Degenerate minimal surfaces in $\mathbb{R}^4$

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§0. Introduction

Recently, Hoffman and Osserman [6] have studied degenerate minimal surfaces in $\mathbb{R}^4$ in great details. In this paper we continue the study from a different point of view. We first analyse the geometry of the complex quadric Q_2 in CP^3 by looking at its intersections with hyperplanes in CP^3, as studied in [6], but we emphasize on their intrinsic aspects and their relations with the euclidean geometry in $\mathbb{R}^4$ since each point in Q_2 represents an oriented plane in $\mathbb{R}^4$. Most important of all, we will show that there are natural ways to assign normal directions to each intersection so that when we study degenerate minimal surfaces in $\mathbb{R}^4$ there are natural normal vector fields to facilitate understanding the second fundamental form. Finally, we relate the generalized Gauss map to the curvature ellipse which is an important tool for the study of surfaces of codimension 2, so that we give an alternative proof of a classical theorem of Eisenhart [5] which gives a characterization of 2-degenerate minimal surfaces.

§1. Preliminary

The quadric in CP^3 is defined to be

$$Q_2 = \{Z \in CP^3 / Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0\} \tag{1.1}$$

Each point of Q_2 can be viewed naturally as an oriented plane in $\mathbb{R}^4$, generated by its real and imaginary parts:

$$X = (x_1, x_2, x_3, x_4), \quad Y = (y_1, y_2, y_3, y_4) \tag{1.2}$$

where $x_j + i\, y_j = z_j$, $1 \le j \le 4$.

To each orientable minimal surface in $\mathbb{R}^4$ given by the immersion

$$x : M^2 \to \mathbb{R}^4, \tag{1.3}$$

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the generalized Gauss map

(1.4) $$[\phi] : M^2 \to Q_2$$

is defined by

(1.5) $$\phi = \frac{\partial x}{\partial \xi} - i \frac{\partial x}{\partial \eta},$$

where $\zeta = \xi + i\eta$ is the conformal structure on M defined by isothermal parameters (ξ, η); x is said to be degenerate if its Gauss image lies in some hyperplane in CP^3, and is said to be h-degenerate if h is the largest integer such that the Gaussian image lies in a projective subspace of codimension h. Therefore there are only three kinds of degeneracy: 1-degenerate, 2-degenerate and 3-degenerate. It's known [6] that 3-degenerate minimal surfaces in $\mathbb{R}^4$ are just planes. We will, therefore, only concentrate on 1-degenerate and 2-degenrate minimal surfaces in $\mathbb{R}^4$.

§2. The geometry of the quadric Q_2

1) It's known [6] that the intersection of Q_2 with a hyperplane H in CP^3 is congruent in Q_2 to the one by assuming H to be determined by

(2.1) $$CZ_3 - Z_4 = 0$$

with

(2.2) $$C = it, \ 0 \le t \le 1$$

For $t = 1$, $S = Q_2 \cap H$ is the union of two projective lines:

(2.3) $$\begin{aligned} L_1 &: Z_1 - iZ_2 = 0, \ iZ_3 - Z_4 = 0 \\ L_2 &: Z_1 + iZ_2 = 0, \ iZ_3 - Z_4 = 0 \end{aligned}$$

with only one common point: $[0, 0, 1, i]$, and S has constant curvature $K = 2$.

For $0 \le t \le 1$, S is isometric to the quadric $\widetilde{Q}_1$:

(2.4) $$\widetilde{Z}_1^2 + \widetilde{Z}_2^2 + k\widetilde{Z}_3^2 = 0$$

in CP^3, where

(2.5) $$k = \frac{1 - t^2}{1 + t^2}$$

Now set

(2.6) $$Z_1 = \widetilde{Z}_1, \ Z_2 = \widetilde{Z}_2, \ Z_3 = \sqrt{k}\ \widetilde{Z}_3$$

which identifies $\widetilde{Q}_1$ with $Q_1 = \{Z \in CP^2 / Z_1^2 + Z_2^2 + Z_3^2 = 0\}$. Through the identification,

(2.7) $$\begin{aligned} \zeta \in C - \{0\} &\leftrightarrow \left[\frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right), \frac{-i}{2}\left(\zeta + \frac{1}{\zeta}\right), 1\right] \\ \zeta = 0 &\leftrightarrow [1, i, 0] \\ \zeta = \infty &\leftrightarrow [1, -i, 0] \end{aligned}$$

betwen $\hat{C}$ and Q_1, we see that the Fubini-Study metric on CP^2

(2.8) $$d\tilde{s}^2 = 2\frac{|\widetilde{Z} \wedge d\widetilde{Z}|^2}{|\widetilde{Z}|^4}$$

induces a metric in C given by

(2.9) $$ds^2 = \frac{4}{k} \frac{|\zeta|^4 + 2k|\zeta|^2 + 1}{\left(|\zeta|^4 + \frac{2}{k}|\zeta|^2 + 1\right)^2} |d\zeta|^2 \equiv \lambda^2 |d\zeta|^2.$$

