Submanifolds with constant mean curvature

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0. General data.

Let M^n be an n-dimensional $(n \geq 2)$, riemannian C^∞ manifold isometrically C^∞ immersed into an $(n+p)$-dimensional riemannian C^∞ manifold $\widetilde{M}^{n+p}$ of constant sectional curvature c. Let $\| B \|$ denote the norm of the second fundamental form B and H the mean curvature normal (vector field) of this immersion.

1. Introduction.

In [14] Simons proved the following inequality in case $\widetilde{M}^{n+p} = S^{n+p}$ (= standard sphere) and M^n oriented, minimal and compact:

$$\int_{M^n} \left\{ \left(2 - \frac{1}{p}\right) \| B \|^4 - n \| B \|^2 \right\} dv \geq 0 \tag{1.1}$$

where dv is the volume element of M^n. If follows that if M^n is not totally geodesic and $\| B \|^2 = \left(2 - \frac{1}{p}\right)^{-1} n$, then $\| B \|^2 = \left(2 - \frac{1}{p}\right)^{-1} n$. Using (1.1) Chern, do Carmo and Kobayashi determined in [5] all compact minimal submanifolds of S^{n+p} satisfying

$$\| B \|^2 = \left(2 - \frac{1}{p}\right)^{-1} n; \tag{*}$$

the condition (*) was subsequently generalized by Braidi and Hsiung (see [2]).

A submanifold M^n of a riemannian manifold $\widetilde{M}^{n+p}$ is said to have *constant mean curvature* if H is parallel as a section of the normal bundle.

The purpose of the present paper is to determine all isometric immersions of M^n into $\widetilde{M}^{n+p}$ (where the constant c is 0 or -1) with constant mean curvature, such that $\| B \|$ is constant (this condition is

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automatically satisfied if M^n is compact) and which satisfy a condition analogous to (*).

2. The main integral formula.

Suppose we are given the Data of §0. We choose locally an "M^n--adapted" orthonormal frame field $(e_1, \ldots, e_{n+p})$ of $\widetilde{M}^{n+p}$ which means that the vector fields $e_1, \ldots, e_n$, restricted to M^n, are tangent to M^n and hence $e_{n+1}, \ldots, e_{n+p}$, restricted to M^n, are normal to M^n. In this situation we make the following conventions: the ranges of the indices

$$A, B, C, \ldots, \quad i, j, k, \ldots, \quad \alpha, \beta, \gamma, \ldots,$$

are

$$1 \le A, B, C, \ldots \le n+p, \quad 1 \le i, j, k, \ldots \le n, \quad n+1 \le \alpha, \beta, \gamma, \ldots \le n+p,$$

and all sums extend always over the respective ranges of repeated indices.

Thus, for all vector fields X, Y on M^n we have by definition of the second fundamental form B:

$$B(X, Y) = \sum_\alpha \langle \widetilde{\nabla}_X Y, e_\alpha \rangle e_\alpha = \sum_\alpha \langle H_\alpha(X, Y) \rangle e_\alpha \tag{2.1}$$

where H_α denotes the self adjoint tensorfield of type (1.1) on M^n characterized by

$$\langle H_\alpha(X), Y \rangle = -\langle \widetilde{\nabla}_X e_\alpha, Y \rangle = \langle \widetilde{\nabla}_X Y, e_\alpha \rangle = \langle B(X, Y), e_\alpha \rangle. \tag{2.2}$$

Hence the matrix $(h^\alpha_{ij})_{i,j=1,\ldots,n}$ of H_α with respect to $e_1, \ldots, e_n$ satisfies

$$h^\alpha_{ij} = \langle H_\alpha(e_i), e_j \rangle = \langle B(e_i, e_j), e_\alpha \rangle. \tag{2.3}$$

