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A construction of $\mathscr{L}_\infty$-spaces and related Banach spaces

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Abstract.

Let $\lambda > 1$. We prove that every separable Banach space E can be embedded isometrically into a separable $\mathscr{L}_\infty^\lambda$-space X such that X/E has the RNP and the Schur property. This generalizes a result in [2]. Various choices of E allow us to answer several questions raised in the literature. In particular, taking $E = \ell_2$, we obtain a $\mathscr{L}_\infty^\lambda$-space X with the RNP such that the projective tensor product $X \hat{\otimes} X$ contains c_0 and hence fails the RNP. Taking $E = L^1$, we obtain a $\mathscr{L}_\infty^\lambda$-space failing the RNP but nevertheless not containing c_0.

0. Introduction and background.

In the paper [2], the first example of a $\mathscr{L}_\infty$-space not containing c_0 was constructed. This space has the RNP and the Schur property. In this paper, we present another approach to such examples in a more general framework: for any $\lambda > 1$, we embed isometrically each separable Banach space E into a separable $\mathscr{L}_\infty^\lambda$-space, which we denote by $\mathscr{L}_\lambda[E]$ in such a way thet the quotient space $\mathscr{L}_\lambda[E]/E$ has the RNP and the Schur property. The underlying idea is that, since this quotient can be viewed as "small" (roughly), the space $\mathscr{L}_\lambda[E]$, although it is a $\mathscr{L}_\infty$-space, still does inherit many of the properties of E such as weak sequential completeness, not containing c_0 or L^1, the RNP or the Schur property.

Various choices of E give us several interesting examples of $\mathscr{L}_\infty$-spaces. Taking $E = \{0\}$ (and noting that we can ensure that $\mathscr{L}_\infty[E]$ is always infinite dimensional), we obtain a $\mathscr{L}_\infty$-space with RNP and the Schur property, which is one of the main results of [2]. Taking $E = L^1$, we obtain a $\mathscr{L}_\infty$-space without the RNP but still not containing c_0. This answers a question raised in [1].

Taking $E = \ell_2$, we obtain a $\mathscr{L}_\infty$-space X with the RNP such that the projective tensor product $X \hat{\otimes} X$ contains c_0 and hence fails the RNP. This answers a question raised in [3].

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The general approach of our paper is to show that a certain class of inductive limits of finite dimensional spaces have the RNP and the Schur property (cf. theorem 1.6). This class includes also the examples constructed in [7]; it is this observation, by the first author, which was the starting point of this paper.

Let us recall some background.

A Banach space X is called a $\mathscr{L}_\infty^\lambda$-space if there is a filtering increasing family of finite dimensional subspaces $X_i \subset X$ such that $X = \overline{\bigcup X_i}$ and $d(X_i, \ell_\infty^{\dim X_i}) \leqq \lambda$.

We refer to [1] for more information on $\mathscr{L}_\infty$-spaces as well as the Schur property and the Radon-Nikodym property (in short RNP). For the RNP, the standard reference is [3].

Let us briefly recall how an inductive limit of Banach spaces is defined. Let $(E_n)_{n \geq 0}$ be a sequence of spaces, given together with a sequence of isometric embeddings $j_n : E_n \to E_{n+1}$. Then, the inductive limit X of the system (E_n, j_n) is defined as follows. We consider the subspace of ΠE_n formed by all the sequences (x_n) such that $j_n(x_n) = x_{n+1}$ for all n sufficiently large. We equip this space with the semi-norm $\|(x_n)\| = \lim \|x_n\|$. Let $\mathscr{X}$ be the normed space obtained after passing to the quotient by the kernel of that semi-norm. The space X is then defined as the completion of the space $\mathscr{X}$. Clearly, there is a system of isometric embeddings $J_n : E_n \to X$ such that if $X_n = J_n(E_n)$ we have $X_n \subset X_{n+1}$ and the union $\bigcup X_n$ is dense in X.

In practice, this construction shows that we may always do as if the spaces E_n formed an increasing sequence of subspaces of some larger space, and we may then identify X simply with $\overline{\bigcup E_n}$. We will need the following result.

Proposition 0.1. *For a Banach space X, let P be any of the following properties:*

a) The Schur property.
b) The space X does not contain an isomorphic copy of c_0.
c) Weak sequential completeness.
d) The RNP.

Now let E be a closed subspace of X.

If both E and X/E have the property P, then the same is true for X.

Proof. For a) and b), this is quite easy to prove.
For c) it was pointed to the second author by Gilles GODEFROY.
For d) it was proved by Edgar (cf. [3] p. 211).