Using the formula for the Gaussian curvature

(2.10) $$K = -\frac{2\partial\bar{\partial}\log\lambda^2}{\lambda^2}$$

where $\partial = \dfrac{\partial}{\partial\zeta}$, $\bar{\partial} = \dfrac{\partial}{\partial\bar{\zeta}}$, we get

(2.11) $$K = 2 - k^2 \frac{\left(|\zeta|^4 + \frac{2}{k}|\zeta|^2 + 1\right)^3}{(|\zeta|^4 + 2k|\zeta|^2 + 1)^3}.$$

From (2.11) and the fact that

(2.12) $$\begin{aligned} |\widetilde{Z}_1|^2 + |\widetilde{Z}_2|^2 + |\widetilde{Z}_3|^2 &= \frac{1}{2|\zeta|^2}\left(|\zeta|^4 + \frac{2}{k}|\zeta|^2 + 1\right) \\ |\widetilde{Z}_1|^2 + |\widetilde{Z}_2|^2 + k^2|\widetilde{Z}_3|^2 &= \frac{1}{2|\zeta|^2}(|\zeta|^4 + 2k|\zeta|^2 + 1) \end{aligned}$$

for $\zeta \ne 0$, we therefore obtain the Ness' formula [7, p 60] for the Gaussian curvature on $\widetilde{Q}_1$:

(2.13) $$K = 2 - k^2 \frac{(|\widetilde{Z}_1|^2 + |\widetilde{Z}_2|^2 + |\widetilde{Z}_3|^2)^3}{(|\widetilde{Z}_1|^2 + |\widetilde{Z}_2|^2 + k^2|\widetilde{Z}_3|^2)^3}$$

To determine the extremum of K, note that K depends only on $r = |\zeta|^2$. Studing the function

$$f(r) = \frac{r^2 + \frac{2}{k}r + 1}{r^2 + 2kr + 1}, \quad r \geq 0, \tag{2.14}$$

we find that K achieves its maximum at $|\zeta| = 1$ and its minimum at $\zeta = 0$ or $\zeta = \infty$. And we obtain the result of Hoffman-Osserman [6]:

$$\max K = 2 - k^2 \qquad \min K = 2 - \frac{1}{k}. \tag{2.15}$$

Furthermore, from (2.9) we see that $ds^2|_{|\zeta| = r} = ds^2|_{|\zeta| \frac{1}{r}}$ and therefore, $K(\zeta) = K\left(\frac{1}{\zeta}\right)$.

Calculating the area for $D = \{\zeta \in \widehat{C} / |\zeta| \leq 1\}$ we find

$$A(D) = 2\pi. \tag{2.16}$$

Hence S can be viewed, intrinsically, as in Figure 1.

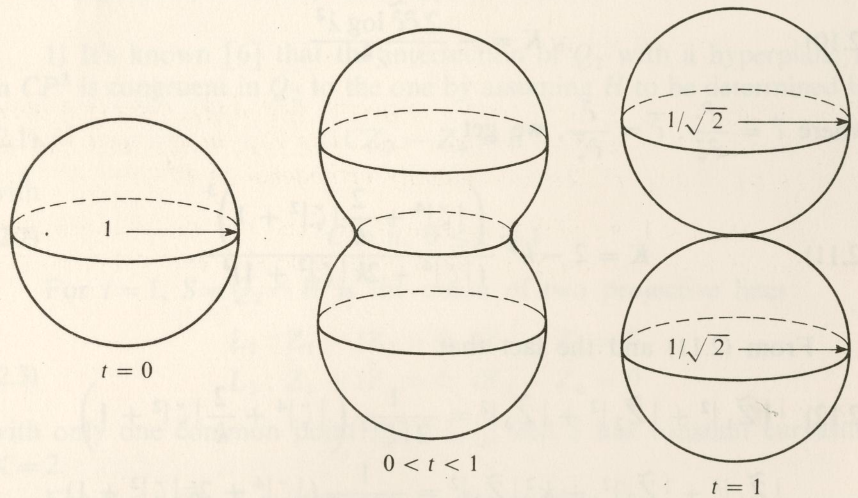

Fig. 1

2) Since each point in Q_2 represents an oriented plane in $\mathbb{R}^4$ and isometries on Q_2 are induced by isometries on $\mathbb{R}^4$, 0(4), we now study S in terms of the 4-dimensional euclidean geometry.