Furthermore, the covariand derivate ∇B of the second fundamental form is (of type $T^* \otimes T^* \otimes T^* \otimes \perp$) defined as usual (see e.g. Kobayashi--Nomizu II, p. 25) by

$$(\nabla B)(X, Y, Z) = (\nabla_Z B)(X, Y) =$$
$$= \nabla^\perp_Z(B(X, Y)) - B(\nabla X, Y) - B(X, \nabla_Z Y) \tag{2.4}$$

where $\nabla^\perp$ is the normal connection and one defines:

$$h^\alpha_{ijk} = \langle (\nabla B)(e_i, e_j, e_k), e_\alpha \rangle = \langle (\nabla_{e_k} B)(e_i, e_j), e_\alpha \rangle \tag{2.5}$$

(which is in concordance with the notation (2.10) of [2]). Moreover we define the "Hessian" $\nabla^2 B = \nabla\nabla B$ of B (which is of type $T^* \otimes T^* \otimes T^* \otimes T^* \otimes \perp$) by (see 2.4.)):

$$(\nabla^2 B)(X, Y, Z, W) = (\nabla_W(\nabla_Z B))(X, Y) - (\nabla_{\nabla_W Z} B)(X, Y) \tag{2.6}$$

and the "Laplacian" ΔB of B as the trace of the Hessian $\nabla^2 B$ of B (which is again of the same type $T^* \otimes T^* \otimes \perp$ as B) by

$$(\Delta B)(X, Y) = \sum_k (\nabla^2 B)(X, Y, e_k, e_k) \tag{2.7}$$

which is a contraction of $\nabla^2 B$ independent of the special choice of the orthonormal frame field $e_1, \ldots, e_n$. If we define (again in concordance with (2.13) and (2.18) of [2]):

$$h^\alpha_{ijk\ell} = \langle (\nabla^2 B)(e_i, e_j, e_k, e_\ell), e_\alpha \rangle$$

and

$$\Delta h^\alpha_{ij} = \langle (\Delta B)(e_i, e_j), e_\alpha \rangle \tag{2.8}$$

then

$$(\Delta B)(e_i, e_j) = \sum_\alpha (\Delta h^\alpha_{ij}) e_\alpha = \sum_{\alpha, k} h^\alpha_{ijkk} e_\alpha. \tag{2.9}$$

For tensor fields A, $\widetilde{A}$ of type $T^* \otimes T$ on M we define their inner product as usual (with $A^* =$ adjoint of A) by

$$\langle A, \widetilde{A} \rangle = \operatorname{tr}(A^* \circ \widetilde{A}) = \sum_{i,j} \langle A(e_i), e_j \rangle \langle \widetilde{A}(e_i), e_j \rangle = \sum_{i,j} a_{ij} \widetilde{a}_{ij} \tag{2.10}$$

and the norm by

$$\| A \| = \sqrt{\langle A, A \rangle}.$$

For tensor fields B, $\widetilde{B}$ of type $T^* \otimes T^* \otimes \perp$ (i.e. of the type of the second fundamental form) we define the inner product, resp. the norm, by

$$\langle B, \widetilde{B} \rangle = \sum_{i,j,\alpha} \langle B(e_i, e_j), e_\alpha \rangle \langle \widetilde{B}(e_i, e_j), e_\alpha \rangle, \text{ resp. } \| B \| = \sqrt{\langle B, B \rangle}. \tag{2.11}$$

(again independent of the special choice of the orthonormal frame $e_1, \ldots, e_{n+p}$).

Analogous definitions yield inner products for tensor fields of other types and the canonically induced connections for these tensor fields are *metric* with respect to these inner products.

