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Green function behaviour of critical Galton-Watson processes with immigration

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1. Introduction

Let $\{X_n\}$ be a Galton-Watson branching process allowing immigration (GWI) and $\{p_0, p_1, \ldots\}$ and $\{q_0, q_1, \ldots\}$ its offspring and immigration distributions respectively. In this paper we are concerned only with the critical, aperiodic and irreducible case, i.e. we assume throughout that

$$\sum_{k=1}^{\infty} k p_k = 1,$$

$$\text{g.c.d. } \{k \in \mathbb{N} \mid p_k > 0\} = 1$$

and

every state of the state space $\mathbb{N}_0$ of $\{X_n\}$ can be reached from every other state,

where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The two latter conditions are fulfilled, if, e.g., $0 < p_1 < 1$ and $0 < q_0 < 1$ (see [8]). Furthermore we assume that

$$\sum_{k=2}^{\infty} k^2 p_k \log k < \infty,$$

$$\sum_{k=2}^{\infty} k q_k \log k < \infty. \tag{1}$$

Finally, let

$$\alpha = \frac{1}{2} \sum_{k=2}^{\infty} k(k-1) p_k, \quad \gamma = \frac{1}{\alpha} \sum_{k=1}^{\infty} k q_k, \quad \beta = [\alpha(\gamma - 1)]^{-1},$$

$P_n(i,j)$ be the n-step transition probability from i to j (written $P(i,j)$ if $n = 1$). and for $i \neq j$ let (see e.g. [1], p. 31)

$${}_iP_n(i,j) = P(X_n = j,\ X_k \neq i,\ k = 1, \ldots, n-1 \mid X_0 = i), \quad n \in \mathbb{N}.$$

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In Section 2 we shall study the asymptotic behaviour of the Green function of the GWI

$$G(i,j) = \sum_{n=0}^{\infty} P_n(i,j),$$

which exists if $\gamma > 1$ (in this case the GWI is transient [6]), giving the expected number of visits of $\{X_n\}$ to j, starting in state i.

If $\gamma \leq 1$ the GWI is null-recurrent and it makes sense to investigate

$${}_iG(i,j) = \sum_{n=1}^{\infty} {}_iP_n(i,j), \; i \neq j, \; i,j \in \mathbb{N}_0,$$

which is the expected number of visits to state j between successive visits to state i. In Section 3 we will obtain some information on this quantity.

To get a fairly complete description of the asymptotic behaviour of the Green function of the GWI let us recall that Pakes [7] has demonstrated the following results (relation (2) is stated incorrectly in the cited paper):

$$\text{(2)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} G(i,j) = \beta, \; i \in \mathbb{N}_0$$

and

$$\lim_{i\to\infty} i^{\gamma-1} G(i,j) = \alpha^{\gamma-1}\Gamma(\gamma-1)\theta_j, \; j \in \mathbb{N}_0,$$

where $\gamma > 1$ and $\{\theta_j \mid j \in \mathbb{N}_0\}$ denotes the invariant measure of $\{X_n\}$, satisfying

$$0 \leq \theta_j = \sum_{k=0}^{\infty} \theta_k P(k,j), \; j \in \mathbb{N}_0,$$

and being uniquely determined by

$$\text{(3)} \qquad \theta_j = \lim_{n\to\infty} n^{\gamma} P_n(i,j), \; i,j \in \mathbb{N}_0.$$

Finally, in Section 4 we shall be concerned with the corresponding functionals of the time-reversed GWI.

2. Asymptotic properties of the Green function

In this section we will prove

Theorem 1. *Let* $\gamma > 1$, $0 \leq \lambda < \infty$. *Then*

$$\lim_{\substack{i\to\infty,\, j\to\infty \\ i \neq j,\, j/i \to \lambda}} G(i,j) = \begin{cases} \beta & \text{if } \lambda \geq 1 \\ \beta\lambda^{\gamma-1} & \text{if } 0 \leq \lambda \leq 1 \end{cases}$$

and

$$\lim_{j\to\infty} G(j,j) = 1 + \beta.$$

The proof is based on the following local limit theorem, given in [4].

$$\text{(4)} \qquad P_n(i,j) \sim \frac{1}{\alpha n} \{j/i\}^{(\gamma-1)/2} \exp\left\{-\frac{i+j}{\alpha n}\right\} I_{\gamma-1}\left(\frac{2}{\alpha n}\sqrt{ij}\right)$$

$$(j, n \to \infty; \; \sup i/n, \; \sup j/n < \infty),$$

where I_ρ is the modified Bessel function of order ρ, and the following estimate established in [3]

$$\text{(5)} \qquad \sup_{j\geq 0} P_n(i,j) \leq C(in)^{-\xi/2}, \; i, n \in \mathbb{N},$$

where $\xi = \min(1, \gamma)$ and C is a finite constant.

