On relative stability of function-germs

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Introduction.

This work was firstly suggested by observing very simple examples:

1. The following unstable function germs have quite different behavior:

$$f : (R,0) \to (R,0) \qquad f(x) = x^3$$
$$g : (R,0) \to (R,0) \qquad g(x) = 0$$

If we compare f with the germ of a function h whose restriction to $R_- = \{x \in R \mid x \leq 0\}$ coincides with the restriction of f to R_-, it is possible to conjugate f and h through the germ of a diffeomorphism ϕ whose restriction to R_- coincides with the identity. The same doesn't happen with g, since it is always possible to obtain a function germ which vanishes when restricted to R_-, but which behaves (even topologically) quite differently from g.

2. Such an observation gets more interesting in higher dimensions. In dimension two, working again with germs at the origin, one can check that:

(a) The germ $f_1(x,y) = x^2 + y^2$ is (right) finitely determined, stable and it is clearly possible to conjugate f_1 and h through the germ of a diffeomorphism ϕ of R^2, whose restriction to $R^2_- = \{(x,y) \in R^2 \mid x \leq 0\}$ is the identity, if we assume that f_1 and h coincide when restricted to R^2_-.
(b) The germ $f_2(x,y) = (x^2 + y^2)^2$ is not (right) finitely determined but it is determined by its infinite jet and it also has the same property as the germ f_1 with respect to the mentioned conjugation and for the same subespace R^2_-.
(c) The germ $f_3(x,y) = x^2$ is not (right) finitely determined neither infinitely determined but it also satisfies the same conjugation property (with respect to R^2_-) that f_1 and f_2 satisfy.
(d) The germ $f_4(x,y) = 0$ is not (right) finitely or infinitely determined. However, it is possible to obtain a germ h, whose restriction to R^2_- coincides

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with the restriction of f_4 to R^2_-, but behaves (even topologically) different from f_4. In other words, f_4 doesn't satisfy the conjugation property with respect to R^2_-.

This suggests us to define "S-relative equivalence", where S is a submanifold with non empty boundary, having the same dimension as the manifold and consequently, to introduce the "S-stability" concept (def. 1.3).

After the introduction of basic terminology, we define "infinitesimal stability relative to S" (S-infinitesimal stability). The main theorem in this section shows that S-infinitesimal stability implies S-stability (th. 1.7).

Then, we introduce the "Jacobian-Lojasiewicz condition relative to S" and we prove that it is a sufficient condition for S-infinitesimal stability (th. 2.5).

Finally, it is proved that S-stability implies Jacobian-Lojasiewicz condition relative to S (th. 2.7).

These results, alltogether, give us a final one: S-infinitesimal stability is equivalent to S-stability (corol. 2.8).

1.

Since this paper is concerned with a local study, we usually consider germs and jets at 0, of mappings between euclidean spaces and suitable sets of R^n containing the origin. The following notations are used:

We denote by S a submanifold of R^n with the same dimension and with non empty boundary

$$(\text{w. } \ell.\text{g. } S = R^n_- = \{(x^1, x^2, \ldots, x^n) \in R^n \mid x^1 \leq 0\}).$$

1.1 Let $f: (R^n, 0) \to (R^p, 0)$ be a differentiable germ. $\varepsilon(f, S; n, p)$ denotes the set of germs at $0 \in R^n$ of smooth mappings from R^n to R^p, whose restriction to S coincides with the restriction of f to S.

If $p = 1$ or $f \equiv 0$, we omit it in this notation. Hence, $\varepsilon(S; n)$ is the set of germs in $\mathcal{M}(n)$ which vanishes when restricted to S.

1.2 $\mathcal{R}_S(n) = \mathcal{R}_S$ is the set of germs at 0, of local diffeomorphisms of $\mathcal{R}^n$, whose restriction to S coincides with the identity. We also observe that $\mathcal{R}_S$ is a subgroup of $\mathcal{R}$, which acts on the right, on $\varepsilon(f, S; n)$, in a natural way:

If $g \in \varepsilon(f, S; n)$ and $\phi \in \mathcal{R}_S$, then $g \circ \phi$ is the germ, at 0, of the composition:

$$g \circ \phi : R^n \to R.$$

Of course, a germ $f \in \mathcal{M}(n)$ is S-equivalent to another germ $g \in \varepsilon(f, S; n)$ (equivalent relative to $\mathcal{R}_S$) if g belongs to the $\mathcal{R}_S$-orbit of f.