As an example we obtain from (2.3), (2.4), (2.5), (2.9):

$$\langle H_\alpha, H_\beta \rangle = \sum_{i,j} h^\alpha_{ij} h^\beta_{ij} \tag{2.12}$$

$$\| B \|^2 = \sum_{i,j,\alpha} (h^\alpha_{ij})^2 = \sum_\alpha \| H_\alpha \|^2 \tag{2.13}$$

$$\| \nabla B \|^2 = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 = \sum_k \langle \nabla_{e_k} B, \nabla_{e_k} B \rangle \qquad 4)$$

$$\langle B, \Delta B \rangle = \sum_{i,j,k,\alpha} h^\alpha_{ij} h^\alpha_{ijkk} = \sum_{i,j,\alpha} h^\alpha_{ij} \Delta h^\alpha_{ij}. \tag{2.15}$$

Moreover the metricity of the corresponding connections yields for the Laplacian of the function $\| B \|^2$ as usual (using (2.7), (2.14)).

$$\Delta(\| B \|^2) = \Delta\langle B, B\rangle = \langle \Delta B, B\rangle + 2\langle \nabla B, \nabla B\rangle + \langle B, \Delta B\rangle,$$

i.e.

$$\| \nabla B \|^2 = \frac{1}{2}\Delta(\| B \|^2) - \langle B, \Delta B\rangle. \tag{2.16}$$

Remark. The left-hand side, resp. the right-hand side, of (2.16) is a differential operator applied to B of order one, resp. two, (the left-hand side usually called the "1.st Beltrami operator of B"). We want to show now that the part of $\langle B, \Delta B\rangle$ which involves derivatives of B of order 2 (or 1) does *only* depend on the Hessian of the mean curvature normal field H of B, all the rest depending in a purely algebraic way on B, involving some selfadjoint tensor field C of type $\perp^* \otimes \perp$ (i. e. an endomorphism field of the normal bundle $\perp$ of M) derived from B. So we start with some definitions: the mean curvature normal (vector field) H is defined as

$$H = \frac{1}{n}\sum_i B(e_i, e_i) \underset{(2.1)}{=} \frac{1}{n}\sum_\alpha (\mathrm{tr}\, H_\alpha) e_\alpha = \frac{1}{n}\sum_{\alpha, i} h^\alpha_{ii} e_\alpha, \tag{2.17}$$

thus

$$n^2 \| H \|^2 = \sum_\alpha (\mathrm{tr}\, H_\alpha)^2 = \sum_{\alpha, i} (h^\alpha_{ii})^2. \tag{2.18}$$

Moreover, using the fact that covariant differentiation commutes with contractions, we get from (2.17) by covariant differentiation $\nabla^\perp$ with respect to the normal connection:

$$(\nabla^\perp H)(X) = \frac{1}{n}\sum_i \nabla^\perp_X (B(e_i, e_i)) = \frac{1}{n}\sum_i (\nabla B)(e_i, e_i, X) \tag{2.19}$$

and for the Hessian $\nabla^\perp \nabla^\perp H$ of H which is of type $T^* \otimes T^* \otimes \perp$ (like B):

$$(\nabla^\perp \nabla^\perp H)(X, Y) = \frac{1}{n}\sum_i (\nabla^2 B)(e_i, e_i, X, Y). \tag{2.20}$$

Thus we get from (2.5), (2.19), resp. from (2.8), (2.20):

$$\left.\begin{aligned} (\nabla^\perp H)(e_k) &= \nabla^\perp_{e_k} H = \frac{1}{n}\sum_{i,k,\alpha} h^\alpha_{iik}\, e_\alpha, \\ (\nabla^\perp \nabla^\perp H)(e_k, e_\ell) &= \frac{1}{n}\sum_{i,k,\ell,\alpha} h^\alpha_{iik\ell}\, e_\alpha \end{aligned}\right\} \tag{2.21}$$

Consequently we get from (2.11), (2.3), (2.21):

$$n\langle B, \nabla^\perp \nabla^\perp H\rangle = \sum_{i,k,\ell,\alpha} h^\alpha_{k\ell} h^\alpha_{iik\ell}. \tag{2.22}$$