For $t = 1$, S decomposes in two complex projective lines. We take one of them, say,

$$L\colon Z_2 = iZ_1, \; Z_4 = iZ_3, \tag{2.17}$$

write

$$Z_1 = \alpha + i\beta, \; Z_3 = \gamma + i\delta. \tag{2.18}$$

Then the homogeneous vector

$$[Z_1, Z_2, Z_3, Z_4] = X + iY \tag{2.19}$$

satisfies

$$X = (\alpha, -\beta, \gamma, -\delta), \; Y = (\beta, \alpha, \delta, \gamma) \tag{2.20}$$

Set

$$V = (\gamma, \delta, -\alpha, -\beta), \; W = (\delta, -\gamma, -\beta, \alpha) \tag{2.21}$$

natural orthogonal complements to X, Y in $\mathbb{R}^4$. Then

$$V + iW = [Z_3, -iZ_3, -Z_1, iZ_1] \tag{2.22}$$

represents the oriented plane normal to (2.19) and describes a complex projective line in Q_2:

$$\widetilde{L}\colon \widetilde{Z}_2 = -i\widetilde{Z}_1, \; \widetilde{Z}_4 = -i\widetilde{Z}_3 \tag{2.23}$$

We can conclude now that

Proposition 2.1. *For each complex projective line in Q_2, there is a natural correspondence to another complex projective line in Q_2 such that any two corresponding planes are mutually orthogonal in $\mathbb{R}^4$.*

For $0 \leq t < 1$, we will show that there exist two natural orthogonal normal fields defined on S. We start with some algebraic considerations.

Let

$$Z = (Z_1, Z_2, Z_3, Z_4) \in C^4 - \{0\}. \tag{2.24}$$

Write

$$Z = X + iY \tag{2.25}$$

with

$$X = (x_1, x_2, x_3, x_4), \; Y = (y_1, y_2, y_3, y_4) \tag{2.26}$$

where $Z_j = x_j + iy_j$, $1 \leq j \leq 4$.

Define

$$N_1 = \operatorname{Im}(\overline{Z}_2 Z_3, \overline{Z}_3 Z_1, \overline{Z}_1 Z_2, 0) \qquad N_2 = \operatorname{Im}(\overline{Z}_2 Z_4, \overline{Z}_4 Z_1, 0, \overline{Z}_1 Z_2) \tag{2.27}$$

It's trivial to see that

Lemma 2.2. $\operatorname{Im}(\overline{Z}_1 Z_2)=0 \Leftrightarrow \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2\end{pmatrix}=0.$

Let π be the canonical projection from $\mathbb{R}^4$ to the x_1x_2 – plane. Then we have

Lemma 2.3. *Im* $(\overline{Z}_1 Z_2)=0 \Leftrightarrow \pi(X)$ *e* $\pi(Y)$ *are linearly dependent.*

Since

$$\begin{aligned} N_1 &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1, 0), \\ N_2 &= (x_2y_4 - x_4y_2, x_4y_1 - x_1y_4, 0, x_1y_2 - x_2y_1), \end{aligned} \tag{2.28}$$

it can be proved that if $x_1y_2 - x_2y_1 = 0$ then N_1 and N_2 are linearly dependent.. Together with Lemmas 2.2, 2.3 we obtain

Lemma 2.4. N_1 *and* N_2 *are linearly dependent if and only if* $\pi(X)$ *and* $\pi(Y)$ *are linearly dependent.*

Furthermore, by straightforward calculation, we have

Lemma 2.5. $\langle N_j, X\rangle = 0 = \langle N_j, Y\rangle$ *for* $j=1,2$.

Now in case

$$Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0,\ Z_4 = it\,Z_3 \tag{2.29}$$

for $0 \le t < 1$, we have

$$Z_1^2 + Z_2^2 + (\sqrt{1-t^2}\,Z_3)^2 = 0 \tag{2.30}$$

and

$$(x_1, x_2, \sqrt{1-t^2}\,x_3),\ (y_1, y_2, \sqrt{1-t^2}\,y_3) \tag{2.31}$$

are orthogonal and possess the same positive norm.

Since $\operatorname{Im}(\overline{Z}_2\sqrt{1-t^2}\,Z_3, \sqrt{1-t^2}\,\overline{Z}_3 Z_2, \overline{Z}_1 Z_2) \neq 0$, we have

Lemma 2.6. N_1 *will never vanish if* (2.29) *holds.*

And, in this case, with

$$x_4 = -t\,y_3,\ y_4 = t x_3 \tag{2.32}$$

and

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + t^2y_3^2 &= y_1^2 + y_2^2 + y_3^2 + t^2x_3^2 \\ x_1y_1 + x_2y_2 + x_3y_3 - t^2\,x_3y_3 &= 0 \end{aligned} \tag{2.33}$$

we have

Lemma 2.7. $\langle N_1, N_2\rangle = 0$ *if* (2.29) *holds.*

Combining these results and the fact that the directions of N_1 and N_2 are independent of the choice of the homogeneous coordinates, we conclude that

Proposition 2.8. *For* $0 \le t < 1$, *there are two natural unitary normal vector fields,* n_1, n_2, *defined on S, such that*

a) $n_1 \perp n_2$

b) $n_1 /\!/ N_1$ *and*

c) $n_2 /\!/ N_2$ *when* $Im(\overline{Z}_1 Z_2) \neq 0$.