1.3 **Definition.** $f \in \mathcal{M}(n)$ is S-stable if f is S-equivalent to any $g \in \varepsilon(f, S; n)$. In other words, if the $\mathcal{R}_S$-orbit of f contains $\varepsilon(f, S; n)$.

1.4 **Theorem.** *Let $f \in \mathcal{M}(n)$ and $S = R^n_-$. Suppose that for any $w \in \varepsilon(S; n)$, there exists $\xi \in \varepsilon(S; n, n)$ such that $w(x) = f'(x)(\xi(x))$. Then f is S-stable.*

Proof. It follows the usual procedure and can be found in [2 – § 1.2 – proposition 1.13].

Remark. Let us denote by $\langle df \rangle$ the Jacobian ideal of f. If $f \in \mathcal{M}(n)$ satisfies the hypothesis of theorem 1.4, that is, if:

$$\varepsilon(S; n) \subset \varepsilon(S; n) \langle df \rangle,$$

then we say that f is S-infinitesimally stable.

1.5 **Corollary.** *Let $f \in \mathcal{M}(n)$ be S-infinitesimally stable and $g \in \varepsilon(f, R^n\text{-}S; n)$. Then g is S-stable.*

1.6 **Theorem.** *Let $f \in \mathcal{M}(n)$ be (right) finitely determined and $S = R^n_-$. Then f is S-stable.*

Remark. This Theorem is also proved in [2 – 1.2] by using results of relative finite determinacy. We prove it here, directly.

Proof. It is enough to show that f is S-infinitesimally stable. Since f is (right) finitely determined, we have:

$$\mathcal{M}^\ell(n) \subset \mathcal{M}(n) \langle df \rangle \text{ for some positive integer } \ell.$$

Now, let $w \in \varepsilon(S; n)$. One can check that $w(x) = x_1^k w_k(x)$, with $w_k \in \varepsilon(S; n)$, for any positive integer k. We let $k = \ell$, to get that:

$$x_1^\ell = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \xi_i(x), \ \xi_i(0) = 0$$

and

$$w(x) = w_\ell(x) \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \xi_i(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \eta_i(x),$$

Where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ satisfies $\eta|_S = 0$.

Therefore, $w \in \varepsilon(S; n) \langle df \rangle$.

Remarks:

1. Theorem 1.6 gives an answer to the problem which was suggested in the introduction. The germs $f(x) = x^3$ and $f_1(x, y) = x^2 + y^2$ are S-stable ($S = R_-$ or R^2_-, respectively), since they are (right) finitely determined.

2. We have just proved that (right) finite determinacy is a sufficient condition for S-stability.

However it is not necessary, since it is possible to check that:

The germ $f_2 \in \mathcal{M}(2)$, $f_2(x, y) = (x^2 + y^2)^2$ is not (right) finitely determined, but it is S-stable, with $S = R^2_-$. It is enough to show that given any $w \in \varepsilon(S; 2)$, one can find $\xi_1, \xi_2 \in \varepsilon(S; 2)$, satisfying the following condition:

$$\frac{w(x, y)}{4(x^2 + y^2)} = x\xi_1(x, y) + y\xi_2(x, y),$$

that gives S-infinitesimal stability for f. But such an equation has a well known solution.

3. We also observe that f_2 is determined by its infinite jet [3 – pg. 101].

Next theorem shows that remark 3 is a general observation and not an isolated fact.

1.7 **Theorem.** *Let $f \in \mathcal{M}(n)$ be (right) infinitely determined and $S = R^n_-$. Then f is S-stable.*

We use the following lemma:

1.8 **Lemma.** *Let $f \in \mathcal{M}^\infty(n)$. Then, $f = gh$, where $h \in \mathcal{M}^\infty(n)$, $g \in \mathcal{M}^\infty(n)$ and $g(x) > 0$ $\forall x \neq 0$.*

Proof of the lemma.

For each non negative integer i, we denote by

$$B_i = B(0, 1/2^i) = \{x \in R^n \mid |x| < 1/2^i\},$$
$$F_i = \overline{B}_i - \overline{B}_{i+1};$$

and for each positive integer i,

$$G_i = (R^n - B_{i-1}) \cup \overline{B}_{i+2}.$$

Clearly, $d(F_i, G_i) > 1/2^{i+2}$.