Moreover we define the following self-adjoint endomorphism C of the normal bundle $\perp$ of M^n in $\widetilde{M}^{n+p}$:

$$C(\xi) = \sum_{i,j} \langle B(e_i, e_j)\, \xi\rangle B(e_i, e_j) \text{ for every normal vector } \xi. \tag{2.23}$$

Evidently the elements of the matrix of C with respect to $e_{n+1}, \dots, e_{n+p}$ are

$$C_{\alpha\beta} = \langle C(e_\alpha), e_\beta\rangle \underset{\substack{(2.13)\\(2.3)}}{=} \sum_{i,j} h^\alpha_{ij} h^\beta_{ij} \underset{(2.10)}{=} \langle H_\alpha, H_\beta\rangle, \tag{2.24}$$

hence

$$\mathrm{tr}\, C = \sum_\alpha \| H_\alpha \|^2 \underset{(2.13)}{=} \| B \|^2 \tag{2.25}$$

and

$$\begin{aligned} \| C \|^2 &= \mathrm{tr}\,(C \circ C) = \\ &= \sum_{\alpha,\beta} C^2_{\alpha\beta} \underset{(2.24)}{=} \sum_{\alpha,\beta} \langle H_\alpha, H_\beta\rangle^2 = \Sigma\, h^\alpha_{ij} h^\beta_{ij} h^\alpha_{k\ell} h^\beta_{k\ell}. \end{aligned} \tag{2.26}$$

Finally we introduce for abbreviation:

$$\delta = \sum_{\alpha,\beta} \mathrm{tr}\,(H_\alpha H_\beta H_\alpha)\, \mathrm{tr}\,(H_\beta) = \sum_{i,j,k,\ell} h^\alpha_{ij} h^\beta_{jk} h^\alpha_{ki} h^\beta_{\ell\ell}, \tag{2.27}$$

a function which is (as a double contraction) independent of the special choice of $e_1, \dots, e_{n+p}$.

Using a computation of Braidi and Hsiung in [2] we get then:

Proposition 1. *Under the hypothesis of* §0 *and if* H_α, C *and* σ *are defined as in* (2.2), (2.23) *and* (2.27) *respectively, then*

$$\begin{aligned} \| \nabla B \|^2 &= \frac{1}{2}\nabla(\| B \|^2) - n\langle B, \nabla^\perp \nabla^\perp H\rangle + \\ &+ (\| C \|^2 + \sum_{\alpha,\beta} \| H_\alpha H_\beta - H_\beta H_\alpha \|^2) + n^2 c \| H \|^2 - (nc \| B \|^2 + \sigma) \end{aligned} \tag{2.28}$$

Proof. This is a direct consequence of (2.16) and the formula (3.2) of Braidi and Hsiung (see [2]) for $\langle B, \Delta B\rangle$ [which in sum is a generalization of a formula of Nomizu and Smyth (see [12]) for $\langle B, \Delta B\rangle$ in the hypersurface case $p = 1$ to the case of a general codimension $p \geqq 1$ of M^n in $\widetilde{M}^{n+p}$]: one simply has to translate formula (3.2) of [2] term by term using our formulas (2.15), (2.22), (2.13), (2.18), (2.27), (2.26) and (2.10) respectively to obtain:

$$\langle B, \Delta B\rangle = n\langle B, \nabla^{\perp}\nabla^{\perp} H\rangle + nc\,\| B\|^2 - n^2 c\,\| H\|^2 + \sigma - \\ -(\| C\|^2 + \sum_{\alpha,\beta} \| H_\alpha H_\beta - H_\beta H_\alpha\|^2),$$

which together with (2.16) implies (2.28).

