BOL. SOC. BRAS. MAT., VOL. 14 N.º 2 (1983), 147-162

Mechanical and gradient systems; local perturbations and generic properties

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1. Introduction.

The Kupka-Smale theorem [1, 2, 3] states that for "generic" dynamical systems, or differential equations, all singularities and closed orbits are hyperbolic and that the stable and unstable manifolds intersect transversally. In generalizations to special classes of dynamical systems, e.g. volume preserving–, Hamiltonian– or geodesic – flows, the main problem is to prove that there are enough "local perturbations", see [4, 5, 6, 7, 8]. In this paper we consider the problem of local perturbations for gradient systems and mechanical systems, which we define as follows.

Let (M, g) be a Riemannian manifold. A *gradient system* on (M, g) is a vector field X on M which is the g-gradient of some function $f : M \to \mathbb{R}$, i.e., $df = g(X, -)$. A *mechanical system* on (M, g) is a vector field X on the co-tangent bundle $T^*(M)$ of M, which can be obtained in the following way from a potential function $V : M \to \mathbb{R}$:

– let ω denote the canonical symplectic form on $T^*(M)$;
– let $K: T^*(M) \to \mathbb{R}$, the kinectic energy, be the function which assigns to each co-vector α the value $\frac{1}{2} \| \alpha \|_g^2$, where $\| \;\; \|_g$ denotes the induced norm in $T^*(M)$;
– let $\pi: T^*(M) \to M$ denote the canonical projection;

then X, or X_V, is the Hamiltonian vector field on $(T^*(M), \omega)$ with Hamiltonian $H_V = K + V_0\pi$. We observe that X, or X_V, is also called the Hamiltonian vector field *of* V.

Our results on mechanical systems only deal with the cases where the curvature is identically zero. I think that without this restriction, the results remain true, but are much harder to prove. Since the main examples of actual mechanical systems, like the n-body problem, satisfy

Part of this research was carried out at IMPA in Rio de Janeiro with financial support of CNPq (Brazil).
Recebido em 01/03/84.

this condition of zero curvature, I decided not to pursue this matter. In this case of zero curvature one can always choose local coordinates with respect to which the Riemannian metric is just the Euclidean inner product. If $x_1, \ldots, x_n$ are such coordinates, the flow of X_V in $T^*(M)$ has trajectories which project on solutions of the second order equation

$$\ddot{x}_i(t) = -\frac{\partial V}{\partial x_i}(x(t)).$$

The difference between the perturbation arguments in the above mentioned generalizations and the perturbation results we prove here is that for the present systems we only show that *generically* they admit enough perturbations; for definitions and statement of the results see section 2. This idea of making a system "pertubable" was used earlier to make exponential maps generic [9]. Our results on local perturbations can be used to prove Kupka-Smale type theorems for gradient systems and mechanical systems, also when they depend on parameters. This is indicated in section 5.

Our results on mechanical systems imply that they are generically as complicated as general Hamiltonian systems.

2. Local perturbations, statement of the main results.

Let M be a manifold and $\mathbf{X}$ a vector space of smooth (i.e. C^∞) vector fields on M. Consider some vector field $X \in \mathbf{X}$ and a point $p \in M$ such that $X(p) \neq 0$. We denote the time t map of X by X_t; since our considerations are local we ignore the fact that the flow of X may be incomplete. Let Σ_0 be a local section of X at p and Σ_t a local section at $X_t(p)$. Poincaré maps $P_{t,X}: \Sigma_0 \to \Sigma_t$ are defined (in a neighbourhood of p in Σ_0) by: $P_{t,X}(q)$ is the first intersection of the X integral curve through q with Σ_t. For X' near X, $P_{t,X'}$ is defined by using X' integral curves instead of X integral curves (but using the same sections Σ_0 and Σ_t).

