On surfaces of right helicoid type in H^3

Hiroshi Mori

1. Introduction.

In a recent paper [3], we have constructed a family of complete minimal surfaces of revolution in the hyperbolic 3-space H^3 of constant sectional curvature -1, which are the first examples of such surfaces in H^3 other than the trivial ones H^2, the hyperbolic 2-planes, and showed that some of them are globally stable (cf. [1]).

In this paper, we shall construct a family of complete surfaces $M_a (a \geqq 0)$ of right helicoid type in H^3 and show that for $0 \leqq a \leqq 3\sqrt{2}/4$, M_a are globally stable, and for $a \geqq \sqrt{(105\pi)}/8$, M_a are not globally stable.

2. On surfaces of right helicoid type in H^3.

It is known (see [2]) that the hyperbolic 3-space H^3 of constant sectional curvature -1 is realized in the Lorentz 4-space L^4 as a hypersurface:

$$H^3 = \{x \in L^4;\ \langle x, x \rangle = -1,\ x_1 \geqq 1\},$$

where $\langle\, ,\, \rangle$ is a non-degenerate symmetric bilinear form defined by $\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \ldots + x_4 y_4$, for x, y in L^4.

It is easily shown that the curve γ defined by

$$\gamma(s) = (\cosh s, 0, \sinh s, 0), \qquad s \in \mathbb{R},$$

is a complete geodesic in H^3 parametrized by arc-length, where $\mathbb{R}$ is the set of all real numbers. For any C^2 function $\theta = \theta(t)$, $t \in J$, an open interval of $\mathbb{R}$, the C^2 mapping f from the product space $J \times R$ into H^3,

$$f(s, t) = (\cosh s \cosh t, \cosh s \sinh t, \sinh s \cos \theta(t), \sinh s \sin \theta(t)),$$

defines a surface M of right conoid type in H^3 by screwing γ. It can be easily shown that the first fundamental form I of f is

$$I = ds^2 + (\cosh^2 s + \sinh^2 s\, \theta'(t)^2) dt^2. \tag{1}$$

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And it is easily checked that $N(s,t)=((\cosh^2 s+\sinh^2 s\,\theta'(t)^2)^{-1/2}\times$ $\times(\sinh s\sinh t\,\theta'(t), \sinh s\cosh t\,\theta'(t), \cosh s\sin\theta(t), -\cosh s\cos\theta(t))$ is a field of unit normal vectores along f and that the second fundamental form II of f is

$$(2)\qquad II=-(\cosh^2 s+\sinh^2 s\,\theta'(t)^2)^{-1/2}(2\theta'(t)ds\,dt+\frac{1}{2}\sinh 2s\,\theta''(t)dt^2).$$

From (1) and (2) it follows that f is an immersion, and that f is a minimal immersion (i.e., M is said to be a right helicoid in H^3) if and only if on the interval J, the following holds.

$$(3)\qquad \theta''(t)=0.$$

A general solution of (3) is represented as

$$(4)\qquad \theta(t)=at+b,\qquad a,b: \text{constants}.$$

From (4) it is obvious that J, the domain of definition of the function $\theta(t)$, can be extended to $\mathbb{R}$. Rotating the surface M around the x_1, x_2 – plane and reversing the orientation of the x_4 – axis if necessary, we may assume that

$$(5)\qquad a\geqq 0,\ b=0.$$

Conversely, for each non-negative constant a, we see that the surface M_a defined by the one – one, analytic mapping $f:\mathbb{R}\times\mathbb{R}\to H^3$,

$$(6)\qquad f(s,t)=(\cosh s\cosh t, \cosh s\sinh t, \sinh s\cos at, \sinh s\cos at)$$

is a complete minimal surface in H^3. Thus we have the following result.