Moreover we get:

Proposition 2. *Under the hypothesis of §0 and if H_α, C and σ are defined as in (2.2), (2.23) and (2.27) respectively, then one has the inequality:*

$$\| C\|^2 + \sum_{\alpha,\beta} \| H_\alpha H_\beta - H_\beta H_\alpha\| \le \left(2 - \frac{1}{p}\right)\| B\|^4, \tag{2.29}$$

and we discuss the equality in (2.29):

(*i*) *if $p = 1$, then equality sign holds in (2.29) always.*

(*ii*) *If $p = 2$ and if the equality sign holds in (2.29), then $H \equiv 0$ (i.e. M^n is a* minimal *submanifold in $\widetilde{M}^{n+p}$), in particular*

$$\sigma = 0. \tag{2.30}$$

(*iii*) *If $p \ge 3$, then the equality sign holds in (2.29) if and only if $B \equiv 0$ (i.e., M^n is* totally geodesic *in $\widetilde{M}^{n+p}$).*

Remark. In the case (ii), in addition, we can see that for every point x in M^n there exists an M^n-adapted orthonormal frame $e_1, \ldots, e_{n+2}$ of $T_x\widetilde{M}^{n+2}$, such that the matrices of H_{n+1}, resp. H_{n+2}, with respect to $e_1, \ldots, e_n$ are (at x):

$$\pm\frac{\| B\|}{2}\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline & 0 & 0 \end{array}\right), \text{ resp. } \pm\frac{\| B\|}{2}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right). \tag{2.31}$$

Proof. In case $p = 1$ the equality (2.29) follows trivially from (2.25) and (2.26). We have to study therefore only

$$p \ge 2 \tag{2.32}$$

Proposition 2 depends then on two inequalities, stated in the following two Lemmas.

Suppose that $e_1, \ldots, e_n$ is any orthonormal frame of T_xM, such that the matrix of H_α with respect to $e_1, \ldots, e_n$ is diagonal (such a frame exists, since H_α is self adjoint).

Lemma 1. *Under the hypothesis of Proposition 2 we have for all α, β:*

$$\| H_\alpha H_\beta - H_\beta H_\alpha\|^2 \le 2\,\| H_\alpha\|^2\,\| H_\beta\|^2. \tag{2.33}$$

Moreover, if the equality sign holds in (2.33) for $x \in M^n$ and $\| H_\alpha\|$ and $\| H_\beta\|$ are both *different from zero (hence $\alpha \ne \beta$), then the matrix of H_α, resp. of H_β, with respect to the above frame equals (after a suitable renumbering of the $e_1, \ldots, e_n$):*

$$\lambda\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline 0 & & 0 \end{array}\right), \textit{ resp. } \mu\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right) \quad \textit{with } \lambda, \mu \ne 0; \tag{2.34}$$

in particular:

$$tr\, H_\alpha = tr\, H_\beta = 0, \tag{2.35}$$

and (observe that $\alpha \ne \beta$):

$$|\lambda| = \frac{\| H_\alpha\|}{\sqrt{2}},\ |\mu| = \frac{\| H_\beta\|}{\sqrt{2}} \textit{ and } \langle H_\alpha, H_\beta\rangle = 0 \tag{2.36}$$

Proof. The proof of Lemma 3.1 in [2] by Braidi and Hsiung yields exactly the statement of our Lemma 1.

Suppose now $x \in M^n$ and that the frame $(e_{n+1}, \ldots, e_{n+p})$ of the normal space $\perp_x$ of M^n at the point x is chosen such that the matrix of C (see (2.23)) with respect to $e_{n+1}, \ldots, e_{n+p}$ is diagonal, i.e. (see (2.24)):

$$C_{\alpha\beta} = \langle H_\alpha, H_\beta\rangle = 0 \quad \text{for} \quad \alpha \ne \beta(*). \tag{2.37}$$

Lemma 2. *Under the hypothesis of Proposition 2, we have at the point x:*

$$\| C\|^2 + 2\sum_{\alpha\ne\beta} \| H_\alpha\|^2\,\| H_\beta\|^2 \le \left(2 - \frac{1}{p}\right)\| B\|^4 \tag{2.38}$$

and the equality sign holds in (2.38) if and only if

$$\| H_\alpha\| = \| H_\beta\| \quad \textit{for all} \quad \alpha, \beta. \tag{2.39}$$

(*) Such a choice of the frame $(e_{n+1}, \ldots, e_{n+p})$ is always possible, since C is a self adjoint endomorphism of $\perp_x$.