In order to get a more precise and useful description of the normal fields we set

$$\begin{aligned} w &= \frac{Z_3 - i Z_4}{Z_1 - i Z_2} = (1+t)\frac{Z_3}{Z_1 - i Z_2}, \\ \widetilde{w} &= \frac{Z_3 + i Z_4}{Z_1 - i Z_2} = (1-t)\frac{Z_3}{Z_1 - i Z_2}. \end{aligned} \tag{2.34}$$

Then $w = \dfrac{1+t}{1-t}\widetilde{w}$ and it can be easily seen that

$$\begin{aligned} &(Z_1, Z_2, Z_3, Z_4) = \\ &= (Z_1 - iZ_2)\left(\frac{1}{2}\left(1 - \frac{1-t}{1+t}w^2\right), \frac{i}{2}\left(1 + \frac{1-t}{1+t}w^2\right), \frac{1}{1+t}w, \frac{it}{1+t}w\right) \end{aligned} \tag{2.35}$$

and, at $Z_1 - i Z_2 \neq 0$, n_1 and n_2 are parallel to

$$\begin{aligned} \widetilde{n}_1 &= (2\operatorname{Re} w,\ 2\operatorname{Im} w,\ (1-t)\,|w|^2 - (1+t),\ 0) \\ \widetilde{n}_2 &= (-2t\operatorname{Im} w,\ 2t\operatorname{Re} w,\ 0,\ (1-t)\,|w|^2 + (1+t)) \end{aligned} \tag{2.36}$$

respectively, which satisfy obviously $\langle \widetilde{n}_1, \widetilde{n}_2\rangle = 0$ and

$$|\widetilde{n}_1| = |\widetilde{n}_2| = (1-t)^2\,|w|^4 + 2(1+t^2)\,|w|^2 + (1+t)^2. \tag{2.37}$$

Remark. The directions of $\widetilde{n}_1$ and $\widetilde{n}_2$ extend naturally over $Z_1 - i Z_2 = 0$ to the directions of $(0,0,1,0)$ and $(0,0,0,1)$, respectively, since w extends naturally to ∞ at $Z_1 = iZ_2$.

§3. 1-degenerate minimal surfaces in $\mathbb{R}^4$

Let $x: M^2 \to \mathbb{R}^4$ be an 1-degenerate minimal surface. Without loss of generality (see [6]), we may assume its generalized Gauss map satisfies:

$$\phi_4 = it\,\phi_3 \tag{3.1}$$

for some $0 \le t < 1$.

Set

$$g = \frac{\phi_3 - i\phi_4}{\phi_1 - i\phi_2} = (1+t)\frac{\phi_3}{\phi_1 - i\phi_2} \tag{3.2}$$

which is a meromorphic function on M. Comparing (2.35), ϕ can be written as

$$\phi = \frac{\phi_1 - i\phi_2}{2}\left(1 - \frac{1-t}{1+t}g^2, i\left(1 + \frac{1-t}{1+t}g^2\right), \frac{2}{1+t}g, \frac{2it}{1+t}g\right). \tag{3.3}$$

Set

$$f(\zeta)\,d\zeta = (\phi_1 - i\phi_2)\,d\zeta \tag{3.4}$$

which is a global holomorphic differential on M. Then the induced metric

$$ds^2 = \lambda^2 |d\zeta|^2 \tag{3.5}$$

is given by

$$\lambda^2 = \frac{1}{2}|\phi|^2 \equiv \frac{1}{2}\sum_{j=1}^{4}|\phi_j|^2 =$$

$$= \frac{|f|^2}{4}\left\{1 + \frac{2(1+t^2)}{(1+t)^2}|g|^2 + \left(\frac{1-t}{1+t}\right)^2 |g|^4\right\}. \tag{3.6}$$

And the Gaussian curvature K given by the formula (see [4])

$$K = -4\frac{|\phi \wedge \phi'|^2}{|\phi|^6} \tag{3.7}$$

can be computed to be

$$K = -\frac{16|g'|^2}{|f|^2}(1+t)^4 \frac{(1+t)^2 + 2(1-t)^2|g|^2 + \dfrac{(1-t)^2(1+t^2)}{(1+t)^2}|g|^4}{\{(1+t)^2 + 2(1+t^2)|g|^2 + (1-t)^2|g|^4\}^3} \tag{3.8}$$

From (2.36), (2.37) and (3.6), setting $g = u + iv$, we get that

$$\begin{aligned} N_1 &= (-2u, -2v, -(1-t)|g|^2 + (1+t), 0) \\ N_2 &= (-2tv, 2tu, 0, (1-t)|g|^2 + (1+t)) \end{aligned} \tag{3.9}$$

are two normal vector fields defined off the isolated set: $\phi_1 - i\phi_2 = 0$, with

$$N_1 \perp N_2, \tag{3.10}$$

$$|N_1| = |N_2| = (1-t)^2|g|^4 + 2(1+t^2)|g|^2 + (1+t)^2 = \frac{4\lambda^2(1+t)^2}{|f|^2}.$$

Furthermore,

$$n_1 = \frac{N_1}{|N_1|},\ n_2 = \frac{N_2}{|N_2|} \tag{3.11}$$

are two mutually orthogonal unitary normal vector fields which extend over all M.

