c-equivalence of embeddings is different from equivalence and bordism of pairs

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(0) Manifolds and maps will be C^{∞}. All manifolds will be compact without boundary. Fix two manifolds N^n and M^m ($n \leq m$ are the dimensions). There are several equivalence relations between embeddings. We shall use:

(a) the embeddings $f, g : N \to M$ are *d*-equivalent (or simply equivalent) iff there is a diffeomorphism $h : M \to M$ such that $h \circ f = g$;

(b) f and g are *B*-equivalent iff their normal bundles are equivalent. This occurs iff there is a bordism of the pairs $(M, f(N))$ and $(M, g(N))$, which, restricted to the submanifolds is a product bordism (see [1]).

(c) In [1], [2], [3] we defined several other relations using surgery on M and studied some of their properties. The most interesting of them seems to be *c*-equivalence (see definition 2 below).

The main result of this paper is

Theorem A. *c-equivalence lies strictly between d-equivalence and B-equivalence.*

We shall show this in the case $N = S^1 \times S^1$ and $M = S^1 \times S^1 \times S^1$.

(1) Let $\psi : S^p \times D^{m-p} \to M$ be an embedding and $M' = \chi(M, \psi)$ be the manifold which is obtained by the surgery (of type p) defined be ψ. Let $f : N \to M$ be an embedding. If $f(N) \cap \psi(S^p \times D^{m-p}) = \varnothing$ we shall say that the surgery is away from f. In this case there is a well determined embedding $f' : N \to M'$ defined by f.

Definition 1. *Two embeddings f, $g : N \to M$ will be c-related if they are d-equivalent or if there is a surgery, away from f and g, such that f' and g' are d-equivalent.*

c-relation is reflexive and symmetric but in general it is not transitive, so we give the:

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Definition 2. *c-equivalence is the equivalence relation which is generated by c-relation: f and g are c-equivalent if there are embeddings* $f = h_1, h_2, \dots, h_r = g$ $(h_i : N \to M)$ *such that* h_i *and* h_{i+1} *are c-related* $(i = 1, 2, \dots, r-1)$.

It should be clear that in general f and g are not c-equivalent if f' and g' are c-related, because all surgeries must be performed on the same M.

If f and g are d-equivalent, they are c-equivalent and it is easy to see that two c-equivalent embeddings are B-equivalent. For embeddings of S^1 into surfaces and into 3-manifolds, c-equivalence is the same as B-equivalence. In these cases we have even more: c-relation coincides with B-equivalence (see [1], [2], [3] for these and others cases).

(2) We now adapt some definitions and results from [5] for our purposes:

Let $\dim N = 2$ and $\dim M = 3$ and $f: N \to M$ be an embedding. If $f(N)$ is 2-sided in M, it will be called a "surface" in M, then $f(N)$ will be "incompressible" in M if $N \neq S^2$ and $\ker(f_{1*} : \pi_1(N) \to \pi_1(M)) = 0$. We say that M is P^2-irreducible iff (a) every embedded 2-sphere bounds a ball and (b) M does not contain a 2-sided projective plane.

Theorem. (Waldhausen) *Let M be* P^2*-irreducible and* $f, g: N \to M$ *be embeddings such that both are incompressible surfaces. If* $g_{1*}(\pi_1(N)) \subset f_{1*}(\pi_1(N))$ *then* $g(N)$ *is isotopic to* $f(N)$.

For the proof see [5].