We will also use the following well known fact:

Proposition 0.2. *Let $\{x_n\}$ be a sequence tending weakly to zero in some Banach space X. Then, for each $\varepsilon > 0$, there are numbers $\alpha_n \geqq 0$ such that $\Sigma \alpha_n = 1$ and such that*

$$\max \{\|\Sigma \varepsilon_n \alpha_n x_n\| \mid \varepsilon_n = \pm 1\} < \varepsilon. \tag{0.1}$$

Proof. We can assume that $X = C(K)$ for compact K. If $x_n \to 0$ weakly, then the functions $|x_n|$ also tend to zero weakly (by dominated convergence) to zero. Therefore, the convex hull of $\{|x_n| \mid n \geqq 1\}$ contains 0 in its norm closure. In other words, for each $\varepsilon > 0$, there are numbers $\alpha_n \geqq 0$ such that $\Sigma \alpha_n = 1$ and

$$\sup \{\Sigma \alpha_n |x_n(\xi)| \mid \xi \in K\} < \varepsilon \tag{0.2}$$

Clearly, (0.2) is equivalent to (0.1).

1. A certain class of inductive limits.

We start be recalling a known construction. This construction has been very fruitful in [6], and more recently in [7]. It was used in [7] repeatedly to construct Banach spaces enjoying certain special extension properties. Since $\mathscr{L}_\infty$-spaces can be characterized in terms of extension properties (cf.e.g. [1]) it is not surprising that we find this point of view useful in this context also.

Lemma 1.1. *Let E, B be Banach spaces, and let $\eta \leq 1$. Let S be a (closed) subspace of B and let $u : S \to E$ be an operator such that $\|u\| \leq \eta$.*

Then, there exist a Banach space E_1, an isometric embedding $j : E \to E_1$, and an operator $\tilde{u} : B \to E_1$ such that $\tilde{u} | S = ju$ and $\|\tilde{u}\| \leq 1$.

Moreover, the spaces E_1/E and B/S are isometric.

Proof. We consider $B \oplus E$, equipped with the norm $\|(b, e)\| = \|b\| + \|e\|$ for all b in B and all e in E. Let $N = \{(s, -us) \mid s \in S\}$. We let $E_1 = (B \oplus E)/N$ and we denote by π the canonical surjection of $B \oplus E$ into $(B \oplus E)/N$. We let, for b in B and e in E,

$$\tilde{u}(b) = \eta(b, 0) \quad \text{and} \quad j(e) = \pi(0, e).$$

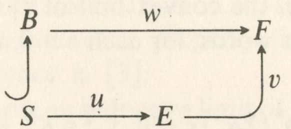
It is then easy to check that $\|\tilde{u}\| \leq 1$ (actually we have always $\|\tilde{u}\| = 1$ if $S \neq B$), that j is an isometric embedding, and that $\tilde{u} | S = ju$.

Finally, it is not hard to check that $\tilde{u} : B \to E_1$ induces (after passing to the quotient over S) an isometry between B/S and E_1/E.

Remark 1.2. We note that if B/S is finite dimensional (in short f.d.) then E_1/E will be of the same finite dimension.

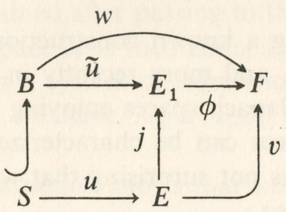
We will use the following remarkable (although simple) property of the space E_1: it is the solution of a universal problem (analogously to amalgamated sums in the category of groups).

Proposition 1.3. i) *The triplet $(E_1, j, \tilde{u})$ constructed above has the following property: consider any commutative diagram*



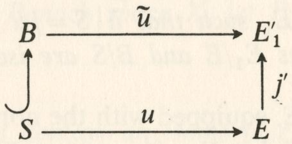
(where F is some Banach space, and $w: B \to F$, $v: E \to F$ are such that $vu = w \mid S$).

Then there is a unique linear map $\phi: E_1 \to F$ such that $w = \phi\tilde{u}$ and $v = \phi j$. Equivalently, we have a commutative diagram



Moreover, we have (1.1) $\|\phi\| \leq \max\{\|v\|, \|w\|\}$.

ii) *The triplet $(E_1, j, \tilde{u})$ is unique in the following sense: suppose $(E'_1, j', \tilde{u}')$ is another triplet such that the diagram*



is commutative, with $j': E \to E'_1$, isometric, $\|\tilde{u}'\| \leq 1$, and satisfying the above property (i), that is to say, for any v,w as above, there is a unique $\phi': E'_1 \to F$ such that $v = \phi' j'$, $w = \phi' u'$ and $\|\phi'\| \leq \max\{\|v\|, \|w\|\}$. Then, necessarily there is an isometric isomorphism $T: E_1 \to E'_1$ such that $Tj = j'$ (hence $T(j(E)) = j'(E)$).