Theorem 1. *For each nonnegative constant a, the one – one analytic mapping $f:\mathbb{R}\times\mathbb{R}\to H^3$ defined by* (6), *defines a complete minimal surface M_a in H^3.*

3. On the stability of the minimal surfaces M_a in H^3.

At first, we recall the definition of the stability. Let $f: M\to H^3$ be a C^∞ minimal immersion of an oriented, connected 2-manifold M into H^3. A domain D on M with compact closure is stable if the second variation of the induced area of D is nonnegative for all variations that leave the boundary ∂D of D fixed. The immersion f is stable (or M is globally stable) if every such domain is stable. The purpose of this section is to prove the following results.

Theorem 2. *Let $M_a (a\geqq 0)$ be as in Theorem 1.*

(i) *For $a\leqq 3\sqrt{2}/4$, M_a is globally stable.*
(ii) *For $a\geqq\sqrt{(105\pi)}/8$, M_a is not globally stable (i.e., there is an unstable, relatively compact domain D on M_a).*

To prove this Theorem we shall prepare some lemmas.

Lemma 1. *Let M_a be as in Theorem 1 and D a relatively compact domain on M_a. Then the first eigenvalue $\lambda_1(D)$ of D with respect to the Laplace--Beltrami operator of M_a satisfies that*

$$\lambda_1(D)>\frac{1}{4}.$$

Proof. Since M_a is analytic diffeomorphic to the Euclidean plane $\mathbb{R}^2$, for a given domain D on M_a, there exists a domain D' on M_a which is simply connected and in which the closure of D is contained. From this it follows that

$$(7)\qquad \lambda_1(D)>\lambda_1(D'),$$

where $\lambda_1(D')$ is the corresponding eigenvalue of D'. And from the equation of Gauss and the fact that M_a is minimally embedded in H^3, the Gaussian curvature K of M_a satisfies that

$$(8)\qquad K\leqq -1.$$

From Mckean's theorem (see [4]) together with (8) it follows that

$$(9)\qquad \lambda_1(D')\geqq\frac{1}{4}.$$

Combining (7) and (9) completes the proof.

Lemma 2 (see [3]). *Let M and $f: M\to H^3$ be as above, and D a domain on M. Then for a normal variation with variable vector field uN, $u\in C^\infty(\bar{D})$, $u\,|\,\partial D\equiv 0$, the second variation of area is given by*

$$A''(0)=\int_D[\,|\nabla u|^2-(\|B\|^2-2)u^2]\,dV\geqq$$
$$\geqq\int_D[\lambda_1(D)+2-\|B\|^2]u^2dV,$$

where N is a field of unit normal vectors along f, $\bar{D}$ is the closure of D and we used the Rayleigh characterization of $\lambda_1(D)$.

Remark. The restriction to normal variations is not essential. In fact, using the equation of Codazzi, Y. Ogawa has shown that for a variation with variable vector field $uN + vX, v \in C^\infty(\bar{D}), v \mid \partial D \equiv 0$, the second variation of area is equal to the one of this Lemma, where X is a C^∞ field of unit tangent vectors along f.

We now compute the length of the second fundamental form $\| B \|$. From the minimality of M_a in H^3 together with (1), (2) and (4) it follows that

$$\| B \|^2 (s, t) = 2a^2 (\cosh^2 s + a^2 \sinh^2 s)^{-2}.$$

Thus we have the inequality

$$\| B \|^2 (s, t) \leqq 2a^2, \qquad \text{for } s, t \text{ in } \mathbb{R}.$$

Combining this estimate of $\| B \|$ with Lemmas 1 and 2 completes the proof of Theorem 2, (i).

We shall now prove the assertion of Theorem 2, (ii). To do this we use the following result.

