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Minimal hypersurfaces of S^4 with non zero Gauss-Kronecker curvature

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Introduction.

This paper deals with minimal hypersurfaces of the 4-sphere with non zero Gauss-Kronecker curvature.

In section 1 we use the associated Gauss map of an immersion $f: M^3 \to S^4$ (i.e., the translation of the unit normal in S^4 to the origin of R^5) to show that if f is minimal and the Gauss-Kronecker curvature is $\neq 0$ then M admits a metric with scalar curvature $\varkappa \equiv 6$. We recall (see [4]) that any compact orientable 3-manifold M admits a unique decomposition $M = M_1 \# \ldots \# M_n$ into a connected sum of prime 3-manifolds. A prime 3-manifold which is not diffeomorphic to $S^1 \times S^2$ is either a $K(\pi, 1)$-manifold or it is covered by a homotopy 3-sphere. In [3], Gromov--Lawson proved that the existence of positive scalar curvature implies that M has no $K(\pi, 1)$ factor in its prime decomposition. We use this to show the following result.

Theorem 1.2. *Let $f: M \to S^4$ be an immersion of a compact orientable 3-manifold having Gauss-Kronecker curvature $K \neq 0$, then*

$$M = X_1 \# \ldots \# X_\ell \# S^1 \times S^2 \# \ldots \# S^1 \times S^2$$

where each X_j is covered by a 3-manifold which is homotopy equivalent to S^3.

We note that in theorem 1.2 the same conclusion holds if the mean curvature H of the immersion satisfy the inequality $HK^{-1} + 3 > 0$. This is a weaker and open condition which is much easier to be verified.

A compact surface Σ embedded in a manifold X is called incompressible if its Euler characteristic $\chi(\Sigma) \leq 0$ and the homomorphism between fundamental groups $\pi_1 \Sigma \to \pi_1 X$ is injective. In [9], Schoen-Yau proved that any 3-manifold which admits an incompressible surface cannot carry a complete metric of positive scalar curvature. Given that $f: M^3 \to S^4$ is a complete immersion with principal curvatures bounded away from

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zero, then its associated Gauss map is also a complete immersion. We therefore get the following result.

Theorem 1.5. *Let $f: M^3 \to S^4$ be a complete minimal immersion of an oriented 3-manifold. If the principal curvatures are bounded away from zero, then M cannot carry an incompressible surface.*

In section 2 we consider compact minimal hypersurfaces of S^4 with Gauss-Kronecker curvature $\neq 0$ and constant scalar curvature. We use the moving frame method to show the following uniqueness result.

Theorem 1.12. *Let $f: M \to S^4$ be an immersion of a compact oriented 3-manifold having constant scalar curvature $\varkappa$. Assume that the Gauss--Kronecker curvature of the immersion is $\neq 0$ everywhere. Then $\varkappa \equiv 3$, i.e., M is the Clifford torus in S^4.*

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§1. The associated Gauss map and positive scalar curvature metrics.

Let $\psi: M \to S^n$ be an immersion of an oriented $(n\text{-}1)$-dimensional manifold M into the n-sphere $S^n \subset \mathbb{R}^{n+1}$. The associated Gauss map $\psi^*: M \to S^n$ is defined pointwise as the image of the unit normal in S^n translated to the origin of $\mathbb{R}^{n+1}$. The square of the length of the second fundamental form of M is given by

$$S = H^2 - \varkappa + n(n-1)$$

where H and $\varkappa$ are respectively the mean and scalar curvature of the immersed manifold M.

Remark. Let v be a unit normal vector at $p \in M$ and let $\lambda_1, \ldots, \lambda_{n-1}$ be the *principal curvatures* of ψ at p, i.e., the eigenvalues of the form II $(V, W) = \langle \nabla_V W, v \rangle$ on T_pM. With this notation we have

$$H = \Sigma \lambda_i$$
$$\varkappa = n(n-1) + 2 \sum_{i<j} \lambda_i \lambda_j$$
$$K = \lambda_1, \ldots, \lambda_{n-1}$$

where H, $\varkappa$, K are respectively the *mean, scalàr and Gauss-Kronecker curvature* of ψ at p.