Proof. The proof is straightforward.

In (i), we define ϕ by

$$\forall b \in B,\ \forall e \in E, \qquad \phi(\pi(b, e)) = wb + ve.$$

ϕ is clearly unique and satisfies (1.1).

The proof of (ii) follows from the unicity property of ϕ. Taking successively

$$w = \tilde{u}',\ v = j' \quad \text{and} \quad w = \tilde{u},\ w = j,$$

we obtain $T: E_1 \to E'_1$ and $T': E'_1 \to E_1$ such that, by the unicity property, TT' and $T'T$ have to coincide with the identity on E'_1 and E_1 respectively.

We will need to abbreviate the terminology: in the preceding situation, we will say that any embedding j' for which there is a $\tilde{u}'$ satisfying proposition 1.3.(ii) is *associated to* (E, u, S, B).

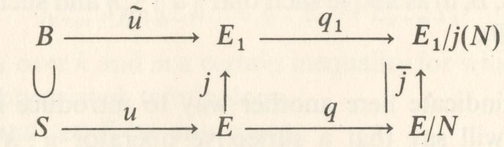
We will need the following simple observation.

Proposition 1.4. *Consider (E, u, S, B) as above and let E_1 be any space associated to (E, u, S, B). Let j be the embedding of E into E_1, let N be a subspace of E and let*

$$q: E \to E/N \quad \text{and} \quad q_1: E_1 \to E_1/j(N)$$

be the canonical quotient maps.

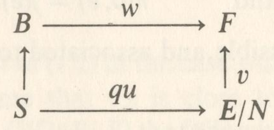
Then $\bar{j}$ is associated to $(E/N, qu, S, B)$, via the following diagram:



where $\bar{j}: E/N \to E_1/j(N)$ is the embedding naturally associated to j.

Proof. This can be proved by directly exhibiting a suitable isometry between $E_1/j(N)$ and $(B \oplus E/N)/\{(s, -qus) \mid s \in S\}$.

We indicate an argument using the preceding proposition: consider a commuting diagram



then, by the property of E_1 we know that there is a unique map $\phi: E_1 \to F$ such that: $\phi j = vq$,

$\phi\tilde{u} = w$

and $\|\phi\| \leq \max\{\|w\|, \|vq\|\}$.

Clearly $\phi|j(N)=0$, so that there is a map $\bar{\phi}: E_1/j(N) \to F$ satisfying $\phi = \bar{\phi} q_1$.

We then check easily that $\bar{\phi}\bar{j} = v$, $\|\bar{\phi}\| \leq \max\{\|w\|, \|v\|\}$, and it is easy to see that the unicity of ϕ implies that of $\bar{\phi}$. This shows by proposition 1.3(ii) that $\bar{j}$ is associated to $(E/N, qu, S, B)$.

We also record the following simple fact.

Remark 1.5. In the above situation, if we have for some $\delta \leq 1$

$$\forall s \in S \qquad \|u(s)\| \geq \delta \|s\|,$$

then necessarily

$$\forall x \in B \qquad \|\tilde{u}(x)\| \geq \delta \|x\|.$$

Indeed, $$\|\hat{u}(x)\| = \inf_{s \in S} \{\|x+s\| + \|u(s)\|\}$$

$$\geq \inf \{\delta\|x+s\| + \delta\|s\|\} = \delta\|x\|.$$

We again introduce more terminology: we will say that an isometric embedding

$$j : E \to E_1 \text{ is } \eta\text{-admissible} \quad (0 \leq \eta \leq 1)$$

if there exists (S, B, u) as above such that $\|u\| \leq \eta$ and such that j is associated to $(E, u S, B)$.

Remark. We indicate here another way to introduce η-admissible embeddings. We will say that a surjective operator $u : X \to Y$ is a metric surjection if the associated isomorphism from $X/_{\mathrm{Ker}\, u}$ into Y is an isometry.

Now, let $j : E \to E_1$ be an isometric embedding. Then j is η-admissible iff the following holds: there exists a Banach space B and a metric surjection $\pi : B \oplus E \to E_1$ such that

$$(*) \qquad \forall b \in B, \quad \forall e \in E \qquad \|\pi(b,e)\| \geq \|e\| - \eta\|b\|$$

$$\text{and} \qquad \pi(0, e) = j(e).$$

Indeed, if j is η-admissible and associated to (E, u, S, B) with $\|u\| \leq \eta$, then

$$\|\pi(b,e)\| = \inf_{s \in S} \|b+s\| + \|e - u(s)\|$$

$$\geq \inf_{s \in S} (\eta(\|s\| - \|b\|) + \|e\| - \eta\|s\|)$$

$$= \|e\| - \eta\|b\|.$$

Conversely, if we assume (*), then we have $\|e\| \leq \eta\|b\|$ for all (b, e) in the kernel of π. Let S be the projection of Ker π onto B. It follows that for all s in S, there is a unique point e in E such that (s, e) is in Ker π. Let us denote it by $e = -u(s)$. It is then easy to check that E_1 is associated to (E, u, S, B).

