in viscoelasticity.

If we define $V : C \to R$ by the relation

(19) $$V(\phi) = H(\phi(0)) - \frac{1}{2} \int_{-r}^{0} \dot{a}(-\theta) \left[\int_{\theta}^{0} h(\phi(s)) ds \right]^2 d\theta$$

then the derivative of V along solutions of (16) is given by

$$\dot{V}(\phi) = \frac{1}{2} \dot{a}(r) \left[\int_{-r}^{0} h(\phi(\theta)) d\theta \right]^2 - \frac{1}{2} \int_{-r}^{0} \ddot{a}(-\theta) \left[\int_{\theta}^{0} h(\phi(s)) ds \right]^2 d\theta,$$

Since the hypotheses on $a(t)$ and

$$\left[\int_{-r}^{0} h(\phi(\theta)) d\theta \right]^2 \ge 0, \quad \left[\int_{\theta}^{0} h(\phi(s)) ds \right]^2 \ge 0$$

imply that $V(\phi) \le 0$.

If $u(x)=\min\{H(x),\ H(-x)\}$, $0\le x<\infty$, then $u(x)$ is continuous, $u(0)=0$, $u(x)>0$ for $x>0$ and $u(x)\to\infty$ with $x\to\infty$. We can see also that

$$V(\phi)\ge H(\phi(0))\ge \min\{H(|\phi(0)|),\ H(-|\phi(0)|)\} = u(|\phi(0)|).$$

Therefore from Onuchic [6-c; p. 43, Thm. 2] it follows that all solutions of (16) are bounded in the future.

We will show now that the hypothesis of Theorem 4.1 are satisfied.

Since $f(t,0)=0$, $x(t)\equiv 0$ is a bounded solution of (16) and (H_1) is satisfied, (H_3) follows from the fact that $h(x)\in C^1$ and the hypothesis on $a(t)$.

To show that (H_2) is satisfied, we do as follows.

Since equation (16) satisfies (i) and (ii), from corollary 5[2-a] it follows that the solution $x=0$ of (16) is globally asymptotically stable. From theorem 4 [3] this solution is globally uniformly asymptotically stable. Thus, we have that:

(*) for every $\alpha\ge 0$ there exist $a(\alpha)\ge 0$ and $A(\alpha)>0$ such that if $\|\phi\|\le\alpha$, then $\|x_t(t_0,\phi)\|\le a(\alpha)$ for all t on $[t_0, t_0+A(\alpha)]$.

Let us consider the variational system of (16) relatively the solution $x_t(t_0,\phi)$,

$$u(t) = F(t,x_t)u_t = -\int_{-r}^{0} a(-\theta)\dot h(x(t+\theta))u(t+\theta)d\theta. \tag{20}$$

Let us show that the solutions of (20) are uniformly bounded.

We have that

$$|F(t,\phi)|\le\int_{-r}^{0} a(-\theta)\,|h(\phi(\theta))|\,d\theta\le \bar b(\alpha)\int_{-r}^{0} a(-\theta)d\theta,$$

where $\bar b(\alpha) = \sup\limits_{\|\xi\|\le\alpha} |\dot h(\xi(\theta))|$.

From (ii) we have $\int_{-r}^{0} a(-\theta)d\theta\le k_1$. Then, if $\|\phi\|\le\alpha$ we have $|F(t,\phi)|\le b(\alpha)$, where $b(\alpha)=\bar b(\alpha).\ k_1$ for all t on (t_0,ϕ).

By using (*) we have that $|\dot u(t)|\le b(a(\alpha))\cdot\|u_t\|$ for all t on $[t_0, t_0+A(\alpha)]$.

Integrating the last inequality and using Gronwall's Inequality we obtain

$$\|u_t(t_0,\phi)\|\le e^{b(a(\alpha))(t-t_0)}\|\phi\|\le e^{b(a(\alpha))A(\alpha)}.\ \alpha=\bar\beta(\alpha).$$

Therefore,

(I) $\|u_t(t_0,\phi)\|\le\bar\beta(\alpha)$ for all t on $[t_0,t_0+A(\alpha)]$ provided $\|\phi\|\le\alpha$.

We consider now $t\ge t_0+A(\alpha)$.