Warning. The sum $\sum_{\alpha \neq \beta} \| H_\alpha \|^2 \| H_\beta \|^2$ (since it is *not* a full contraction of the tensor field $\langle H_{\xi_1}, H_{\xi_2} \rangle \langle H_{\xi_3}, H_{\xi_4} \rangle$ of type $\perp^* \otimes \perp^* \otimes \perp^* \otimes \perp^*$, the sum being extended only over $\alpha \neq \beta$) does not have an invariant geometric meaning but does depend on the choice of the frame $(e_{n+1}, \dots, e_{n+p})$ of $\perp_x$, as the trivial example with $n = p = 2$ of the circular cylinder

$$M^2 = \{ x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1 \text{ and } x_4 = 0 \} \text{ in } \widetilde{M}^4 = \mathbb{R}^4$$

really shows. Therefore the conclusions (2.38), (2.39) are in general bound to the special choice of $(e_{n+1}, \dots, e_{n+p})$ with the property (2.37).

Proof. We can express the symmetric polynomial $\sum_{\alpha < \beta} (x_\alpha - x_\beta)^2$ in the p indeterminates $x_{n+1}, \dots, x_{n+p}$ as usual in terms of the elementary symmetric functions of the x_α and one finds:

$$\sum_{\alpha < \beta} (x_\alpha - x_\beta)^2 = (p-1)\Big(\sum_\alpha x_\alpha\Big)^2 - 2p \sum_{\alpha < \beta} x_\alpha x_\beta. \tag{2.40}$$

Therefore we get, using (2.26) and our hypothesis (2.37):

$$\begin{aligned} \| C \|^2 + 2 \sum_{\alpha \neq \beta} \| H_\alpha \|^2 \| H_\beta \|^2 &= \sum_\alpha \| H_\alpha \|^4 + 2 \sum_{\beta < \alpha} \| H_\alpha \|^2 \| H_\beta \|^2 + \\ + 2 \sum_{\alpha < \beta} \| H_\alpha \|^2 \| H_\beta \|^2 &= \Big(\sum_\alpha \| H_\alpha \|^2\Big)^2 + 2 \sum_{\alpha < \beta} \| H_\alpha \|^2 \| H_\beta \|^2 = \\ &= \left(1 + \frac{p-1}{p}\right)\Big(\sum_\alpha \| H_\alpha \|^2\Big)^2 - \frac{1}{p}(p-1)\Big(\sum_\alpha \| H_\alpha \|^2\Big)^2 - \\ &\quad - 2p \sum_{\alpha < \beta} \| H_\alpha \|^2 \| H_\beta \|^2. \end{aligned}$$

The last equation yelds therefore together with (2.13) and (2.40):

$$\begin{aligned} \| C \|^2 + 2 \sum_{\alpha \neq \beta} \| H_\alpha \|^2 \| H_\beta \|^2 &= \left(2 - \frac{1}{p}\right) \| B \|^4 - \\ &\quad - \frac{1}{p} \sum_{\alpha < \beta} (\| H_\alpha \|^2 - \| H_\beta \|^2), \end{aligned} \tag{2.41}$$

from where the statement of Lemma 2 follows immediately.