To study the induced second fundamental form we set

$$e_1 = \frac{1}{\lambda}\frac{\partial x}{\partial \xi},\ e_2 = \frac{1}{\lambda}\frac{\partial x}{\partial \eta} \tag{3.12}$$

which form a local tangent frame for x. Note that

$$\left(\frac{\partial N_1}{\partial \xi}\right)^t = \left\langle \frac{\partial N_1}{\partial \xi}, e_1 \right\rangle e_2 + \left\langle \frac{\partial N_1}{\partial \xi}, e_2 \right\rangle e_2 =$$

$$= \frac{1}{\lambda}\left\{\left(1 + \frac{1-t}{1+t}|g|^2\right)\left(u_\xi \operatorname{Re} f + u_\xi \operatorname{Im} f\right)e_1 + \right.$$

$$\left. + \left(1 + \frac{1-t}{1+t}|g|^2\right)\left(-u_\xi \operatorname{Im} f - v_\xi \operatorname{Re} f\, e_2\right)\right\},$$

$$\left(\frac{\partial N_2}{\partial \xi}\right)^t = \frac{1}{\lambda}\left\{-t\left(1 - \frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Re} f + u_\xi \operatorname{Im} f\right)e_1 + \right.$$

$$\left. + t\left(1 - \frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Im} f - u_\xi \operatorname{Re} f\right)e_2\right\},$$

(3.13)

$$\left(\frac{\partial N_1}{\partial \eta}\right)^t = \frac{1}{\lambda}\left\{-\left(1 + \frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Re} f + u_\xi \operatorname{Im} f\right)e_1 + \right.$$

$$\left. + \left(1 + \frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Im} f - u_\xi \operatorname{Re} f\right)e_2\right\},$$

$$\left(\frac{\partial N_2}{\partial n}\right)^t = \frac{1}{\lambda}\left\{t\left(1 - \frac{1-t}{1+t}|g|^2\right)\left(-u_\xi \operatorname{Re} f + v_\xi \operatorname{Im} f\right)e_1 + \right.$$

$$\left. + t\left(1 - \frac{1-t}{1+t}|g|^2\right)\left(u_\xi \operatorname{Im} f + v_\xi \operatorname{Re} f\right)e_2\right\}.$$

Therefore we obtain

$$\langle A^{n_1}, A^{n_1}\rangle = \langle A^{n_1}e_1, A^{n_1}e_1\rangle + \langle A^{n_1}e_2, A^{n_1}e_2\rangle =$$

$$= \frac{1}{\lambda^2 |N_1|^2}\left\{\left\langle\left(\frac{\partial N_1}{\partial\xi}\right)^t, \left(\frac{\partial N_1}{\partial\xi}\right)^t\right\rangle + \left\langle\left(\frac{\partial N_1}{\partial\eta}\right)^t, \left(\frac{\partial N_1}{\partial\eta}\right)^t\right\rangle\right\} =$$

$$(3.14)\qquad = \frac{|f|^4 |g'|^2}{2\lambda^6(1+t)^2}\left(1 + \frac{1-t}{1+t}|g|^2\right)^2,$$

$$\langle A^{n_2}, A^{n_2}\rangle = \frac{t^2 |f|^4 |g'|^2}{2\lambda^6(1+t)^2}\left(1 - \frac{1-t}{1+t}|g|^2\right)^2,$$

$$\langle A^{n_1}, A^{n_2}\rangle = \langle A^{n_1}e_1, A^{n_2}e_1\rangle + \langle A^{n_1}e_2, A^{n_2}e_2\rangle = 0,$$

$$\langle A, A\rangle = \langle A^{n_1}, A^{n_1}\rangle + \langle A^{n_2}, A^{n_2}\rangle =$$

$$= \frac{|f|^4 |g'|^2}{2\lambda^6(1+t)^2}\left\{\left(1 + \frac{1-t}{1+t}|g|^2\right)^2 + t^2\left(1 - \frac{1-t}{1+t}|g|^2\right)^2\right\}.$$

From (3.14) we see immediately: $\langle A^{n_1}, A^{n_1}\rangle - \langle A^{n_2}, A^{n_2}\rangle \geq 0$ and for any unitary normal vector

$n = n_1 \cos\alpha + n_2 \operatorname{sen}\alpha$, $\langle A^n, A^n\rangle = \cos^2\alpha\,\langle A^{n_1}, A^{n_1}\rangle + \operatorname{sen}^2\alpha\,\langle A^{n_2}, A^{n_2}\rangle$.