Theorem 1.1. *Let $\psi: M^3 \to S^4$ be an immersion with Gauss-Kronecker curvature $K \neq 0$. The associated Gauss map is an immersion and ψ is minimal $\Leftrightarrow \varkappa^* \equiv 6$ where $\varkappa^*$ is the scalar curvature of ψ^*.*

Proof. We choose a local orthonormal frame field e_A in S^4 such that when restricted to M, e_1, e_2, e_3 are tangent to M and $e_4 = \psi^*$. We denote θ_A its dual coframe and by θ_{AB} its connection forms. We will assume that the second fundamental form is diagonalized, i.e.,

$$\theta_{i4} = \lambda_i \theta_i, \quad i = 1, 2, 3.$$

It follows that

$$d\psi^* = de_4 = \theta_{41} e_1 + \theta_{42} e_2 + \theta_{43} e_3$$
$$= -\lambda_1 \theta_1 e_1 - \lambda_2 \theta_2 e_2 - \lambda_3 \theta_3 e_3$$

and therefore ψ^* is an immersion. We may also choose a local orthonormal frame e_A^* in S^4 such that $e_i^* = e_i$, $i = 1, 2, 3$, are tangent to $\psi^*(M)$ and $e_4^* = \psi$. Since

$$d\psi^* = \theta_1^* e_1^* + \theta_2^* e_2^* + \theta_3^* e_3^*$$

it follows that for $i = 1, 2, 3$

$$\theta_i^* = \theta_{4i} = -\lambda_i \theta_i .$$

Similarly,

$$\theta_i = \theta_{4i}^* .$$

Comparing the two equations, we may write

$$\theta_{i4}^* = \lambda_i^* \theta_i^*$$

where $\lambda_i^* = \lambda_i^{-1}$. Therefore,

$$\varkappa^* = 6 + 2[\lambda_1^* \lambda_2^* + \lambda_1^* \lambda_3^* + \lambda_2^* \lambda_3^*]$$
$$= 6 + 2\left[\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3}\right]$$
$$= 6 + \frac{2}{\lambda_1\lambda_2\lambda_3}[\lambda_3 + \lambda_2 + \lambda_1]$$
$$= 6 + 2H/K.$$

Remark. Since $S^* = (H^*)^2 - \varkappa^* + 6$ we may replace the condition $\varkappa^* \equiv 6$ in Lemma 3.1 by $S^* = (H^*)^2$. This implies that $\psi^*: M \to S^4$ is never minimal, i.e., $H^* \neq 0$, in contrast with the 2-dimensional case, where if $\psi: M^2 \to S^3$

is a minimal immersion with Gauss curvature $\neq 0$, then the associated Gauss map, $\psi^*: M^2 \to S^3$ is alawys a minimal immersion (c.f. [5]).

We now recall (see [4]) that each compact 3-manifold M can be expressed in a unique way as a connected sum of a finite number of prime 3-manifolds, i.e.,

$$M = X_1 \# \dots \# X_\ell \# (S^1 \times S^2) \# \dots \# (S^1 \times S^2) \# K_1 \# \dots \# K_n$$

where each $\pi_1(X_j)$ is finite and K_j is a $K(\pi, 1)$-manifold. In particular one has the following nice consequence of Theorem 1.1.

Theorem 1.2. *Let M be a compact orientable 3-manifold. If for some immersion $f: M \to S^4$ the Gauss-Kronecker curvature, K, is always $\neq 0$ and $-3 < HK^{-1}$, then M carries no $K(\pi, 1)$ factor in its prime decomposition.*

Proof. By theorem 1.1 we know M carries a metric with scalar curvature $k^* = 6 + 2HK^{-1} > 0$. Now assume M has a $K(\pi, 1)$ factor in its prime decomposition. By a result of Gromov-Lawson [3], any metric with scalar curvature ≥ 0 on M is flat. This proves the theorem.

In the following theorem we need the notion of an incompressible surface.

Definition 1.3. *A compact surface Σ embedded in a manifold X is called incompressible if $\chi(\Sigma) \leq 0$ and if the homomorphism $\pi_1 \Sigma \to \pi_1 X$ is injective.*

Theorem 1.4. (Schoen-Yau) *Any 3-manifold which admits an incompressible surface cannot carry a complete metric of positive scalar curvature.*

Proof. See [9].

Theorem 1.5. *Let $f: M \to S^4$ be a complete minimal immersion of an oriented 3-manifold. If the principal curvatures are bounded away from zero, then M cannot carry an incompressible surface.*

Proof. The scalar curvature of the metric induced by the associated Gauss map $f^*: M \to S^4$ is identically 6. The principal curvatures of f beeing bounded away from zero implies that f^* is complete. The theorem follows.