Proof of Theorem 1. Let $\varepsilon > 0$ and $1 \leq i \neq j$. We break the sum defining G into the form

$$G(i,j) = S(i,j,\varepsilon) + T(i,j,\varepsilon),$$

where

$$S(i,j,\varepsilon) = \sum_{n=1}^{[\varepsilon i]} P_n(i,j) \text{ and } T(i,j,\varepsilon) = \sum_{n=[\varepsilon i]+1}^{\infty} P_n(i,j)$$

and estimate, using (5),

$$\text{(6)} \qquad S(i,j,\varepsilon) \leq C \sum_{n=1}^{[\varepsilon i]} (in)^{-1/2} \leq C'\sqrt{\varepsilon}, \; i,j \in \mathbb{N}.$$

Since $j, n \to \infty$ and $j/i \to \lambda$, the following conditions are satisfied by the indices in the sum $T(i,j,\varepsilon)$:

$$i/n < \varepsilon^{-1} \quad \text{and} \quad j/n < j/i\varepsilon < (1+\lambda)/\varepsilon$$

if i and j are sufficiently large. Equation (4) then yields, as $i, j \to \infty$, $j/i \to \lambda$,

$$T(i,j,\varepsilon) = (1+o(1)) \frac{1}{\alpha} \left\{\frac{j}{i}\right\}^{(\gamma-1)/2} \sum_{n>\varepsilon i} \frac{i}{\alpha n} \exp\left\{-\frac{i}{\alpha n}\left(1+\frac{j}{i}\right)\right\} I_{\gamma-1}\left(\frac{2i}{\alpha n}\sqrt{j/i}\right)\frac{\alpha}{i}$$

$$\to \frac{1}{\alpha}\lambda^{(\gamma-1)/2} \int_{\alpha\varepsilon}^{\infty} x^{-1} \exp\{-(\lambda+1)/x\} I_{\gamma-1}\left(\frac{2}{x}\sqrt{\lambda}\right) dx.$$

Now recall (6) to deduce, as $\varepsilon \downarrow 0$,

$$\lim_{\substack{i\to\infty,\, j\to\infty \\ i\neq j,\, j/i\to\lambda}} G(i,j) = \frac{1}{\alpha}\,\lambda^{(\gamma-1)/2} \int_0^\infty x^{-1} \exp\{-x(1+\lambda)\} I_{\gamma-1}\,(2x\sqrt{\lambda})\,dx$$

$$= \frac{\lambda^{(\gamma-1)/2}}{\alpha(\gamma-1)} \left\{\frac{1+\lambda-\{(1+\lambda)^2-4\lambda\}^{1/2}}{2\sqrt{\lambda}}\right\}^{\gamma-1}.$$

But the latter expression (found e.g. in [5]) coincides with the limit in the first part of the theorem. The second part being evident, the theorem is proved.

The case $j\to\infty$, $i/j\to 0$, excluded in Theorem 1, will be treated in Theorem 2. It requires some preliminaries.

Lemma 1. *Let $\varepsilon > 0$. There exists $j_0 = j_0(\varepsilon)$ such that, for all integer $j \geq j_0$.*

$$P(r,k) \leq \varepsilon/j, \text{ for all } 0 \leq r \leq j/3 \text{ and } k \geq j.$$

Proof. Let $Q(n,s)$ be the coefficient of x^s in $\left(\sum_{m=0}^{\infty} p_m x^m\right)^n$ (i.e. the one--step transition probability of the imbedded Galton-Watson process), and j, k and r be as above. In view of the facts

$$Q(s,i) \leq M(i-s)^{-2}\sqrt{s}, \quad i > s,$$

([2], p. 538), and

$$\lim_{s\to\infty} sP(0,s) = 0,$$

which is obvious from (1), one sees, in combination with

$$P(n,s) = \sum_{m=0}^{s} P(0,m)\,Q(n,s-m), \quad n,s \in \mathbb{N}_0,$$

that

$$P(r,k) = \sum_{s=0}^{[j/3]} P(0,s)\,Q(r,k-s) + \sum_{s=[j/3]+1}^{k} P(0,s)\,Q(r,k-s)$$

$$\leq \sup_{(k-j/3)\leq i\leq k} Q(r,i) + \sup_{j/3\leq s\leq k} P(0,s)$$

$$\leq M' j^{-3/2} + \sup_{j/3\leq s\leq k} P(0,s) \leq \varepsilon/j,$$

for all $j \geq j_0$, $j_0 = j_0(\varepsilon)$ appropriately chosen.

Exactly as Kesten, Ney & Spitzer [2] we now define the stopping times

$$T(j) = \begin{cases} \min\{k \mid k\in\mathbb{N},\ X_k \geq j\} \\ \infty, \qquad \text{if no such } k \text{ exists} \end{cases}, \quad j\in\mathbb{N}.$$

Since there are no absorbing states it follows that $\overline{\lim}_{n\to\infty} X_n = \infty$ a.s. and hence that $T(j)$ is a.s. finite.