From this, we get for each $t > 0$ and $k \in \mathbb{N}$ a map $J_{p,t,k}$, defined on a "neighbourhood" of X in $\mathbf{X}$ which assigns to X' near X the k-jet of $P_{t,X'}$ at p. We say that, for some $t > 0$ and $k \in \mathbb{N}$ this map has maximal rank if there is a finite dimensional linear subspace $\tilde{\mathbf{X}} \subset \mathbf{X}$ such that each $\tilde{X} \in \tilde{\mathbf{X}}$ is zero on a neighbourhood of both p and $X_{t(p)}$ and such that the map, from $\tilde{\mathbf{X}}$ to $J^k((\Sigma_0, p), \Sigma_t)$ which assigns to $\tilde{X} \in \tilde{\mathbf{X}}$ the k-jet of $P_{t,X+\tilde{X}}$ at p, has maximal rank at zero; note that this maximal rank does not depend on the choice of Σ_0 and Σ_t. We call X *k-perturbable* (*in* $\mathbf{X}$) *at the point* p if for some $t_0 > 0$, $J_{p,t,k}$ has maximal rank for all $t \in (0, t_0)$. We say that X is *k-perturbable* if X is k-perturbable at each point p with $X(p) \neq 0$.

We observe that if $\mathbf{X}$ is a vector space of Hamiltonian vector fields on a symplectic manifold, then no $X \in \mathbf{X}$ is k-perturbable for $k \geq 1$. This is a consequence of the fact that X_t has to preserve the symplectic structure. In fact, for $X \in \mathbf{X}$ with Hamiltonian H_X we consider Poincaré maps $P_{t,X}: \Sigma_0 \to \Sigma_t$ as above. These Poincaré maps preserve H_X, i.e.,

$$H_X \mid \Sigma_0 = (H_X \mid \Sigma_t) \circ P_{t,X}.$$

Also on each level of $H_X \mid \Sigma_0$ or $H_X \mid \Sigma_t$, there is an induced symplectic structure; $P_{t,X}$ preserves these symplectic structures. We say that a map from Σ_0 to Σ_t is *parameter symplectic* if it both preserves the levels of H_X and the symplectic structure on these levels. In order to modify the definition of k-perturbable for the Hamiltonian context, we require that $\tilde{\mathbf{X}} \subset \mathbf{X}$, as above, consists of vector fields $\tilde{X}$ for which the Hamiltonian $H_{\tilde{X}}$ can be chosen to be zero in a neighbourhood of both p and $X_t(p)$ (so that the levels of H_X and $H_X + H_{\tilde{X}}$ are the same in Σ_0 and Σ_t); in the Hamiltonian case we also replace $J^k((\Sigma_0, p), \Sigma_t)$ by $J^k_{P.S}((\Sigma_0, p), \Sigma_t)$, the space of k-jet of parameter symplectic maps from Σ_0 to Σ_t at p. In this way one defines *k-perturbable* (*in the Hamiltonian sense*).

Our main results on local perturbations are:

Theorem A. *Let (M, g) be a Riemannian manifold. For each $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that the set of functions $f: M \to \mathbb{R}$, for which $\operatorname{grad}_g f$ is k-perturbable, within the class of g-gradient vector fields, is residual in the C^N-topology. In fact, for each compact $K \subset M$, the set F_K of functions $f: M \to \mathbb{R}$ which* (a) *have no critical point in K and* (b) *have k-perturbable g-gradient in each point of K, is open in the C^N-topology; F_K is open and dense in the* (*open*) *subset of functions having no critical point in K.*

Theorem B. *Let (M, g) be a Riemannian manifold with zero curvature. For each $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that the set of functions $V: M \to \mathbb{R}$, for which the corresponding Hamiltonian vector field X_V is k-perturbable in the Hamiltonian sense* (*within the class of mechanical systems*), *is residual in the C^N-topology. In fact, for each compact set $K \subset T^*(M)$, disjoint from the zero section, the set H_K of functions $V: M \to \mathbb{R}$, such that X_V is k-perturbable in all points of K, is open and dense.*

An example of a gradient system which is not k-perturbable, $k \geq 1$, can easily be given: $M = \mathbb{R}^n$, $n \geq 3$, g is the Euclidean metric and $f = x_1$, so that $X = \operatorname{grad} f = \dfrac{\partial}{\partial x_1}$. We can take $\Sigma_0 = \{x_1 = 0\}$ and $\Sigma_t = \{x_1 = t\}$.