The origin, in system (16), is a globally asymptotically stable equilibrium point. Since $h(0)>0$, from corollary 5[2-a] the solution $u=0$ of the linear variational equation

$$\dot u(t) = -\int_{-r}^{0} a(-\theta)\,\dot h(0)\,u(t+\theta)d\theta$$

is globally asymptotically stable. Therefore the solution $u=0$ of (20) is globally asymptotically stable, i.e., given $\alpha>0$ and for every $\varepsilon>0$, there exists $\bar A(\varepsilon,\alpha)>0$ such that if $\|\phi\|\le\alpha$, then $\|u_t(t_0,\phi)\|\le\varepsilon$ for all t, $t\ge t_0+\bar A(\varepsilon,\alpha)$.

Therefore,

(II) For every $\alpha\ge 0$ there exist $\varepsilon>0$ and $A=A(\varepsilon,\alpha)$ such that if $\|\phi\|\le\alpha$ implies $\|u_t(t_0,\phi\|\le\varepsilon$ for all t, $t\ge t_0+A$.

In fact:

a) If $\bar A(\varepsilon,\alpha)\le A(\alpha)$ the assertion (II) is correct.
b) If $\bar A(\varepsilon,\alpha)\ge A(\alpha)$, we can repeat the same argument in (I) for the interval $[A(\alpha),\bar A(\varepsilon,\alpha)]$ and then the assertion (II) is also true.

From (I) and (II) we have that, for every $\alpha\ge 0$ there exists $\beta(\alpha)= =\max\{\bar\beta(\alpha),\varepsilon\}$ such that if $\|\phi\|\le\alpha$, then $\|u_t(t_0,\phi)\|\le\beta(\alpha)$ for all t, $t\ge t_0$, i.e, the solutions of (20) are uniformly bounded.

We also know that

$$u_t(t_0,\phi) = T(t,t_0:\psi)\phi \text{ for all } t,\ t\ge t_0.$$

Let us consider $\phi,\psi\in C$ such that $\|\phi\|\le\alpha$ and $\|\psi\|\le\alpha$. Then, we have $\|T(t,t_0:\psi)\phi\|=\|u_t(t_0,\phi)\|\le\beta(\alpha)$ for all t, $t\ge t_0$.

Since $T(t,t_0:\psi)$ is a linear operator, by using the Uniformly Boundedness Principle, there exists a constant $N(\alpha)>0$ such that

$$\|T(t,t_0:\psi)\|\le N(\alpha) \text{ for all } t,\ t\ge t_0.$$

Therefore, for every $\alpha>0$ there exists a constant $N(\alpha)>0$ such that if $\|\psi\|\le\alpha$, then $\|T(t,t_0:\psi)\|\le N(\alpha)$ for all t, $t\ge t_0$.

Thus the hypothesis H_2 of the Theorem 4.1 is satisfied. Therefore any solution of (16) is uniformly stable.

Let $g(t,y_t)$ be a perturbation of system (16) satisfying the hypothesis H_3 of the theorem 4.2. Then, by Theorem 4.3, every solution, bounded in the future, of the perturbed system is uniformly stable.

References

[1] Halanay, A.; *Differential equations: stability oscillations, time lags*, Academic Press, New York, 1966.

[2-a] Hale, J.K.; *Sufficient conditions for stability and instability of autonomous functional differential equations*, Journal of Differential Equations 1, pp. 452-482, 1965.

[2-b] ———; *Functional differential equations*, Springer-Verlag, Vol. 3, New York, 1971.

[3] Izé, A. F. and Reis, J. G.; *Contributions to stability of neutral functional differential equations*, Notas de Matemática do I.C.M. de São Carlos-USP, n.º 1, 1977.

[4] Krasovskii, N.; *Stability of motion*, Stanford University Press, Stanford, California, 1963.

[5] Molfetta, N.A.; *Alekseev's integral formula and applications in stability problems*, An. Acad. brasil. Ciênc., pp. 303-308, 1969, 41(3).

[6-a] Onuchic, N.; *On the uniform stability of a perturbed linear system*, J. Math. Anal. Appl, Vol. 6, n.º 3, pp. 457-464, 1963.

[6-b] ———; *On the uniform stability of a perturbed linear functional differential equation*, Proceedings of the American Mathematical Society, Vol. 19, n.º 3, pp. 528-532, 68.

[6-c] ———; *Equações diferenciais com retardamento*, 8.º Colóquio Brasileiro de Matemática, Poços de Caldas, 1971.

[7] Shanholt, G. A.; *A nonlinear variation – of – constants formula for functional differential equations*, Mathematical Systems Theory, Vol. 6, n.º 4, pp. 343-352, 1973.

[8] Strauss, A.; *On the stability of a perturbed nonlinear system*, Proceedings of the American Mathematical Society, Vol. 17, n.º 4, pp. 803-807, 1966.

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