Theorem 1.6. *Let $f: M \to S^4$ be an immersion as in Theorem 1.5, then M has no covering by a manifold $\widetilde{M}^3$ where $\widetilde{M}^3$ is diffeomorphic to the interior of a compact 3-manifold X such that $\pi_1(\partial X) \neq 0$.*

Proof. If we assume the conclusion of the theorem false, then we may apply a result of Gromov-Lawson (cf. [3]) to conclude that M carries no complete metric of uniformly positive scalar curvature. But the metric induced by the associated Gauss map is a complete metric of positive scalar curvature. This is a contradiction.

In general for minimal immersion $M^3 \to S^m$, $m > 4$, we still have similar results about the scalar curvature. For this we need the following well-known fact.

Lemma 1.7. *Let M be an n-manifold minimally immersed in the sphere S^m, $m > n+1$. Let N_1 be the unit normal sphere bundle of the immersion. If at $(x, v) \in N_1$ the principal curvature $\lambda_1(x, v), \dots, \lambda_n(x, v)$, i.e., the eigenvalues of the second fundamental form $A^v \equiv -(\nabla v)^T$, are non-zero, then the polar mapping*

$$\ell: N_1 \to S^m$$

given by $\ell(x, v) = v$ is an immersion near (x, v). At such a point the principal curvatures of the immersion ℓ are

$$1/\lambda_1(x, v), \dots, 1/\lambda_n(x, v), 0, \dots, 0.$$

Proof. That ℓ is an immersion is easily seen. To prove the other part we first extend v_x to a unit normal vector field v so that

$$(\nabla v)_x^N = 0.$$

Here $(\;)^N \equiv 1 - (\;)^T$, where $(\;)^T$ denotes orthogonal projection onto $T_x M$. Now we choose a local orthonormal frame $e_1, \dots, e_n$ on M satisfying

$$-(\nabla_{e_i} v)^T = \lambda_i(x, v) e_i \text{ at } x.$$

Then we complete the e_i's to a basis e_i, e_α for N_1. Obviously the position vector X of M is orthogonal to e_A, $A = 1, \dots, m-1$ and also to v. Hence, X is the unit normal to the immersion ℓ. To obtain the principal curvatures of the immersion ℓ we observe that

$$\nabla_{\ell * e_i} X = e_1[X] = dX(e_i) = e_i$$

where, at x, $\ell * e_i = (\nabla_{e_i} v)^T = -\lambda_i(x, v)\, e_i$. It follows then that

$$\nabla_{\ell * e_i} X = -\frac{1}{\lambda_i(x, v)} \ell * e_i.$$

We also have

$$\nabla_{\ell * e_\alpha} X = e_\alpha[X] = 0.$$

Example 1.8. Let Σ^2 be the Veronese surface embedded in S^4. It is well known that the principal curvatures are all constant $1/\sqrt{3}$, $-1/\sqrt{3}$ for all directions $\nu \in N_1$. Therefore $\ell : N_1 \to S^4$ is an immersion with principal curvature

$$-\sqrt{3},\ 0,\ \sqrt{3}.$$

Remark. It is a well known fact that the veronese is the focal variety of an isoparametric family of hypersurfaces M_t^4 in S^4 (c.f. [6]). Among this family there is one that has constant principal curvatures $-\sqrt{3}, 0, \sqrt{3}$. The above example, due to the work of Cartan, may be pictured as the polar mapping of the Veronese surface $\Sigma \to S^4$.

Lemma 1.9. *Let $M^3 \to S^m$ be a minimal immersion and suppose the conditions of the lemma 1.7 are satisfied. Then we have an immersion*

$$\ell : N_1 \to S^m$$

with constant scalar curvature $\varkappa \equiv m(m-1)$.