Lemma 1. $M = S^1 \times S^1 \times S^1$ *is* P^2*-irreducible.*

Proof. (i) Let $p: R^3 \to M$ be the universal covering and $f: S^2 \to M$ be an embedding. Hence f lifts to an embedding $\tilde{f}: S^2 \to R^3$, so there is a ball $B \subset R^3$ such that $\delta B = \tilde{f}(S^2)$. From $\pi_2(M) = 0$ it follows that $f(S^2)$ separates M into two components. Let γ be a small arc which crosses $f(S^2)$ transversally. Then γ lifts to an arc $\tilde{\gamma}$ which is transversal to $\tilde{f}(S^2)$. One of its end points, say $\tilde{a}$, lies in the interior of B. Let C be the component of M which contains $a = p(\tilde{a})$. Let us show that $\overline{C} = C \cup f(S^2)$ is simply connected. Since $\pi_1(S^2) = 0$, by van Kampen's theorem it follows that $\pi_1(\overline{C}) \to \pi_1(M)$ is a monomorphism. If there would be a $[\mu] \neq 0$ in $\pi_1(\overline{C}, a)$ it could be represented by a loop μ in the interior of C. Let n be a sufficiently large positive integer. Then $n \cdot \mu$ would be lifted, starting from $\tilde{a}$, to an arc in R^3 which joins $\tilde{a}$ to a point outside of B without crossing $\tilde{f}(S^2)$. Then μ cannot exist. It follows that $\overline{C}$ lifts to B and $p \mid B$ is a diffeomorphism onto the ball $\overline{C}$ with boundary $f(S^2)$.

(ii) Suppose there is an embedding $g: RP^2 \to M$. Since $\pi_1(RP^2) = Z_2$, we have $g_{1*}(\pi_1(RP^2)) = 0$ and g would lift to an embedding $\tilde{g}: RP^2 \to R^3$, which is impossible.

(3) **Lemma 2.** *Let* $f: T^2 \to T^3$ *be an embedding, then* $f(T^2)$ *is incompressible in* T^3 *iff it does not bound in* T^3.

Proof. $\pi_1(T^2)$ and $\pi_1(T^3)$ are isomorphic to $H_1(T^2)$ and $H_1(T^3)$. Let ε_1 and ε_2 be the generators of $H_1(T^2)$ which correspond to the factors of $T^2 = S^1 \times S^1$ and let analogously e_1, e_2, e_3 be the generators of $H_1(T^3)$. For both manifolds, H_2 may be identified with $H_1 \wedge H_1$ with basis $(\varepsilon_1 \wedge \varepsilon_2)$ and $(e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3)$ respectively. If f_{1*} is given by $f_{1*}(\varepsilon_i) = a_i e_1 + b_i e_2 + c_i e_3$ then $f_{2*}(\varepsilon_1 \wedge \varepsilon_2) = (a_1 e_1 + b_1 e_2 + c_1 e_3) \wedge (a_2 e_1 + b_2 e_2 + c_2 e_3)$. Hence $f(T^2)$ does not bound iff $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ has rank 2, iff $\ker f_{1*} = 0$, iff $f(T^2)$ is incompressible in T^3.

(4) Now let G be a free abelian group of finite rank n and let H be a subgroup. Then there is a basis $(e_1, e_2, \dots, e_n)$ of G and there are positive integers $k_1, k_2, \dots, k_p$, $p \leq n$, such that $(k_1 e_1, k_2 e_2, \dots, k_p e_p)$ is a basis of H (see [4]).

Lemma 3. *Let* $f: T^2 \to T^3$ *be the embedding given by* $f(x, y) = (x, y, 0)$. *Another embedding* $g: T^2 \to T^3$ *is incompressible iff g is d-equivalent to f.*

Proof. Let (b_1, b_2, b_3) be the generators of $\pi_1(T^3)$, then $f_{1*}(\pi_1(T^2))$ is generated by (b_1, b_2). By assuming that $g(T^2)$ is incompressible it follows that $g_{1*}(\pi_1(T^2)) = Z \oplus Z$. Then there is a new basis (e_1, e_2, e_3) of $\pi_1(T^3)$ and there are numbers k_1, k_2 such that $(k_1 e_1, k_2 e_2)$ is a basis for $g_{1*}(\pi_1(T^2))$. There is also a diffeomorphism $h: T^3 \to T^3$ such that $h_{1*}(b_i) = e_i$, $i = 1, 2, 3$. For the embedding $h \circ f: T^2 \to T^3$ we have $g_{1*}(\pi_1(T^2)) \subset (hf)_{1*}(\pi_1(T^2))$. By the theorem above $g(T^2)$ is isotopic to $(hf)(T^2)$ and therefore it is d-equivalent to $(hf)(T^2)$ and $f(T^2)$. On the other hand if $g(T^2)$ is d-equivalent to $f(T^2)$ it is also incompressible.