Proof of proposition 2 *under the hypothesis* (2.32): $p \geqq 2$. For the following we fix an arbitrary point $x \in M^n$ and all considerations about the tensors involved happen at x: since (e.g. due to (2.28)) the sum

$$\sum_{\alpha \neq \beta} \| H_\alpha H_\beta - H_\beta H_\alpha \|^2 = \sum_{\alpha, \beta} \| H_\alpha H_\beta - H_\alpha H_\beta \|^2$$

is independent of the special choice of the frame $e_{n+1}, \dots, e_{n+p}$, we might assume that the orthonormal frame $(e_{n+1}, \dots, e_{n+p})$ of $\perp_x$ is chosen such that (2.37) is fulfilled. Then the inequality (2.29) follows by composing the inequalities (2.33), (2.38). That the equality sign holds in (2.29) *if* $B \equiv 0$, is trivial. Assume oppositely that the equality sign holds in (2.29) *on* M. Then the equality sign must hold in (2.38) and in (2.33) for all α, β with $\alpha \neq \beta$ at $x \in M$. We distinguish two cases:

1st CASE: $B_x \neq 0$: then, due to (2.1), there exists $\alpha \in \{n+1, \dots, n+p\}$ such that $H_\alpha \neq 0$, consequently, according to (2.39):

$$\| H_\beta \| = \| H_{n+1} \| \neq 0 \quad \text{for all } \beta \in \{n+1, \dots, n+p\}. \tag{2.42}$$

Choose now the orthonormal frame $(e_1, \dots, e_n)$ of $T_x M$ such that H_{n+1} has a diagonal matrix with respect to $e_1, \dots, e_n$. Then because of (2.42) and Lemma 1 the matrix of H_β with respect to $e_1, \dots, e_n$ for $\beta > n+1$ must be equal to

$$\mu_\beta \left(\begin{array}{cc|c} 0 & 1 & \\ & & 0 \\ 1 & 0 & \\ \hline & 0 & 0 \end{array} \right) \text{ with } |\mu_\beta| = \frac{\| H_{n+1} \|}{\sqrt{2}} \neq 0$$

and therefore (observe (2.32)): for all

$$\alpha, \beta \in \{n+2, \dots, n+p\}, \ \langle H_\alpha, H_\beta \rangle = \pm \| H_{n+1} \|^2 \neq 0. \tag{2.43}$$

But according to (2.36) we have: for all

$$\alpha, \beta \in \{n+2, \dots, n+p\} \text{ with } \alpha \neq \beta, \ \langle H_\alpha, H_\beta \rangle = 0. \tag{2.44}$$

So (2.43) and (2.44) imply (together with (2.32)):

$$p = 2. \tag{2.45}$$

Furthermore we get from (2.35): $\operatorname{tr} H_{n+1} = \operatorname{tr} H_{n+2} = 0$, which yields by (2.17), (2.45): $H = 0$ resp. by (2.27), (2.45): $\sigma = 0$. Moreover according to (2.13), (2.45) we have

$$\| B \|^2 = \| H_{n+1} \|^2 + \| H_{n+2} \|^2,$$

thus by (2.42):

$$\| H_{n+1} \| = \| H_{n+2} \| = \frac{\| B \|}{\sqrt{2}}$$

and therefore (2.34) and (2.36) imply (2.31).

2^{nd} CASE: $B_x = 0$: then it follows trivially from (2.17), (2.27), that $H = 0$ and $\sigma = 0$ at x, and hence for $p = 2$ the statement (ii) is trivially true.

Thus we have shown: if there exists at least some point $x \in M^n$ with $B_x \neq 0$, then necessarily $p = 2$ and the statement (ii) holds at *all* points of M^n. If therefore $p \geqq 3$, we must have $B \equiv 0$.

This ends the proof of Proposition 2.