The preceding definition (*) of η-admissibility has the advantage to make more evident the following observation:

If $j_0 : E \to E_1$, $j_1 : E_1 \to E_2 \ldots$
$j_n : E_n \to E_{n+1}$ are η-admissible embeddings, then the composition $j_n j_{n-1} \ldots j_0 : E \to E_{n+1}$ is also η-admissible.

This can be checked easily by induction once the case $n = 1$ has been verified.

The main result of this section can now be stated:

Theorem 1.6. *Let η be such that $0 \leq \eta < 1$.*

Let $E_0, E_1, \ldots$ be a sequence of finite dimensional (in short f.d.) Banach spaces and let $j_0 : E_0 \to E_1, \ldots, j_n : E_n \to E_{n+1} \ldots$ be a sequence of η-admissible isometric embeddings. Let us denote by X the inductive limit of the system (E_n, j_n). Then X has the R.N.P. and the Schur property.

The proof will use the fact that the embeddings

$$j_{k+m} \circ j_{k+m-1} \ldots \circ j_k : E_k \to E_{k+m+1}$$

satisfy uniformly over k and m a certain inequality for which we introduce the following abbreviated terminology.

Let $\delta > 0$ and let E be any space.

We will say that a subspace N of E is δ-well placed in E if the following property holds.

(1.2) For any probability space $(\Omega, \mathcal{A}, P)$ and for any z in $L^1(P; E)$ such that $\mathbb{E}z \in N$ we have $\mathbb{E}\|z\| \geq \|\mathbb{E}z\| + \delta\mathbb{E}\|q(z)\|_{E/N}$ where $q : E \to E/N$ is the quotient map.

We will use a variant of (1.2) in the case when $\mathbb{E}z$ is not assumed to be in N, but we only assume that $\mathbb{E}z$ is close to N.

Precisely, for all z in $L^1(\Omega, P; E)$ the following is a consequence of (1.2).

$$(1.3) \qquad \mathbb{E}\|z\| \geq \|\mathbb{E}z\| + \delta\mathbb{E}\|q(z)\| - (2+\delta)\|q(\mathbb{E}z)\|.$$

This is easy to check: consider an arbitrary $\varepsilon > 0$, we can find y in N such that

(1.4) $$\| \mathbb{E}z - y \| \le \| q(\mathbb{E}z) \| + \varepsilon.$$

Now we apply (1.2) to the modified variable $\tilde{z} = z - \mathbb{E}z + y$.

This yields to

(1.5) $$\mathbb{E}\| \tilde{z} \| \ge \| y \| + \delta\, \mathbb{E}\| q(z) - q(\mathbb{E}z) \|.$$

On the other hand, we have by the triangle inequality

(1.6) $$\mathbb{E}\| z \| \ge \mathbb{E}\| \tilde{z} \| - \| \mathbb{E}z - y \|$$

and

(1.7) $$\| y \| \ge \| \mathbb{E}z \| - \| \mathbb{E}z - y \|.$$

(1.8) $$\mathbb{E}\| q(z) - q(\mathbb{E}z) \| \ge \mathbb{E}\| q(z) \| - \| q(\mathbb{E}z) \|$$

Combining (1.6), (1.5), (1.7), (1.4) and (1.8) we obtain

$$\mathbb{E}\| z \| \ge \| \mathbb{E}z \| + \delta\, \mathbb{E}\| q(z) \| - (2+\delta)\, \| q(\mathbb{E}z) \| - 2\varepsilon$$

which establishes the announced claim (1.3). Actually, we need to record one more variation of (1.3) involving the conditional expectation with respect to a σ-subalgebra B of $\mathcal{O}$. Indeed, if z is in $L^1(\Omega, \mathcal{O}, P; E)$, we have a.s.

$$\mathbb{E}^B \| z \| \ge \| \mathbb{E}^B z \| + \delta \mathbb{E}^B \| q(z) \| - (2+\delta)\, \| q(\mathbb{E}^B z) \|.$$

Indeed, this is trivial when B is finite and the general case follows easily from the finite case.

Finally, we may integrate the preceding inequality and obtain

(1.9) $$\mathbb{E}\| z \| \ge \mathbb{E}\| \mathbb{E}^B z \| + \delta \mathbb{E}\| q(z) \| - (2+\delta)\, \mathbb{E}\| q(\mathbb{E}^B z) \|.$$

The main technical lemma that we use in this paper is the following.