Therefore we have

Proposition 3.1. *The two natural unitary normal vector fields n_1, n_2 defined in* (3.11) *satisfy*

$$(3.15)\qquad \langle A^{n_2}, A^{n_2}\rangle \leq \langle A^n, A^n\rangle \leq \langle A^{n_1}, A^{n_1}\rangle$$

for any unitary vector field n normal to x.

And since $\langle A, A\rangle = -\frac{1}{2}K$ we have

Proposition 3.2. *The two orthogonal unitary normal vector fields*

$$(3.16)\qquad V = \frac{\sqrt{2}}{2}(n_1 + n_2),\quad W = \frac{\sqrt{2}}{2}(n_1 - n_2)$$

satisfy

$$(3.17)\qquad \langle A^v, A^v\rangle = \langle A^w, A^w\rangle = -K.$$

Now, if we look at the problem of stability intrinsically, following the arguments of Barbosa do Carmo [1,2], we get:

Theorem 3.3. *Let D be a simply connected domain of an 1-degenerate minimal surface in $\mathbb{R}^4$. If the area of the generalized Gauss map on $\overline{D}$ is less than $\frac{4\pi}{3-k^2}$, where k is given by* (2.5), *then D is stable.*

Proof. Since the generalized Gauss map is a branching covering on its image and it's known [4] that $d\hat{s}^2 = -K\,ds^2$ is the induced Fubini – Study metric on the image, the Gaussian curvature

$$\hat{K} \leq 2 - k^2,$$

by (2.15). From (3.4) and (3.10) in [2], we compute that if the Gauss image has area less than $\frac{4\pi}{3-k^2}$, then the first eigenvalue λ_1 with respect to $d\hat{s}^2$ on D is greater than 2. The rest of the proof then follows the same argument in [1].

Remarks.

1 – For $t = 0$, we have $k = 1$. This gives the same result in [1], which is sharp. It would be interesting to know whether our result is sharp for arbitrary general t. If it were true, then the result obtained by Barbosa – do Carmo [2] for minimal surfaces in $\mathbb{R}^4$ would therefore be sharp also.

2 – From our discussions in this section, we see clearly that 1-degenerate minimal surfaces in $\mathbb{R}^4$ possess many properties similar to those in $\mathbb{R}^3$, as observed first by Hoffman-Osserman [6].

§4. 2-degenerate minimal surfaces in $\mathbb{R}^4$

It's known [6] that any 2-degenerate minimal surface in $\mathbb{R}^4$ is a regular complex analytic curve lying in $C^2 = \mathbb{R}^4$, with respect to some orthogonal complex structure on $\mathbb{R}^4$. Now let

$$(4.1)\qquad \psi = (f, g) : M^2 \to C^2 = \mathbb{R}^4$$

be a regular holomorphic curve, where M is a Riemann surface and $C^2 = \mathbb{R}^2 \oplus i\,\mathbb{R}^2$ is the canonical identification to $\mathbb{R}^4$. Then the real coordinates of ψ are given by

$$(4.2)\qquad x = \operatorname{Re}(f, -if, g, -ig).$$

And with respect to a local complex parameter $\zeta = u + iv$, the generalized Gauss map is given by

$$\phi(\zeta) = (f'(\zeta), -if'(\zeta), g'(\zeta), -ig'(\zeta)). \tag{4.3}$$

Writing

$$f'(\zeta) = \alpha + i\beta,\ g'(\zeta) = \gamma + i\delta \tag{4.4}$$

in real and imaginary parts, then $\phi = X + iY$ with

$$X = (\alpha, \beta, \gamma, \delta),\ Y = (\beta, -\alpha, \delta, -\gamma), \tag{4.5}$$

which satisfy

$$\langle X, Y \rangle = 0,\ |X|^2 = |Y|^2 = |f'|^2 + |g'|^2 = \lambda^2. \tag{4.6}$$

It's easy to see that

$$N_1 = (-\delta, -\gamma, \beta, \alpha),\ N_2 = (\gamma, -\delta, -\alpha, \beta) \tag{4.7}$$

are normal to x and are mutually orthogonal. Then

$$e_1 = \frac{1}{\lambda} X,\ e_2 = \frac{1}{\lambda} Y, \qquad n_1 = \frac{1}{\lambda} N_1,\ n_2 = \frac{1}{\lambda} N_2, \tag{4.8}$$

form a local adapted frame for x. To calculate the second fundamental form, using the Cauchy-Riemann equations

$$\begin{aligned} \alpha_u &= B_v, & \alpha_v &= -\beta_u \\ \gamma_u &= \delta_v & \gamma_v &= -\delta_u \end{aligned} \tag{4.9}$$

we get

(4.10)