If $\widetilde{X} = \operatorname{grad} \widetilde{f}$ and $\widetilde{f}$ has support between Σ_0 and Σ_t, then $\frac{\partial}{\partial\varepsilon}(P_{t,X+\varepsilon\widetilde{X}})|_{\varepsilon=0}$, as a vector field along the map $P_{t,X}: \Sigma_0 \to \Sigma_t$ can be presented by a *gradient* vector field on Σ_t. This implies that X is not k-perturbable.

I think the following is an example of a non-k-perturbable mechanical system (for k sufficiently big): $M = \mathbb{R}^n$, with n sufficiently big, g is the Euclidean metric and $V \equiv 0$. In any case, this example is not k-general in the way defined in the next section.

3. On perturbality.

In this section we analyse the condition of perturbability, introduced in the last section. In the next section we prove the theorems A and B. Since we have to analyse the effect of perturbations on the Poincaré maps we need to know how a time t map depends on perturbations. For general results we refer to [10; Chapter V].

Let X_ε be a vector field on a manifold M (or on $\mathbb{R}^n$) depending smoothly on ε. We want to study the dependence of $\varphi_{\varepsilon,t}$, the time t map of X, on ε, especially near points where X_0 is non-zero. Without loss of generality we may assume that, with respect to local coordinates, X_0 has the form $\frac{\partial}{\partial x_1}$, so that $\varphi_{0,t}(x_1, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n)$. It is immediate that

$$\frac{\partial}{\partial\varepsilon}\varphi_{\varepsilon,t}(x)\,|_{\varepsilon=0} = \int_0^t (\varphi_{t-s})_* \left(\frac{\partial X_\varepsilon}{\partial\varepsilon}(\varphi_s(x))\,|_{\varepsilon=0}\right) ds,$$

where we use φ_s, or φ_{t-s}, as shorthand for $\varphi_{0,s}$, or $\varphi_{0,t-s}$. One should notice that $\frac{\partial}{\partial\varepsilon}\varphi_{\varepsilon,t}|_{\varepsilon=0}$ is a vector field along φ_t, i.e., $\left(\frac{\partial}{\partial\varepsilon}\varphi_{\varepsilon,t}|_{\varepsilon=0}\right)(x)$ is a vector in $T_{\varphi_t(x)}(M)$. The above formula follows by differentiating and integrating $\frac{\partial}{\partial t}\varphi_{\varepsilon,t}(x) = X_\varepsilon(\varphi_{\varepsilon,t}(x))$.

Next we take sections Σ_0 and Σ_t: we choose $p \in M$ so that $X_0(p) \neq 0$, take Σ_0 a section for X_0 through p and Σ_t a section for X_0 through $\varphi_t(p)$. Without loss of generality we way even assume that $\varphi_t(\Sigma_0) = \Sigma_t$. Let $P_\varepsilon: \Sigma_0 \to \Sigma_t$ be the Poincaré map of X_ε. Then

$$\frac{\partial}{\partial\varepsilon}P_\varepsilon(x)\,|_{\varepsilon=0} = \int_0^t (\varphi_{t-s})_* \left(\frac{\partial X_\varepsilon^n}{\partial\varepsilon}(\varphi_s(x))\,|_{\varepsilon=0}\right) \mathrm{d}s,$$

where, for $Y \in T_y(M)$, $y \in \Sigma_t$, Y^n denotes the projection of Y into $T_y(M)/\langle X_0(y)\rangle$; for y in Σ_t we identify this quotient with $T_y(\Sigma_t)$. Strictly speaking this last formula holds only if $\varphi_t(\Sigma_0) = \Sigma_t$, or if X_ε is independent of ε in the points of Σ_0 and Σ_t.