Lemma 1.7. *Let $\eta \le 1$ be given, let $\delta = \dfrac{1-\eta}{1+\eta}$.*

If N is δ-well-placed in E and if $j: E \to E_1$ is an η-admissible embedding, then $j(N)$ is again δ-well-placed in E_1.

Proof. We can clearly assume (w.l.o.g.) that

$$E_1 = B \oplus E/_{\{(s,\,-us)\,|\, s \in S\}} \text{ with } u: S \to E$$

such that $\| u \| \le \eta$ as before, with $j(e) = \pi((0, e))$ for all e in E, where $\pi: B \oplus E \to E_1$ denotes the quotient map.

Let z_1 in $L^1(\Omega, P; E_1)$ be such that

(1.10) $$\mathbb{E}z_1 \in j(N)$$

For any $\varepsilon > 0$, we can clearly find z' in $L^1(B)$ and z'' in $L^1(E)$ such that, for all w in Ω, we have

$$z_1(w) = \pi(z'(w), z''(w))$$

and

(1.11) $$\| z'(w) \| + \| z''(w) \| \le (1+\varepsilon)\, \| z_1(w) \|.$$

We have $\mathbb{E}z_1 = \pi(\mathbb{E}z', \mathbb{E}z'')$, therefore, we deduce from (1.10) that there exists γ in N such that

$$\pi(\mathbb{E}z', \mathbb{E}z'') = j(\gamma) = \pi((0, \gamma)).$$

In other words, for some s in S we have

$$\mathbb{E}z' = s$$
$$\mathbb{E}z'' = \gamma - u(s)$$

Note that $z'' + u(s) \in E$ and also that

$$\mathbb{E}(z'' + u(s)) = \gamma \in N.$$

Therefore we may apply our hypothesis (1.2) to $z = z'' + u(s)$. This yields

(1.12) $$\mathbb{E}\| z'' + us \| \ge \| \gamma \| + \delta \mathbb{E}\| q(z'' + us) \|.$$

On the other hand, we have clearly: $\| s \| \le \mathbb{E}\| z' \|$ and also:

(1.13) $$\begin{aligned} \mathbb{E}\| z'' \| &\ge \mathbb{E}\| z'' + us \| - \| us \| \\ &\ge \mathbb{E}\| z'' + us \| - \eta \| s \| \\ &\ge \mathbb{E}\| z'' + us \| - \eta \mathbb{E}\| z' \|. \end{aligned}$$

Similarly, we have

(1.14) $$\begin{aligned} \mathbb{E}\| q(z'' + us) \| &\ge \mathbb{E}\| q(z'') \| - \eta \| s \| \\ &\ge \mathbb{E}\| q(z'') \| - \eta\, \mathbb{E}\| z' \| \end{aligned}$$

Combining (1.13) with (1.12) and (1.14), we obtain

$$\mathbb{E}\| z'' \| \ge \| \gamma \| + \delta\, \mathbb{E}\| q(z'') \| - (\eta + \delta\eta)\, \mathbb{E}\| z' \|.$$

This implies by (1.11):

$$\frac{1}{(1+\varepsilon)} \mathbb{E}\| z_1 \| \ge \| \gamma \| + \delta\, \mathbb{E}\| q(z'') \| + [1 - \eta - \delta\eta]\, \mathbb{E}\| z' \|$$

and since $1 - \eta - \eta\delta = \delta$ and $\| \gamma \| = \| j(\gamma) \| = \| \mathbb{E}z_1 \|$, we have finally

$$\frac{1}{1+\varepsilon} \mathbb{E}\| z_1 \| \ge \| \mathbb{E}z_1 \| + \delta \mathbb{E}[\| z' \| + \| q(z'') \|].$$

To conclude, it remains to observe that if we denote by $q_1 : E_1 \to E_1/j(N)$ the quotient map, we have obviously

$$\| z' \| + \| q(z'') \| \geq \| q_1(z_1) \|,$$

so that we reach the announced result

$$\mathbb{E} \| z_1 \| \geq \| \mathbb{E} z_1 \| + \delta \mathbb{E} \| q_1(z_1) \|.$$

Proof of theorem 1.6. Let $E_0 = E$.

Without loss of generality, we may assume that $E_0 \subset E_1 \subset \ldots \subset E_n \subset X$ with $\bigcup E_n$ dense in X, and by Lemma 1.7 we may assume that E_k is δ-well--placed in E_{k+n} for all $k, n \geq 0$. By an obvious approximation argument, if follows that E_k is δ-well-placed in X for all $k \geq 0$.

We will use the following well known characterization of the RNP in terms of martingales (cf. [3] chap. V).