By using [3 – lemma IV – 3.3], we conclude that there exist $\alpha_i \in \varepsilon(n)$ such that:

$$\alpha_i|_{F_i} = 1, \ \alpha_i|_{G_i} = 0, \ \alpha_i \geq 0$$

and there also exist constants C_r, independent of $i \in N$, such that:

$$|\alpha_i|_r \leq C_r 2^{ir}$$

where $|\alpha_i|_r = \sup_{|k| \leq r} |D^k \alpha_i(x)|$.

Since $f \in \mathcal{M}^\infty(n)$, for each positive integer r one can associate a positive integer $\mu(r)$ such that:

$$|f|^x_r \leq |x|^{r^2} \ \forall x \in B_{\mu(r)}. \tag{1}$$

We can assume that the sequence $\mu(r)$ is increasing, and $\lim_{r\to\infty} \mu(r) = \infty$.

We also construct a sequence $\beta(i)$ of positive integers, by:

$$\beta(i) = r, \text{ if } \mu(r) \leq i < \mu(r + 1).$$

From (1), it turns out that:

$$\sum_{i=\mu(1)}^{\infty} (1/2^i)^{\beta(i)} \alpha_i$$

converges uniformly as well as its derivatives. Let g be its limit ($g \in \mathcal{M}^\infty(n)$).

Clearly, $g(x) > 0$ if $x \neq 0$.

The quotient f/g is well defined and infinitely differentiable, for $x \neq 0$. It remains to prove that we can extend this quotient to $h \in \mathcal{M}^\infty(n)$.

Observe that:

$$\text{if } x \in B_{\mu(r)} - B_{\mu(r+1)}, \text{ then } x \in F_j, \ \mu(r) \leq j < \mu(r + 1).$$

This implies that:

$$g(x) \geq (1/2^j)^{\beta(j)} = (1/2^j)^r \geq |x|^r. \tag{2}$$

Also, for any non negative integer s, there exist constants C'_s such that:

$$|f/g|^x_s \leq C'_s \frac{|f|^x_s}{(g(x))^{s+1}} \ \forall x \in B_0 - \{0\}. \tag{3}$$

Hence, for any integer $r \geq s + 2$ and $x \in B_{\mu(r)} - B_{\mu(r+1)}$,

$$|f/g|^x_s \leq C'_s \frac{|f|^x_s}{(g(x))^{s+1}} \leq \frac{C'_s |f|^x_r}{(g(x))^{s+1}} \leq C'_s |x|^{r(r-s-1)}$$

and this implies that:

$$\text{if } |x| \to 0, \text{ then } |f/g|^x_s \to 0;$$

thus we can extend the quotient to $h \in \mathcal{M}^\infty(n)$.

Proof of theorem 1.7.

Since f is (right) infinitely determined,

$$\mathcal{M}^\infty(n) \subset \langle df \rangle \quad [4. \text{ th. } 1.2]$$

and then:

$$\mathcal{M}^\infty(n)\varepsilon(S; n) \subset \varepsilon(S; n)\langle df \rangle. \qquad (1)$$

Observe that:

$$\mathcal{M}^\infty(n)\varepsilon(S; n) = \varepsilon(S; n). \qquad (2)$$

For this, take $f \in \varepsilon(S; n)$. By lemma 1.8, since $f \in \mathcal{M}^\infty(n)$, $f = gh$, where $g, h \in \mathcal{M}^\infty(n)$ and $g(x) > 0$ if $x \neq 0$.

Since $f|_S = 0$, it follows easily that $h|_S = 0$, and the other inclusion of (2) is obvious.

Now, from (1) and (2), we have:

$$\varepsilon(S; n) \subset \varepsilon(S; n)\langle df \rangle.$$

So, f is S-stable.

2.

2.1 Definitions:

(*a*) Let S be a closed set of R^n containing the origin and $f \in \mathcal{M}(n)$. Then f verifies a Lojasiewicz inequality with respect to S (f satisfies $\mathcal{L}(S; n)$) if for each germ of a compact set $K \subset R^n$ containing the origin, there exist constants $c > 0$ and $\alpha \geq 0$ such that:

$$|f(x)| \geq c d(x, S)^\alpha \ \forall x \in K.$$

(*b*) Let $\mathcal{I}$ be an ideal finitely generated of $\varepsilon(n)$ and S be a closed subset of R^n, containing 0. Then $\mathcal{I}$ is a Lojasiewicz ideal with respect to S if there exists $f \in \mathcal{I}$ such that f satisfies $\mathcal{L}(S; n)$. In this case, if $\{f_1, f, \ldots, f_r\}$ is an arbitrary system of generators of $\mathcal{I}$, then

$$\sum_{i=1}^r |f_i| \text{ or } \sum_{i=1}^r f_i^2 \text{ satisfies } \mathcal{L}(S; n) \text{ (see } [3 - \text{V}.4]).$$