Next we need formulas for the dependence of the k-jet of P_ε on ε. Observe that if Y is a vector field on a manifold M, whose $(k-1)$-jet is zero in some $q \in M$, then there is a unique homogeneous polynomial vector field $\widetilde{Y}_q$ of degree k on $T_q(M)$ such that Y and $(\mathrm{Exp}_q) * \widetilde{Y}_q$ have the same k-jet in q if Exp_q: $T_q(M) \to M$ is any local diffeomorphism with $\mathrm{d}(\mathrm{Exp}_q)(0) = \mathrm{id}$.

If moreover the above vector field Y has its $(k-1)$-jet zero in all points of a curve γ: $(-\varepsilon, +\varepsilon) \to M$ with $\gamma(0) = q$ and $\dot\gamma(0) \neq 0$, then there is a unique homogeneous polynomial vector field $\widetilde{Y}_q^n$ of degree k on $T_q(M)/\langle\dot\gamma(0)\rangle$, such that for any linear sections $s: T_q(M)/\langle\dot\gamma(0)\rangle \to T_q(M)$, $s_*(\widetilde{Y}_q^n)$ agrees with $\widetilde{Y}_q$ modulo $\dot\gamma(0)$.

Finally, if in the above situation γ is an integral curve of X_0, then the flow φ_t of X_0 induces maps $(\varphi_t)_*: T_p(M)/\langle X_0(p)\rangle \to T_{\varphi_t(p)}(M)/\langle X_0(\varphi_t(p))\rangle$.

Now we come back to the k-jet of $\frac{\partial}{\partial\varepsilon}P_\varepsilon|_{\varepsilon=0}$ at p. This is a k-jet of a vector field along the Poincaré map P_0. Assuming that the $(k-1)$-jet of $\frac{\partial}{\partial\varepsilon}X_\varepsilon|_{\varepsilon=0}$ is zero along the X_0 integral curve through p, the $(k-1)$-jet of $\frac{\partial}{\partial\varepsilon}P_\varepsilon|_{\varepsilon=0}$ in p is also zero and hence the k-jet can be given by a homogeneous polynomial vector field of degree k on $T_{\varphi_t(p)}(\Sigma_t) \simeq T_{\varphi_t(p)}(M)/\langle X_0(\varphi_t(p))\rangle$.

To be more precise, the k-jet of $\frac{\partial}{\partial\varepsilon}P_\varepsilon|_{\varepsilon=0}$ in p is the k-jet of the composition of $P_0: \Sigma_0 \to \Sigma_t$ with a vector field on Σ_t whose $(k-1)$-jet in $P_0(p)$ is zero. This last vector field on Σ_t, which we denote by $\partial_\varepsilon P$, has a k-jet in $P_0(p)$, which can be identified with a homogeneous polynomial vector field of degree k on $T_{\varphi_t(p)}(M)/\langle X_0(\varphi_t(p))\rangle$, denoted by $\partial_\varepsilon\widetilde{P}$. Since the $(k-1)$-jet of $\frac{\partial X\varepsilon}{\partial\varepsilon}$ is zero in all points of the X_0 integral curve from p to $\varphi_t(p)$, we can replace in the above definitions Y by $\frac{\partial X\varepsilon}{\partial\varepsilon}$ and $\gamma(t)$ by $\varphi_t(p)$ and obtain $\partial_\varepsilon\widetilde{X}^n_{\varphi_s(p)}$ for $0 \leq s \leq t$. With this notation we have

$$\partial_\varepsilon\widetilde{P} = \int_0^t (\varphi_{t-s})_*(\partial_\varepsilon\widetilde{X}^n_{\varphi_s(p)})\mathrm{d}s.$$