The combination of Proposition 1 and Proposition 2 allows to get the following:

Theorem. *Let M^n be an n-dimensional ($n \geqq 2$) riemannian C^∞ manifold isometrically C^∞ immersed into an $(n+p)$-dimensional riemannian C^∞ manifold $\widetilde{M}^{n+p}$ of constant curvature c. Let $\| B \|$ denote the norm of the second fundamental form B and H the mean curvature normal vector field of this immersion. Then*:

(i) *On M^n the following inequality holds*:

$$\| \nabla B \|^2 \leqq \frac{1}{2} \Delta (\| B \|^2) - n \langle B, \nabla^\perp \nabla^\perp H \rangle + \left(2 - \frac{1}{p}\right) \| B \|^4 + + n^2 c \| H \|^2 - (nc \| B \|^2 + \sigma), \tag{2.46}$$

which in case $p = 1$ *is always true as the* equality, *and if* $p \geqq 2$ *and the equality sign holds in* (2.46), *we have for* $p = 2$:

$$M^n \text{ is a minimal immersed submanifold of } \widetilde{M}^{n+2}; \tag{2.47}$$

$$p \geq 3: M^n \text{ is a totally geodesic immersed submanifold of } \widetilde{M}^{n+p} \tag{2.48}$$

(ii) *If* $\| B \|$ *is constant,* M^n *has constant mean curvature and*

$$\left(2 - \frac{1}{p}\right) \| B \|^4 + n^2 c \| H \|^2 = nc \| B \|^2 + \sigma, \tag{2.49}$$

then

$$\left.\begin{array}{l}\nabla B = 0, \text{ i.e. } M^n \text{ is immersed in } \widetilde{M}^{n+p} \text{ with} \\ \text{parallel second fundamental form,}\end{array}\right\} \tag{2.50}$$

in particular for

$$p = 1: M^n \text{ is immersed in } \widetilde{M}^{n+1} \text{ as an isoparametric hypersurface} \tag{2.51}$$

and in case $p \geqq 2$ *we have in addition to* (2.50) *that* (2.47) *and* (2.48) *are true.*

Remark. If M^n is compact and oriented, the integral

$$\int_{M^n} \Delta(\| B \|^2) . dv = \int_{M^n} \text{div grad} (\| B \|^2) \, dv$$

vanishes according to the divergence (= STOKES') theorem and therefore parallelity of H gives, integrating (2.46):

$$\int_{M^n} \left(\left(2 - \frac{1}{p}\right) \| B \|^4 + n^2 c \| H \|^2 - nc \| B \|^2 - \sigma \right) dv \geqq 0 \tag{2.49'}$$

and the equality sign in (2.49'), which e.g. follows from (2.49), implies then by (2.46): $\nabla B = 0$, in particular the constancy of $\| B \|$.

Proof. For (i): one gets (2.46) by composing the equation (2.28) with the inequality (2.29) and the equality sign implies the conclusions (i), (ii), (iii) of Prop. 2 which give (2.47), (2.48).

For (ii): in this case the hypothesis imply the vanishing of the right-hand side of (2.46), in particular imply therefore (see (2.46)) $\nabla B = 0$ and that the equality sign holds in (2.46), thus proving (2.47) and (2.48).

Suppose now that $p = 1$. Then it follows from (2.50) that H_{n+1} is parallel and therefore (see e.g. Satz 1 a) of Walden [15]) that all the principal curvatures of (H_{n+1}, i.e. of) M^n in $\widetilde{M}^{n+1}$ are *constant*, i.e. M^n is an *isoparametric hypersurface of* $\widetilde{M}^{n+1}$ (see e.g. Nomizu [11]).

This ends the proof of the Theorem.

Final Remarks.

(i) The problem of determining *all isoparametric hypersurfaces* of $\widetilde{M}^{n+1}$ of constant sectional curvature c is
   (a) *completely solved for* $c \leq 0$, see [3], [4], [10] and [13];
   (b) *not completely solved for* $c > 0$. For the status of that problem see Ferus, Karchen and Münzner [9].

(ii) For immersions of n-dimensional riemannian manifolds M^n into $(n+p)$-dimensional, riemannian manifolds $\widetilde{M}^{n+p}$ of constant curvature c with *parallel second fundamental form* and arbitrary codimension $p \geq 1$ see for the case
   (a) $c = 0$, the papers of Ferus [7], [8] and Walden [15],
   (b) $c < 0$, the thesis of Baches [1].

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