A Banach space X has the RNP if every martingale $(M_n)_{n \geq 0}$ with values in X such that

$$\sup_n \mathbb{E} \| M_n \| < \infty$$

converges almost surely in X.

To prove this, we consider an X-valued martingale $(M_n)_{n \geq 0}$ adapted to an increasing sequence of σ-algebras $(\mathcal{A}_n)_{n \geq 0}$ and such that

$$\sup \mathbb{E} \| M_n \| = c < \infty.$$

We denote by $q_m : X \to X/_{E_m}$ the quotient map. We claim that

(1.15) $$\lim_{m \to \infty} \lim_{n \to \infty} \mathbb{E} \| q_m(M_n) \|_{X/E_m} = 0.$$

Since the spaces E_m are f.d. it is easy to deduce from (1.15) that (M_n) converges almost surely in X.

Indeed, we have by DOOB'S maximal inequality (cf. [3] p. 128):

$$\sup_n \| M_n \| < \infty \quad \text{almost surely and also}$$

$$\lim_{m \to \infty} \downarrow \sup_n \| q_m(M_n) \| = 0 \quad \text{almost surely,}$$

therefore, the random sequence $\{M_n(\omega) \mid n \geq 0\}$ is almost surely a relatively compact subset of X, on which the strong topology agrees with the topology $\sigma(X, D)$ where D is a countable subset of X^*, dense in X^* for $\sigma(X^*, X)$.]

To prove (1.15), we observe first that

$$\| q_m(M_n) \| \quad \text{is non-increasing in } m$$

and that $\mathbb{E} \| q_m(M_n) \|$ is non-decreasing in n, so that both limits in (1.15) are monotone.

Moreover, we have clearly (by definition of the BOCHNER integrability), for each p

(1.16) $$\lim_{m \to \infty} \mathbb{E} \| q_m(M_p) \| = 0.$$

Now, since M_n is a martingale we have, for all $p \leq n$, $M_p = \mathbb{E}(M_n \mid \alpha_p)$ and therefore we deduce from (1.9) that for each m

(1.17) $$\mathbb{E} \| M_n \| \geq \mathbb{E} \| M_p \| + \delta \, \mathbb{E} \| q_m(M_n) \| - (2 + \delta) \, \mathbb{E} \| q_m(M_p) \|.$$

Taking the limit first in n and then in m in (1.17) and using (1.16) we obtain

$$\lim_n \mathbb{E} \| M_n \| \geq \mathbb{E} \| M_p \| + \delta \lim_m \lim_n \mathbb{E} \| q_m(M_n) \|.$$

Now we can let $p \to \infty$, and we obtain

$$\delta \lim_m \lim_n E \| q_m(M_n) \| \leq 0$$

which proves the above claim (1.15) and hence that X has the RNP.

Now we prove that X posseses the Schur property which means (by definition) that weak and strong convergence are equivalent for sequences in X.

We will prove it as follows: we assume the existence of a sequence (x_n) in X which tends weakly to zero and is such that $\| x_n \| > 1$ for all n, and we will reach a contradiction.

By the density of $\cup E_m$ in X, we can assume without loss of generality that $x_n \in \cup E_m$ for all n, or equivalenty that there is an increasing sequence $m_1 < m_2 < \ldots$ such that $x_n \in E_{m_n}$ for each n.

We claim that: (1.18) $\lim_n \| q_m(x_n) \| > \frac{1}{4}$ for each m.

Indeed, if not, we would have for some m, and for n large enough, $\| q_m(x_n) \| < \frac{1}{3}$, hence, for some y_n in E_m, $\| x_n + y_n \| < \frac{1}{3}$; since $\{y_n\}$ is bounded in E_m, we can pass to a subsequence $\{n_k\}$ and obtain y_{n_k} strongly convergent to some element y. Since $\| y_n \| \geq \| x_n \| - \| x_n + y_n \| > \frac{2}{3}$, we have: $\| y \| > \frac{2}{3}$ and, since $\| x_n + y_n \| < \frac{1}{3}$ and $x_{n_k} + y_{n_k} \to y$ weakly, we must have $\| y \| \leq \frac{1}{3}$, which is the desired contradiction, establishing our claim (1.18).