(5) In the lemmas 4 and 5 we put $T^3 = M$.

Lemma 4. *Let* $f: T^2 \to M$ *be an embedding and let* $f(T^2)$ *be incompressible in M. If M' is obtained by a surgery away from f, then* $f'(T^2)$ *does not bound in M'.*

Proof. A surgery of type 0 produces a new connection, so possibly a bounding surface might be transformed into a non bounding one in M', but the opposite will never happen. By a surgery of type 2 we obtain $M' = M \cup S^3$ and $f'(T^2)$ lies in M in the same way as $f(T^2)$ does. So we only need to analyse surgeries of type 1. By Lemma 3 we may suppose that f is given by $f(x, y) = (x, y, 0)$. Take the neighborhood $U = S^1 \times S^1 \times (-\varepsilon, \varepsilon)$ of $f(T^2) = S^1 \times S^1 \times 0$, then $f(T^2)$ divides U in $U_- = S^1 \times S^1 \times (-\varepsilon, 0)$ and

$U_+ = S^1 \times S^1 \times (0, \varepsilon)$. Take a path λ in $M - f(T^2)$ which joins a point $a \in U_-$ to a point $a_+ \in U_+$. Any surgery of type 1 away from $f(T^2)$ can be chosen away from λ. This shows that $M' - f'(T^2)$ is connected and $f(T^2)$ is 2-sided in M', so it does not bound in M'.

We shall say that an embedding f bounds if $f(T^2)$ bounds in M.

(6) **Lemma 5.** *Let $f, g : T^2 \to M$ be two c-related embeddings. If f bounds so does g.*

Proof. If f and g are d-equivalent this is obvious. If not, introduce any surgery which produces the c-relation between f and g. Suppose that g does not bound, so by Lemma 4 g' does not bound either. Then the surgery must transform the bounding f into a non bounding f'. This is only possible by using a surgery of type 0. By van Kampen's theorem we see that $\pi_1(M')$ is the free product of $\pi_1(M)$ and Z, $ker\ f'_{1*} = ker\ f_{1*} \neq 0$ and $ker\ g'_{1*} = ker\ g_{1*} = 0$. Then f' and g' cannot be d-equivalent.

Proposition. *Let $f, g : T^2 \to T^3$ be embeddings. If f is bounding and g is non bounding then they are not c-equivalent.*

Proof. This follows immediately from Lemma 5 and from the definition of c-equivalence.

(7) *Proof of Theorem A*: It suffices to show the results for $N = T^2$ and $M = T^3$.

(a) Take $f(x, y) = (x, y, 0)$ and $g = G \mid T^2$ where $G : S^1 \times D^2 \to T^3$ is any embedding of the solid torus. By the proposition of (6) f and g are not c-equivalent, but both have trivial normal bundles so they are B-equivalent.

(b) Now we shall sketch how one can obtain two embeddings which are c-equivalent but not d-equivalent. Let $D \subset T^3$ be an open cell. Consider two disjoint embeddings $F, G : S^1 \to D$. Let γ be a simple arc joining $P \in F(S^1)$ to $Q \in G(S^1)$, such that $\gamma \in F(S^1) = P$ and $\gamma \in G(S^1) = Q$. Let U, V and W be tubular neighborhoods of $F(S^1)$, $G(S^1)$ and γ respectively, which lie in D. Now take an embedding $H : S^1 \to U \cup V \cup W$ which runs first in U following $F(S^1)$, then it goes through W to V where it follows $G(S^1)$ and finally it goes back in W to U. We may choose H such that $H(S^1)$ does not touch $F(S^1) \cup G(S^1) \cup \gamma$.

Let $f, g : T^2 \to T^3$ be embeddings such that $f(T^2)$ and $g(T^2)$ are the boundaries of smaller tubular neighborhoods of $F(S^1)$ and $G(S^1)$ which are disjoint from $H(S^1)$. It is easy to see that if the knots $F(S^1)$ and $G(S^1)$ are not equivalent than f and g cannot be d-equivalent. But choosing conveniently f and g it results that after a surgery of type 1 along $H(S^1)$

f' and g' are isotopic in $(T^3)'$ so f and g are c-equivalent. For more details see [1].

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