Proof. The principal curvatures of the immersion ℓ are in this case

$$\frac{1}{\lambda_1(x,\nu)},\ \frac{1}{\lambda_2(x,\nu)},\ \frac{1}{\lambda_3(x,\nu)},\ 0, \ldots, 0$$

and the scalar curvature is given by

$$\varkappa = 2\left[\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3}\right] + m(m-1)$$

$$= \frac{1}{\lambda_1\lambda_2\lambda_3}\,[\lambda_1 + \lambda_2 + \lambda_3] + m(m-1)$$

$$= m(m-1).$$

Theorem 1.10. *Let M be a compact orientable 3-manifold and let $f : M \to S^5$ be an immersion as in Lemma 1.9. If M has trivial normal bundle, then M cannot have a $K(\pi, 1)$ factor in its prime decomposition.*

Proof. Let N_1 be the unit normal sphere bundle of the immersion f. The polar mapping induces a metric of positive scalar curvature on N_1. By minimizing mass in $[M] \in H_3(N_1)$ we get a smooth stable hypersurface $\mathfrak{M}^3 \subset N_1 = M \times S^1$. By projection of N_1 onto M we obtain a degree-1 mapping of $\mathfrak{M}^3$ onto M. It follows by a theorem of Shoen-Yau (c.f. [8])

that $\mathfrak{M}^3$ has a metric of scalar curvature $\varkappa > 0$. Now assume M has a $K(\pi, 1)$-factor in its prime decomposition, i.e., $M^3 = M_0^3 \# K^3$, where K^3 is a $K(\pi, 1)$ 3-manifold. The composition

$$\mathfrak{M}^3 \xrightarrow[\text{degree-1}]{} M^3 = M_0^3 \# K^3 \to K^3$$

gives a degree-1 map of $\mathfrak{M}^3$ onto K^3. To complete the proof of the theorem we need the following lemma.

Lemma 1.11. *A compact orientable 3-manifold $\mathfrak{M}$ which admits a degree-1 map onto a compact $K(\pi, 1)$ 3-manifold cannot carry a metric of positive scalar curvature.*

Proof. $\mathfrak{M}$ must have $K(\pi, 1)$ factors in its prime decomposition. If not,

$$\mathfrak{M} = X_1 \# \ldots X_n \# (S^1 \times S^2) \# \ldots \# (S^1 \times S^2)$$

and $\Gamma \equiv \pi_1(\mathfrak{M})$ is the free product

$$F_1 * \ldots * F_n * \mathbb{Z} \ldots * \mathbb{Z}$$

where $F_j = \pi_1(X_j)$ is finite for $j = 1, \ldots, n$. It follows that

$$H^3(\Gamma)/\text{mod. torsion} = 0.$$

Now let $F : \mathfrak{M}^3 \to K$ be a degree-1 map of $\mathfrak{M}^3$ onto a $K(\pi, 1)$ 3-manifold. The map F is determined by the homomorphism

$$\pi_1\,\mathfrak{M}^3 \xrightarrow{F_*} \pi_1\,K = \pi$$

and the composition

$$\mathfrak{M}^3 \xrightarrow{\text{universal}} K(\pi_1\,\mathfrak{M}^3, 1) \xrightarrow{F_*} K$$

is homotopic to the map F. Let g be a generator of $H^3(\pi) = \mathbb{Z}$, then $F^* g \neq 0$ and therefore $\mathbb{Z} \subset H^3(\Gamma)$. This proves the lemma.

§2. A uniqueness theorem.

In this section we want to prove the following uniqueness theorem:

Theorem 1.12. *Let $f : M^3 \to S^4$ be a minimal immersion of a compact oriented 3-manifold having constant scalar curvature $\varkappa$. Assume that the Gauss-Kronecker curvature of the immersion is $\neq 0$ everywhere. Then $\varkappa \equiv 3$, i.e., M is the Clifford torus in S^4.*

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be the principal curvatures of the immersion. Since f is minimal the λ_i's satisfy the equation

$$(*) \qquad \lambda_i^3 - \frac{1}{2} S \lambda_i - K = 0$$

for $i = 1, 2, 3$. Without loss of generality we may as well assume $K > 0$. For a given x on M, we consider the polynomial $P(\lambda) = \lambda^3 - (1/2)S\lambda - K(x)$. From (*) it follows that the equation $P(\lambda) = 0$ has three real roots. This implies

$$0 < K(x) \leq (S^3/54)^{1/2}.$$

The quality is reached only at points where two of the principal curvatures are equal. Since M is compact we may choose a point $p \in M$ so that

$$0 < K(p) \leq K(x) \;\forall\; x \in M.$$

If two of the principal curvatures are equal at p, then K is constant on M and M is an isoparametric hypersurface. Using Cartan's fundamental equation for isoparametric hypersurfaces (c.f. [6]), we obtain that the two distinct principal curvatures satisfy

$$1 + \lambda_1 \lambda_2 = 0.$$

This implies $\lambda_1 = \sqrt{2}, \lambda_2 = \lambda_3 = -\sqrt{2/2}$. Therefore, M is the Clifford torus. We may then assume that the principal curvatures are all distinct at p. Therefore, we may choose a local orthonormal frame, say e_A, on S^4, such that restricted to M, e_1, e_2, e_3 give the principal directions. We need the laplacian of K. For this we denote by ω_A the dual coframe and write the structure equation of S^4 as