Lemma 2. Let $\varepsilon > 0$, $\gamma > 1$. There exists $j_0 = j_0(\varepsilon)$ such that, for all $j \geq j_0$,

$$P(X_{T(j/3)} = k \mid X_0 = i) \leq \varepsilon, \text{ for all } i \leq j/3,\ k \geq j.$$

Proof. Choose K such that for all $i\in\mathbb{N}_0$, $\sum_{j=1}^{n} G(i,j) \leq Kn$. That this is possible follows from Theorem 1 and the observation that $G(i,j) \leq G(j,j)$, $i,j\in\mathbb{N}_0$. Now let $j_0 = j_0(\varepsilon/K)$ as determined in Lemma 1 and i,j,k be as in the proposition. An obvious modification of a result in [2], p. 538, gives

$$P(X_{T(j/3)} = k \mid X_0 = i) \leq \sum_{r=0}^{[j/3]} G(i,r)P(r,k).$$

Recalling the choice of K and j_0 and applying Lemma 1 leads to the desired bound.

Obviously, the following result is a refinement and generalization of (2).

Theorem 2. *Let $\gamma > 1$. Then*

$$\lim_{j\to\infty,\, i/j\to 0} G(i,j) = \beta.$$

Proof. For $\varepsilon > 0$ and $i \leq j/3$ we decompose G into

$$G(i,j) = S(i,j,\varepsilon) + T(i,j,\varepsilon)$$

where

$$S(i,j,\varepsilon) = \sum_{n=1}^{[\varepsilon j]} P_n(i,j) \text{ and } T(i,j,\varepsilon) = \sum_{n=[\varepsilon j]+1}^{\infty} P_n(i,j)$$

and first turn to an estimation of $S(i,j,\varepsilon)$, once more (to obtain the first of the following inequalities) applying a result of Kesten, Ney & Spitzer [2], p. 539. Setting $U(i,j,r) = P(X_{T(j/3)} = r \mid X_0 = i)$ and using the Kronecker $\delta(\ldots)$,

$$\begin{aligned} S(i,j,\varepsilon) &\le \sum_{r\ge j/3} U(i,j,r)\,[\delta(r,j)+S(r,j,\varepsilon)] \\ &= U(i,j,j) + \sum_{r\ge j/3} U(i,j,r)\,S(r,j,\varepsilon) \\ &\le U(i,j,j) + \sup_{r\ge j/3} S(r,j,\varepsilon) \\ &\le U(i,j,j) + C''\sqrt{\varepsilon}, \end{aligned}$$

the last inequality following from

$$S(r,j,\varepsilon) \le \sum_{n=0}^{[\varepsilon j]} \sup_{s\ge 0} P_n(r,s) \le C \sum_{n=1}^{[\varepsilon j]} (rn)^{-1/2} \le C'\sqrt{\varepsilon j/r}, \quad r,j\in\mathbb{N},$$

where we have used (5). Lemma 2 now yields

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{\substack{j\to\infty \\ i/j\to 0}}} S(i,j,\varepsilon) = 0.$$

Finally, the behaviour of the summands of $T(i,j,\varepsilon)$ is dealt with as in (4), hence, as $j\to\infty$, $i/j\to 0$,

$$\begin{aligned} T(i,j,\varepsilon) &= \frac{1+o(1)}{\alpha\Gamma(\gamma)} \sum_{n>\varepsilon j} \frac{\alpha}{j}\left(\frac{j}{\alpha n}\right)^{\gamma} \exp\left\{-\frac{j}{\alpha n}\right\} \\ &\to [\alpha\Gamma(\gamma)]^{-1}\int_{\varepsilon\alpha}^{\infty} x^{-\gamma}\exp\{-1/x\}dx, \text{ as } j\to\infty,\ i/j\to 0 \\ &\to [\alpha\Gamma(\gamma)]^{-1}\int_{0}^{\infty} x^{\gamma-2}e^{-x}\,dx, \text{ as } \varepsilon\downarrow 0 \\ &= [\alpha\Gamma(\gamma)]^{-1}\Gamma(\gamma-1) = \beta. \end{aligned}$$

Corollary. *Let* $\gamma > 1$. *Then*

$$\lim_{j\to\infty} P(\text{There is a } n\in\mathbb{N} \text{ with } X_n = j \mid X_0 = i) = \beta/(1+\beta),\ i\in\mathbb{N}_0.$$

Proof. Observe that $G(i,j) = P(\text{There is a } n\in\mathbb{N} \text{ with } X_n = j \mid X_0 = i)G(j,j)$ and apply Theorem 2.