By an obvious inductive selection, we can now find a sequence

$$m'_1 < m'_2 < \dots$$

and a subsequence $\{x'_n\}$ extracted from $\{x_n\}$ such that:

$$\{x'_1, \dots, x'_n\} \in E_{m'_n}$$

and

$$\| q_{m'_n}(x'_{n+1}) \| > \frac{1}{4}.$$

Now, let $r_1, r_2, \dots$ be the RADEMACHER functions on $[0,1]$, and let (α_n) be positive scalars such that $\Sigma\alpha_n < \infty$. We let

$$S_n = \int \| \sum_{i=1}^{n} \alpha_i r_i(t)\, x'_i \| \, dt.$$

Applying (1.17) with $m = m'_n$, we obtain:

$$S_{n+1} \geq S_n + \delta\, |\alpha_{n+1}|\, \| q_{m'_n}(x'_{n+1}) \|,$$

hence

$$S_{n+1} \geq S_n + \frac{\delta}{4} |\alpha_{n+1}|.$$

Hence, we obtain $S_{n+1} \geq \frac{\delta}{4} \Sigma_1^{n+1} \alpha_i$ for all n, and this contradicts (by proposition 0.2) the fact that x'_n tends weakly to 0.

Remark. The preceding argument shows actually that X possesses the strong Schur property in the sense of [8].

2. Applications to $\mathscr{L}^\infty$-spaces

Our main application is the following result.

Theorem 2.1. *Let $\lambda > 1$ and let E be any separable Banach space. Then there is a separable $\mathscr{L}_\lambda^\infty$-space which we will denote by $\mathscr{L}_\lambda[E]$ which contains E isometrically and is such that the quotient space $\mathscr{L}_\lambda[E]/E$ has the RNP and the Schur property.*

Proof. Let $(F_n)_{n\geq 0}$ be an increasing sequence of f.d. subspaces of E s.t. $\bigcup_{n\geq 0} F_n$ is dense in E. Fix $\eta < 1$ such that $\frac{1}{\lambda} < \eta < 1$. We will construct by induction a sequence of η-admissible embeddings

$$j_0 : E \to E_1, \dots j_n : E_n \to E_{n+1}, \dots$$

together with a sequence of f.d. subspaces $G_n \subset E_n$ such that $G_0 = \{0\}$ and

(2.1) $$(j_{n-1} \cdots j_0)(F_{n-1}) \cup j_{n-1}(G_{n-1}) \subset G_n \quad \text{for all } n \geq 1$$

and

(2.2) $$d(G_n, \ell^\infty_{\dim G_n}) \leq \lambda \qquad \text{for all } n \geq 0.$$

Here is how we start: let us fix $\varepsilon > 0$ such that $1 + \varepsilon = \lambda\eta > 1$. We use the fact that, for any $\varepsilon > 0$, any f.d. space is $(1+\varepsilon)$-isomorphic to a subspace of ℓ_∞^m for some suitable m.

Therefore, we can find a subspace S of ℓ_∞^m and an operator $u : S \to E$ such that $u(S) = F_0$, $\| u \| \leq \eta$ and $\| u^{-1}|_{F_0} \| \leq \lambda$. Applying the construction described in lemma 1.1, we find $j_0 : E \to E_1$ and an extension $\tilde{u} : \ell_\infty^m \to E_1$ such that $\tilde{u}_{|S} = j_0 u$, $\| \tilde{u} \| \leq 1$, and if we let $G_1 = \tilde{u}(\ell_\infty^m)$ we have $G_1 \supset j_0(F_0)$ and (cf. remark 1.5) $d(G_1, \ell_\infty^m) \leq \lambda$.

We can then complete the argument by induction on n. Assume that $E_0, \dots, E_n$, $j_0, \dots j_{n-1}$ and $G_1, \dots, G_n$ have been constructed with the required properties. Then, we consider the subspace of E_n spanned by $(j_{n-1} \circ \dots \circ j_0)(F_n) \cup G_n$ and we denote this subspace by H.

By the same argument, as above, we can find a subspace S of ℓ_∞^m (for some suitable m) and an operator $u : S \to E_n$ such that $\| u \| \leq \eta$, $u(S) = H$ and $\| u^{-1}{}_{|H} \| \leq \lambda$. By repeating the same construction as before, we obtain an η-admissible embedding $j_n : E_n \to E_{n+1}$ (associated to $(E_n, u, S, \ell_\infty^m)$) and $\tilde{u} : \ell_\infty^m \to E_{n+1}$ such that if we let $G_{n+1} = u(\ell_\infty^m)$, we have: $d(G_{n+1}, \ell_\infty^m) \leq \lambda$ (by remark 1.5), and, since $u_{|S} = j_n u$ we have: $j_n(H) \subset G_{n+1}$, which shows that both (2.1) and (2.2) are satisfied by G_{n+1}.

This completes the induction argument.