$$\begin{aligned}
A^{n_1}e_1 &= -\frac{1}{\lambda^3}\Big(-\alpha\delta_u - \beta\gamma_u + \gamma\beta_u + \delta\alpha_u\Big)e_1 - \frac{1}{\lambda^3}\Big(\beta\delta_u - \alpha\gamma_u - \delta\beta_u + \alpha_u\Big)e_2, \\
A^{n_1}e_2 &= -\frac{1}{\lambda^3}\Big(-\alpha\gamma_u + \beta\delta_u + \gamma\alpha_u - \delta\beta_u\Big)e_1 - \frac{1}{\lambda^3}\Big(\beta\gamma_u + \alpha\delta_u - \delta\alpha_u - \gamma\beta_u\Big)e_2, \\
A^{n_2}e_1 &= -\frac{1}{\lambda^3}\Big(\alpha\gamma_u - \beta\delta_u - \gamma\alpha_u + \delta\beta_u\Big)e_1 - \frac{1}{\lambda^3}\Big(-\beta\gamma_u - \alpha\delta_u + \delta\alpha_u + \gamma\beta_u\Big)e_2, \\
A^{n_2}e_2 &= -\frac{1}{\lambda^3}\Big(-\alpha\delta_u - \beta\gamma_u + \gamma\beta_u + \alpha_u\Big)e_1 - \frac{1}{\lambda^3}\Big(\beta\delta_u - \alpha\gamma_u - \delta\beta_u + \gamma\alpha_u\Big)e_2,
\end{aligned}$$

and

$$\begin{aligned}
\langle A^{n_1}, A^{n_1} \rangle &= \langle A^{n_2}, A^{n_2} \rangle = \\
&= \frac{2}{\lambda^6}\Big\{|f'|^2|g''|^2 + |g'|^2|f''|^2 - \bar{f}'f''g'\bar{g}'' - f'\bar{f}''\bar{g}'g''\Big\} = \\
&= \frac{2}{\lambda^6}|(f', g') \wedge (f'', g'')|^2, \\
\langle A^{n_1}, A^{n_2} \rangle &= 0.
\end{aligned} \tag{4.11}$$

Therefore, we have the Gaussian curvature

$$K = -\langle A^n, A^n \rangle, \tag{4.12}$$

for any unitary normal vector n. Summing up, we have

Theorem 4.1. *Let $x : M^2 \to \mathbb{R}^4$ be a 2-degenerate minimal surface in $\mathbb{R}^4$. Then, with respect to any orthonormal normal vectors, v, w, the second fundamental form satisfies*

$$\begin{aligned} \langle A^v, A^v \rangle &= \langle A^w, A^w \rangle = -K, \\ \langle A^v, A^w \rangle &= 0, \end{aligned} \tag{4.13}$$

where K is the Gaussian curvature of the surface.

Remarks.

1 – From (4.7) we see that $\psi = (f, g)$ is orthogonal and isometric to $\tilde{\psi} = (\bar{g}, -\bar{f})$ as 2-degenerate minimal surfaces in $\mathbb{R}^4$. We thus conclude that $|K| = |K^\perp|$, where $K^\perp$ is the normal curvature of x. In fact, this is a characteristic property for holomorphic curves in C^2.

2 – If M is simply connected, then, using a fixed uniform parameter, we can construct global unitary normal vector fields.

3 – The property (4.13) is, in fact, also characteristic. We will explain it in the following discussions.

Finally, we investigate the relation between the generalized Gauss map and the curvature ellipse. Given a minimal immersion $x : M^2 \to \mathbb{R}^4$, for each $p \in M$, the curvature ellipse is defined to be

$$E(p) = \{B(X, X) \mid X \in T_pM,\ |X| = 1\} \tag{4.14}$$

where B is the second fundamental form. If we choose $\{e_1, e_2\}$ an orthonormal base for TpM, then for $X = \cos\theta\, e_1 + \sin\theta\, e_2$, $B(X, X) = \cos 2\theta\, B(e_1, e_1) + \sin 2\theta\, B(e_1, e_2)$. Therefore we get easily:

Lemma 4.2. *$E(p)$ is a circle if and only if $B(X, X) \perp B(X, Y)$ and $|B(X, X)| = |B(X, Y)|$ for any orthogonal base $\{X, Y\}$ for TpM (i.e. $X \perp Y$, and, $|X| = |Y| > 0$).*

For isothermal parameters $\zeta = \xi + i\eta$, $B\left(\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \xi}\right) = \frac{\partial^2 x^\perp}{\partial \xi^2}$ and $B\left(\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta}\right) = \frac{\partial^2 x^\perp}{\partial \xi \partial \eta}$, and

$$\frac{\partial^2 x}{\partial \xi^2} = \frac{1}{\lambda^2}\left\langle \frac{\partial^2 x}{\partial \xi^2}, \frac{\partial x}{\partial \xi}\right\rangle \frac{\partial x}{\partial \xi} + \frac{1}{\lambda^2}\left\langle \frac{\partial^2 x}{\partial \xi^2}, \frac{\partial x}{\partial \eta}\right\rangle \frac{\partial x}{\partial \eta} + B\left(\frac{\partial x}{\partial \eta}, \frac{\partial x}{\partial \xi}\right)$$