Lemma 3 (see [1]). *Let M be a stable, complete minimal surface in H^3. For a fixed point p, we denote by B_r the open geodesic ball in M with center p and radius r. Assume that*

$$\lim_{r \to \infty} \int_{B_r} \| B \|^2 dV / r^2 = 0.$$

Then we have that

$$\limsup_{r \to \infty} \int_{B_r} \| B \|^2 (\| B \|^2 - 6) dV \leqq 0.$$

We apply this Lemma to our surface M_a, which may be identified with $\mathbb{R}^2$ by (6). It follows from (1), (4) and (5) that

$$dV = (\cosh^2 s + a^2 \sinh^2 s)^{1/2} ds \wedge dt. \tag{9}$$

Setting p to be the origin we have that for each positive r,

$$B_r \subset D_r := \{(t, s) \in \mathbb{R}^2;\ |s| \leqq r,\ |t| \leqq r (\cosh^2 s + a^2 \sinh^2 s)^{-1/2}\}.$$

From this together with (9) it can be easily shown that the assumption of Lemma 3 is satisfied for M_a. Defining the function $\Phi(a)$ of a by

$$\Phi(a) = \int_0^\infty [a^2(\cosh^2 s + a^2 \sinh^2 s)^{-4} - 3(\cosh^2 s + a^2 \sinh^2 s)^{-2}] ds, \tag{10}$$

and taking contrapositive of Lemma 3 together with (9) we see that the minimal surface M_a is not stable if $\Phi(a)$ is positive. We now estimate $\Phi(a)$ from below. By the change of variable, $u = \sinh s$, it follows that

$$\Phi(a) = a^2 \int_0^\infty (u^2 + 1 + a^2 u^2)^{-4} (u^2 + 1)^{-1/2} du - 3 \int_0^\infty (u^2 + 1 + a^2 u^2)^{-2} (u^2 + 1)^{-1/2} du = I_1 + I_2, \tag{11}$$

where I_1 (resp. I_2) is the first term (resp. the second term) of the right hand side of (11). Since $-(u^2 + 1)^{-1/2} \geqq -1$ for all non-negative u, we have the following inequality.

$$I_2 \geqq -3 (a^2 + 1)^{-2} \int_0^\infty [u^2 + (a^2 + 1)^{-1}]^{-2} du = -\frac{3}{4} \pi (a^2 + 1)^{-1/2}. \tag{12}$$

On the other hand, by the change of variable, $v = (a^2 + 1)u^2 + 1$, we have the following inequality

$$I_1 = a^2 \int_1^\infty v^{-4} [(a^2 + 1)/(v + a^2)]^{1/2} \frac{1}{2} (a^2 + 1)^{-1/2} (v - 1)^{-1/2} dv \geqq a^2 \frac{1}{2} (a^2 + 1)^{-1/2} \int_1^\infty v^{-9/2} (v - 1)^{-1/2} dv, \tag{13}$$

by virtue of $(a^2 + 1)/(v + a^2) \geqq v^{-1}$ for all $v \geqq 1$. Finally, by the change of variable, $(v - 1)^{1/2} = w$, it follows that

$$\int_1^\infty v^{-9/2} (v - 1)^{-1/2} dv = 2 \int_0^\infty (w^2 + 1)^{-9/2} dw = 2 \frac{6}{7} \frac{4}{5} \frac{2}{3} \int_0^\infty (w^2 + 1)^{-3/2} dw = \frac{32}{35}. \tag{14}$$

Putting (12), (13) and (14) into (11) we have the following estimate:

$$\Phi(a) > \left(\frac{16}{35} a^2 - \frac{3\pi}{4} \right) (a^2 + 1)^{-1/2}.$$

From this it follows that

$$\Phi(a) > 0 \qquad \text{if } a \geqq \sqrt{(105\pi)}/8.$$

Combining this inequality with Lemma 3 completes the proof of Theorem 2, (ii).

Remark. It can also be shown (cf. [1]) that for

$$1/2 < a \leqq (1/2)(1 + 5\pi/96)/(1 - 5\pi/96) = 0.69\ldots,$$

the complete, minimal surfaces M_a in H^3 obtained in [3] are not globally stable.

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Hiroshi Mori
Department of Mathematics
Faculty of Education
Toyama University
9130 Gofuku, Toyama-shi
JAPAN