Now, let X be the inductive limit of the system (E_n, j_n). For simplicity, we now consider (E_n) as an increasing sequence of subspaces of X. With this convention, let Y be the closure of $\bigcup G_n$ in X. Clearly Y is a $\mathscr{L}_\infty^\lambda$-space and, by (2.1), Y contains $\overline{\bigcup F_n} = E$. To complete the proof, it remains to analyse the quotient space $Y/_E$. Clearly $Y/_E$ is naturally embedded isometrically into $X/_E$. Finally, the space $X/_E$ can be viewed as an inductive limit of the spaces $E_n/_E$ and by proposition 1.4, the embedding of $E_n/_E$ into $E_{n+1}/_E$ is η-admissible for all $n \geq 1$. This shows that $X/_E$ satisfies the assumption of theorem 1.6, hence it has the RNP and the Schur property, and so does its subspace $Y/_E$. Therefore, we can take $\mathscr{L}_\lambda[E] = Y$.

Corollary 2.2. *For each $\lambda > 1$, there is a separable $\mathscr{L}_\infty^\lambda$ space which fails the RNP but still does not contain any isomorphic copy of c_0.*

Proof. Take e.g. $E = L^1$ and apply the preceding theorem. By proposition 0.1, $\mathscr{L}_\lambda[E]$ does not contain c_0.

Remark 2.3. i) The preceding corollary answers a question raised in [1], p. 46.

ii) Let E be a space failing the RNP but still not containing c_0 or L^1 (cf.e.g. [9]). Then the space $\mathscr{L}_\lambda[E]$ will be a $\mathscr{L}_\infty^\lambda$-space with similar properties.

From proposition 0.1, we derive immediately the following:

Corollary 2.4. *Let P be any of the properties considered in proposition* 0.1. *Then, for any* $\lambda > 1$, *any separable Banach space E with property P embeds isometrically in a separable* $\mathscr{L}_\infty^\lambda$ *space with property P.*

For the definition and the first properties of the projective tensor product of Banach spaces, we refer to [4], [3] or [7].

Corollary 2.4. *For each* $\lambda > 1$, *there is a* $\mathscr{L}_\infty^\lambda$*-space X which is weakly sequentially complete (in short w.s.c.) and has the RNP (hence it does not contain* c_0*) but the projective tensor product* $X \hat{\otimes} X$ *contains* c_0 *isomorphically.*

Proof. We take $E = \ell_2$ and let $X = \mathscr{L}_\lambda[E]$. By corollary 2.3, X has the RNP and is w.s.c. Let (e_n) be the canonical basis of ℓ_2 considered as a subspace of X. To show $X \otimes X$ contains c_0, we will use a classical theorem of Grothendieck (cf. [4]).

Let v be an element of $\ell_2 \otimes \ell_2$. We may consider v as a finite rank operator on ℓ_2. Let $J : \ell_2 \to L_\infty(\mu)$ be an isometric embedding.

Then the tensor $(J \otimes J)(v)$ in $L^\infty(\mu) \otimes L^\infty(\mu)$ (which corresponds to the composed operator $J v J^*$) satisfies

$$\|(J \otimes J)(v)\|_{L^\infty(\mu) \hat{\otimes} L^\infty(\mu)} \leq K_G \|v\|$$

where K_G is an absolute constant (the so-called Grothendieck's constant).

It follows that for any sequence of scalars (α_n), we have, for any N,

$$\|\sum_1^N \alpha_i e_i \otimes e_i\|_{X \hat{\otimes} X} \leq \lambda K_G \text{ Sup } |\alpha_i|. \tag{2.3}$$

On the other hand,

$$\|\sum_i^N \alpha_i e_i \otimes e_i\|_{X \hat{\otimes} X} \geq \|\sum_i^N \alpha_i e_i \otimes e_i\|_{X \check{\otimes} X} = \text{Sup } |\alpha_i|. \tag{2.4}$$

Therefore, the sequence $\{e_n \otimes e_n \mid n \in \mathbb{N}\}$ spans a subspace isomorphic to c_0 in $X \hat{\otimes} X$.

Remark 2.5. The preceding corollary yields a negative answer to the question of [3] p. 258: is the RNP stable by the projective tensor product?

Remark 2.6. By another application of theorem 1.6, we find that the example constructed in [7], of a Banach space X such that $X \check{\otimes} X = X \hat{\otimes} X$ and X and X^* are both of cotype 2, can be constructed with the RNP and the Schur property.

Remark. The proof of theorem 2.1 can be easily adapted to yield (using remark 1.5 and letting η approach 1) a construction of the Gurarii space (cf. [5]). The possibility of such a construction already has been known for some time to J. LINDENSTRAUSS.

Remark 2.7. Let X be the space considered above in the proof of corollary 2.4. Then, for any norm α on $X \otimes X$ such that $\| \|_\vee \leq \alpha \leq \| \|_\wedge$, the completed tensor product $X \hat{\otimes}_\alpha X$ contains c_0. This follows immediately from (2.3) and (2.4). Therefore, not only the projective tensor product, but any reasonable tensor product, fails to preserve the RNP.

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