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \; \omega_{AB} + \omega_{BA} = 0$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \omega_A \wedge \omega_B.$$

In general we have $\omega_{4i} = \Sigma\, h_{ij}\, \theta_j$, $h_{ij} = h_{ji}$. In our case the second fundamental form

$$h = \Sigma h_{ij}\, \omega_i\, \omega_j$$

is diagonalized, i.e., $h_{ij} = \lambda_i \delta_{ij}$. We now recall that the components h_{ijk} of the covariant derivative, Dh, of h are given by

$$\sum_k h_{ijk}\, \omega_k = dhij - \sum_m h_{im}\, \omega_{mj} - \sum_m h_{mj}\, \omega_{mi}.$$

One easily sees that h_{ijk} is symmetric in all indices. By a long but standard computation (see [7]) one proves that

$$\Delta f_3 = 3[(3-S) f_3 + 2 \sum_{i,j,k} h_{ijk}^2 \lambda_i]$$

where $f_3 \overset{\text{def}}{=\!=} \sum_{i,j,k} h_{ij}\, h_{jk}\, h_{ki} = \Sigma \lambda_i^3$. But since the λ_i's satisfy the equation

$$\lambda_i^3 - \frac{S}{2} \lambda_i \equiv K$$

it follows that $f_3 \equiv 3K$. From this we have

$$\Delta K = 3(3-S)K + 2 \sum_{i,j,k} h_{ijk}^2 \lambda_i.$$

Obs. 1. Differentiating the equality $\Sigma \lambda_i^2 \equiv$ constant, one obtains

$$(\lambda_i - \lambda_k) \nabla \lambda^i + (\lambda_j - \lambda_k) \nabla \lambda_j \equiv 0$$

for $i \neq j$, $i \neq k$, $j \neq k$.

Obs. 2. Let $\tau = (ijk)$ be a permutation of $(1, 2, 3)$. Then we have

$$\begin{aligned} -\nabla K &= \nabla(\lambda_i \lambda_j^2 + \lambda_j \lambda_i^2) \\ &= \lambda_i(2\lambda_j + \lambda_i) \nabla \lambda_j + \lambda_j(2\lambda_i + \lambda_j) \nabla \lambda_i \\ &= \lambda_i(\lambda_j - \lambda_k) \nabla \lambda_j + \lambda_j(\lambda_i - \lambda_k) \nabla \lambda_i \\ &= (\lambda_j - \lambda_i)(\lambda_i - \lambda_k) \nabla \lambda_i. \end{aligned}$$

Since p is a minimum for K we have that

$$\nabla K(p) = 0$$
$$\Delta K(p) \geq 0.$$

Using Obs. 2 we obtain $\nabla \lambda_i(p) = 0$. Also we have

$$\begin{aligned} h_{iik}(p) &= dhii(e_k) - 2 \sum_m h_{im}\, \omega_{mi}(e_k) \\ &= d\lambda_i(e_k) \\ &= 0, \end{aligned}$$

then, at p

$$\Delta K = 3(3-S)\,K + \\ + 2[h_{123}^2 \lambda_1 + h_{132}^2 \lambda_1 + h_{213}^2 \lambda_2 + h_{231}^2 \lambda_2 + h_{312}^2 \lambda_3 + h_{321}^2 \lambda_3].$$

Since the h_{ijk} are symmetric in all indices, it follows that

$$\Delta K(p) = 3(3-S)\,K + 4h_{123}^2(\lambda_1+\lambda_2+\lambda_3)$$

or

$$\Delta K(p) = 3(3-S)\,K(p) \geq 0.$$

This will imply $S \equiv 3$, i.e., M is the Clifford torus in S^4. This shows that the assumption that the principal curvatures are all distinct at p is impossible.

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