3. Expected sojourn times under a taboo.

As mentioned in Section 1, if $\gamma\le 1$, the state space $\mathbb{N}_0$ of the GWI $\{X_n\}$ forms a null-recurrent class, i.e. recurrence to any given state occurs with certainty (hence each state is visited infinitely often with probability one), yet the relative frequency of visits to any given state tends to zero as time approaches infinity. Thus an examination of the quantity ${}_iG(i,j)$ introduced in Section 1 is meaningful.

Theorem 3. *Let* $\gamma\le 1$. *Then*

(7) $$\lim_{j\to\infty} {}_iG(i,j)\,j^{1-\gamma} = [\theta_i\alpha^{\gamma}\Gamma(\gamma)]^{-1}, \quad i\in\mathbb{N}_0,$$

(8) $$\lim_{i\to\infty} {}_iG(i,j)\,i^{\gamma-1} = \theta_j\alpha^{\gamma}\Gamma(\gamma), \quad j\in\mathbb{N}_0$$

(9) $$\lim_{\substack{i,j\to\infty,\ i\neq j \\ i/j\to\lambda>0}} {}_iG(i,j) = \lambda^{1-\gamma}.$$

Proof. Due to irreducibility and (null-) recurrence standard results for Markov chains (see e.g. Karlin & Taylor [1], pp. 35-42) yield ${}_iG(i,j) = \theta_j/\theta_i$ and the theorem follows on observing that

(10) $$\lim_{j\to\infty} j^{1-\gamma}\theta_j = [\alpha^{\gamma}\Gamma(\gamma)]^{-1}$$

(Mellein [4]).

Remark. The GWI $\{X_n\}$ is null-recurrent whenever $\gamma\le 1$, however the cases $\gamma<1$ and $\gamma=1$ exhibit quite different features:

a) Suppose that the process is started in the origin. Relation (7), with $i=0$, then indicates that the expected number of 'visits to state $j\neq 0$ before a first return to the origin' becomes small if $\gamma<1$ and j tends to infinity, it approaches a positive limit if $\gamma=1$.

b) Suppose that the process is started a long way from the origin. Relation (8), with $j=0$ and large i, then states that on the average visits to the origin occuring before a first return to the starting point are "considerably more frequent" when $\gamma<1$ than in the case $\gamma=1$ (the mean number of such visits to the origin is unbounded (as $i\to\infty$) in the former case and bounded in the latter case).

c) Relation (9) indicates a sort of "drift-free oscillatory behaviour" of the process in the case $\gamma=1$ which contrasts with the "attractive nature of the origin region" if $\gamma<1$; started on a high level the process will more often visit states on the origin side of the starting state than those of the opposite side, when $\gamma<1$, while in the case $\gamma=1$ the process is as likely to start wandering out to $+\infty$ as it is to head toward the origin.

4. The time-reversed GWI

Pakes [8] introduced the time-reversed GWI $\{V_n\}$, i.e. the Markov chain on $\mathbb{N}_0$ whose joint probability $P(V_{n_1} = i_1, \ldots, V_{n_k} = i_k \mid V_0 = i_0)$ is defined as the (existing and nondegenerate) limit

$$\lim_{n\to\infty} P(X_{n-n_1} = i_1, \ldots, X_{n-n_k} = i_k \mid X_n = i_0),$$

where $0 < n_1 < \ldots < n_k$ and $i_0, i_1, \ldots, i_k \in \mathbb{N}_0$. It is easily seen that the n-step transition probabilities of $\{V_n\}$ are given by

$$R_n(i,j) = \theta_j P_n(j,i)/\theta_i, \quad i,j \in \mathbb{N}_0,$$

where $\{\theta_j\}$ is the invariant measure (3) of $\{X_n\}$. Clearly, $\{V_n\}$ is nullrecurrent if $\gamma \leq 1$ (as $\{X_n\}$ is) and transient if $\gamma > 1$ and has $\{\theta_i\}$ as its (unique) invariant measure. Furthermore it is known [8] that $\{X_n/n\}$ and $\{V_n/n\}$ have the same limiting distribution.

We observe that the Green function behaviour of $\{V_n\}$ also coincides with that of $\{X_n\}$: when $\gamma > 1$, let $H(i,j) = \sum_{n=0}^{\infty} R_n(i,j)$ and, when $\gamma \leq 1$ define ${}_iH(i,j)$ analogously to ${}_iG(i,j)$. Clearly

$$H(i,j) = \frac{\theta_j}{\theta_i} G(j,i)$$

and hence if i, j behave as in Theorem 1 or as in Theorem 2, then it follows from Theorems 1 and 2 and (10) that $H(i,j) \sim G(i,j)$. Since $\{V_n\}$ and $\{X_n\}$ possess the same invariant measure, ${}_iG(i,j) = {}_iH(i,j)$.

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