(*c*) Let f and S be as in 2.1, (a) and (b). Then f is Jacobian-Lojasiewicz relative to S or f satisfies a Jacobian-Lojasiewicz condition with respect to S, if the Jacobian ideal $\langle df \rangle$ of f is a Lojasiewicz ideal wich respect to S, or equivalently, if $|\nabla f|$ satisfies $\mathcal{L}(S; n)$.

2.2 Definition. *Let $\{b_i\}$ be a sequence of positive real numbers which converges to 0. A sequence of real numbers $\{a_i\}$ is flat along b_i if, for each $r > 0$ there is an N such that $i \geq N$ implies $|a_i| \leq b_i^r$. A sequence of vectors, matrices or ∞-jets is flat along b_i if each entry is, and is flat along x_i in R^n, if it is flat along $|x_i|$.*

We observe that a germ g doesn't satisfy $\mathcal{L}(S; n)$ if and only if there exists a sequence $\{x_i\}$ converging to 0 such that $g(x_i)$ is flat along $d(x_i, S)$.

2.3 Definition. *We denote by $M(S; n)$ the set of functions ϕ which are defined in $R^n - S$ and satisfy the following condition:*

For each germ of a compact set $K \subset R^n$ containing the origin and for each n-uple $k \in N^n$, there exist constants $c > 0$, $\alpha > 0$, such that:

$$|D^k\phi(x)| \leq c d(x, S)^{-\alpha} \ \forall x \in K - S.$$

$M(S; n)$ is the space of the multipliers of the ideal $M_S^\infty(n)$, of function germs which are flat at S.

One proves that:

2.4 Theorem. *Let $\phi \in \mathcal{M}(S; n)$ and f $\mathcal{M}_S^\infty(n)$. Then it is possible to extend the function ϕf on S in such a way that its germ, now also denoted by ϕf, belongs to $\mathcal{M}_S^\infty(n)$.*

The proof follows [3. IV prop. 4.2], adjusted in terms of germs.

We are now able to prove:

2.5 Theorem. *Let $f \in \mathcal{M}(n)$ and $S = R^n_-$. Assume that f satisfies the Jacobian-Lojasiewicz condition relative to S. Then, f is S-infinitesimally stable and therefore S-stable.*

Proof. We firstly observe that:

$$\mathcal{M}_S^\infty(n) = \mathcal{M}_{R^n}^\infty(n) = \varepsilon(S; n).$$

Since $\langle df \rangle = \left\langle \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n} \right\rangle$, we can take $g = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\right)^2$.

We want to prove that $\mathcal{M}_S^\infty(n) \subset \mathcal{M}_S^\infty(n)\langle df \rangle$, or equivalently, that $\varepsilon(S; n) \subset \varepsilon(S; n)\langle df \rangle$.

So, let g' be a representative of g. The proof follows easily, if we assume that $1/g' \in \mathcal{M}(S; n)$:

Let's take $h \in \mathcal{M}_S^\infty(n)$ and let h' be its representative.

By theorem 2.4 (and also by using its notation):

$$h = [h'id] = \left[h'\frac{1}{g'}g'\right] = \left[h'\frac{1}{g'}\right]g = \left(h\frac{1}{g}\right)g \in \mathcal{M}_S^\infty(n)\langle df\rangle.$$

In order to prove that $1/g' \in \mathcal{M}(S;n)$, we observe that by Leibinz's formula, applied to $D^k(1/g')$, there exists a constant c' such that:

$$\left| D^k\left(\frac{1}{g'}\right)(x)\right| \le \frac{c'}{|g'(x)|^{|k|+1}}\ \forall x \in K, \tag{1}$$

and by hypothesis:

$$|g'(x)| \ge c d(x,S)^\alpha\ \forall x \in K. \tag{2}$$

From (1) and (2) it follows that:

$$\left| D^k \frac{1}{g'}(x)\right| \le c d(x,S)^{-\alpha}\ \forall x \in K - S,$$

which implies that $1/g' \in \mathcal{M}(S;n)$.