(4.15)

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{1}{\lambda^2}\left\langle \frac{\partial x^2}{\partial \xi \partial \eta}, \frac{\partial x}{\partial \xi}\right\rangle \frac{\partial x}{\partial \xi} + \frac{1}{\lambda^2}\left\langle \frac{\partial^2 x}{\partial \xi \partial \eta}, \frac{\partial x}{\partial \eta}\right\rangle \frac{\partial x}{\partial \eta} + B\left(\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta}\right)$$

where $\lambda = \left|\frac{\partial x}{\partial \xi}\right| = \left|\frac{\partial x}{\partial y}\right|$. Furthermore, from

$$\frac{\partial x}{\partial \xi} \perp \frac{\partial x}{\partial \eta} \text{ and } \left|\frac{\partial x}{\partial \xi}\right| = \left|\frac{\partial x}{\partial \eta}\right|$$

we get

$$\left\langle \frac{\partial^2 x}{\partial \xi^2}, \frac{\partial x}{\partial \xi}\right\rangle = \left\langle \frac{\partial^2 x}{\partial \xi \partial \eta}, \frac{\partial x}{\partial \eta}\right\rangle \text{ and}$$

(4.16)

$$\left\langle \frac{\partial^2 x}{\partial \xi^2}, \frac{\partial x}{\partial \eta}\right\rangle = -\left\langle \frac{\partial^2 x}{\partial \xi \partial \eta}, \frac{\partial x}{\partial \xi}\right\rangle.$$

Together with Lemma 4.3, we have

Proposition 4.3. *$E(p)$ is a circle if and only if the global holomorphic form $(\phi', \phi')d\zeta^4$ or $\sum_{k=1}^{4} \phi_k'(\zeta)^2 d\zeta^4$ vanishes at p.*

Proof. It's sufficient to note that

$$B\left(\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \xi}\right) \perp B\left(\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta}\right) \text{ and } \left|B\left(\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \xi}\right)\right| = \left|B\left(\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta}\right)\right|$$

$$\text{iff } \frac{\partial^2 x}{\partial \xi^2} \perp \frac{\partial^2 x}{\partial \xi \partial \eta} \text{ and } \left|\frac{\partial^2 x}{\partial \xi^2}\right| = \left|\frac{\partial^2 x}{\partial \xi \partial \eta}\right|$$

iff $(\phi', \phi') = 0$.

Corollary. *The curvature ellipses associated to x are circles everywhere or only at isolated points.*

Combining Prop. 4.3 and the recent results of Hoffman-Osserman [6], we now give an alternative proof of the following.

Theorem 4.4 (Eisenhart [5]). *Let $x: M^2 \to \mathbb{R}^4$ be a minimal surface. Then $x(M)$ is 2-degenerate if and only if all the curvature ellipses are circles.*

Proof. Consider the formula (see [6]):

$$\phi = \frac{f}{2}(1 + g_1 g_2,\ i(1 - g_1 g_2),\ g_1 - g_2,\ -i(g_1 + g_2)).$$

Without loss of generality, we may assume that f never vanishes. Thus we see easily:

$$(\phi', \phi') \equiv 0 \Leftrightarrow g_1' \equiv 0 \text{ or } g_2' \equiv 0$$

This means $g_1 =$ constant or $g_2 =$ constante, i.e. $x(M)$ is 2-degenerate.

On the other hand, let n_1 be a unit normal vector which represents the semi-major axis of the curvature ellipse, and n_2 be the unit orthogonal complement to n_1 in the normal plane. Using an orthonormal base $\{e_1, e_2\}$ for the tangent plane such that

$$n_1 \,/\!/\, B(e_1, e_1) \text{ and } n_2 \,/\!/\, B(e_1, e_2),$$

then, since $B(e_1, e_1) \perp B(e_1, e_2)$ and $|B(e_1, e_1)| \geq |B(e_1, e_2)|$, by straightforward calculation, we obtain

$$\langle A^{n_1}, A^{n_1}\rangle = 2|B(e_1, e_1)|^2,\ \langle A^{n_2}, A^{n_2}\rangle = 2|B(e_1, e_2)|^2$$

and $\langle A^{n_1}, A^{n_2}\rangle = 0$.

Thus we have

Proposition 4.5. *The major directions of the curvature ellipse diagonalize the $\widetilde{A}$ operator defined by $\langle \widetilde{A}(v), w\rangle = \langle A^v, A^w\rangle$.*

Together with Thm. 4.4, we have

Corollary. *For a minimal surface $x: M^2 \to \mathbb{R}^4$, $x(M)$ is 2-degenerate if and only if at each $p \in M$ there exist an orthonormal base $\{v, w\}$ for the normal plane such that (4.13) holds.*

Remark. This investigation of the relation between the generalized Gauss map and the curvature ellipse leads to a generalization of the curvature ellipse for surfaces with higher codimensions. Definitions and some applications are given in Chen [3].

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