Remark. Now it is clear why the germ f_3 in the introduction is S-stable, since it satisfies Jacobian-Lojasiewicz condition with respect to S.

Next lemma is very important for the proof of the last theorem, which completes our work.

2.6 **Lemma.** *Let $f \in \mathcal{M}(n)$, $S = R^n_-$, $w_i \in R \times J^1(n,1)$, $i \in N$ and let also $\{x_i\} = \{(x_i^1, x_i^2, \dots, x_i^n)\}$ be a sequence in R^n, converging to 0, in such a way that:*

$$q_i = w_i - j^1 f(x_i) \text{ is flat along } x_i.$$

Then, there exists $g \in \varepsilon(f,S;n)$ such that $w_i = j^1 g(x_i)$ holds, for a subsequence of $\{x_i\}$.

Proof. It follows [4 – lemma 3.3] and we begin by transforming each w_i into an infinite jet in such a way that all the terms of order greater than 1 of q_i are zero. Thus, we can work with infinite jets.

Let Q be the Taylor field, given by q_i at x_i and by the zero series at points of S.

Similarly as in [4 – lemma 3.3], it can be proved that Q is a C^∞ Whitney field on $S \cup \{x_i\}$, since Q satisfies the following condition (see [3 – IV 1.5 and 1.6]):

For each m and for each multi index λ such that $|\lambda| \le m$,

$$(R_y^m Q)^\lambda(x) = \circ\,(|x-y|^{m-|\lambda|}), \text{ where}$$

$$(R_y^m Q)^\lambda(x) = Q^\lambda(x) - \sum_{L \le m-|\lambda|} Q^{\lambda+L}(y)\frac{(x-y)^L}{L!}$$

for $x, y \in S \cup \{x_i\}$ and $|x-y| \to 0$;

This condition was already checked for $x, y \in \{x_i\} \cup 0$ in [4 – lemma 3.3] and it obviously holds for $x, y \in S$.

Then, by Whitney's Extension Theorem [3 – IV 3.1] there is a $C^\infty q$ with $q(x_i) = q_i$ for a subsequence of $\{x_i\}$ and $j^\infty q|_S = 0$.

It remains to take $g = f + q$, to finish the lemma.

We are now able to prove:

2.7 **Theorem.** *Let $S = R^n_-$ and $f \in \mathcal{M}(n)$, S-stable. Then f satisfies Jacobian--Lojasiewicz condition relative to S.*

Proof. Suppose that f doesn't satisfy Jacobian-Lojasiewicz condition relative to S; we want to prove that f is not S-stable. Since two (right) equivalent germs have identical critical values and R_S-equivalence implies R-equivalence, it is enough to prove that there is $g \in \varepsilon(f,S;n)$ such that reg. val. $(f) \neq$ reg. val. (g).

By hypothesis, there is a sequence $\{x_i\}$ in R^n, converging to 0, such that $|\nabla f(x_i)|$ is flat along $d(x_i, S)$ and consequently

$$|\nabla f(x_i)| \text{ is flat along } x_i. \tag{1}$$

By Sard's Theorem, it is possible to find a sequence $\{y_i\}$ converging to 0, y_i not a critical value of f, such that

$$f(x_i) - y_i \text{ is flat along } x_i. \tag{2}$$

From (1) and (2), it clearly follows that

$$w_i = (y_i, 0_n^1) - (f(x_i), j^1 f(x_i))$$

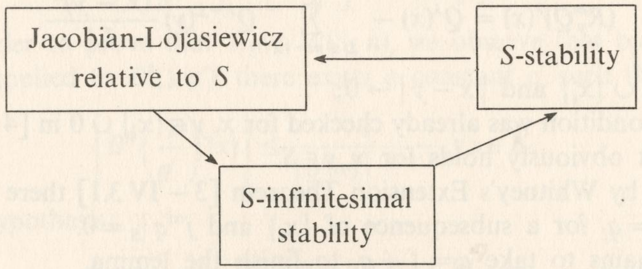
is flat along x_i.

Then, by lemma 2.6 we can find $g \in \varepsilon(f,S;n)$ such that $y_i = g(x_i)$ is a critical value of g.

2.8 **Corollary.** *Let $f \in \mathcal{M}(n)$ and $S = R^n_-$. Then, S-infinitesimal stability is equivalent to S-stability.*

Proof. It follows immediately from theorems 1.4, 2.5 and 2.7.

We finally have the following diagram:



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