We shall use this formula to obtain sufficient conditions for k-perturbability in case we know the system is $(k-1)$-perturbable. We treat gradient systems and mechanical systems separately.

a. Gradient systems. Let (M, g) be a Riemannian manifold, $X = \operatorname{grad}_g f$ a gradient vector field on M, and $p \in M$ a point such that $X(p) \neq 0$. We assume that X is $(k-1)$-perturbable in p, within the class of gradient vector fields. We want to derive sufficient conditions for X to be k-perturbable in p.

We identify $T_{\varphi_t(p)}(M)/\langle X(\varphi_t(p))\rangle$ with $X^{\perp}(\varphi_t(p))$, the orthogonal complement of $X(\varphi_t(p))$ (φ_t denotes, as before, the flow of X). With this identification φ_t induces a map $(\varphi_t)_*: X^{\perp}(p) \to X^{\perp}(\varphi_t(p))$. The Riemannian metric g induces in each $X^{\perp}(\varphi_t(p))$ an inner product which we denote by $\langle\, ,\rangle^n_{\varphi_t(p)}$.

As perturbations of X we take $\widetilde{X} = \operatorname{grad}_g \tilde{f}$, where the k-jet of $\tilde{f}$ is zero in points of the integral curve of X through p (so that the $(k-1)$-jet of $\widetilde{X}$ is zero along this integral curve). For $\tilde{f}$ as above there is for each t (near zero) a homogeneous polynomial $\bar{f}_t$ of degree $(k+1)$ on $X^{\perp}(\varphi_t(p))$ such that the $(k+1)$-jets of $\bar{f}_t$ and of $f \circ (\operatorname{Exp}_{\varphi_t(p)} | X^{\perp}(\varphi_t(p)))$ are equal. Also there is a homogeneous polynomial vector field $\widetilde{X}^n_{\varphi_t(p)}$ on $X^{\perp}(\varphi_t(p))$, representing the "normal" part of the k-jet of $\widetilde{X}$ in $\varphi_t(p)$.

With this notation it is clear that $\widetilde{X}^n_{\varphi_t(p)} = \operatorname{grad} \bar{f}_t$, where the gradient is taken with respect to $\langle\, ,\rangle^n_{\varphi_t(p)}$.

For k-perturbability, we require that

$$\int_0^t (\varphi_{t-s})_*(\widetilde{X}^n_{\varphi_t(p)})\, ds$$

can be made equal to any homogeneous polynomial vector field of degree k by a proper choice of $\tilde{f}$. We compose this integral with $(\varphi_{-t})_*$ and obtain

$$\int_0^t (\varphi_{-s})_*(\widetilde{X}^n_{\varphi_s(p)})\, ds.$$

Note that $(\varphi_{-s})_*(\widetilde{X}^n_{\varphi_s(p)})$ is the gradient of $\bar{f}_s \circ (\varphi_s)_*$ with respect to the inner product $(\varphi_{-s})_*(\langle\, ,\rangle^n_{\varphi_s(p)})$ on $X^{\perp}(p)$.
We denote this last one-parameter family of inner products on $X^{\perp}(p)$ by $\langle\, ,\rangle^n_{p,t}$.

Definition. (compare [9]). *We say that a one-parameter family $\langle\, ,\rangle^n_{p,t}$ of inner products is k-general if for each $s > 0$, there are $0 < t_1 \leq t_2 \leq \ldots \leq t_\ell < s$ and homogeneous polynomials $f_1, \ldots, f_\ell$ on $X^{\perp}(p)$ of degree $k+1$ such that $X_1, \ldots, X_\ell$, with X_i the gradient of f_i with respect to $\langle\, ,\rangle^n_{p.t_i}$, is a basis of the vector space of homogeneous polynomial vector fields of degree k on $X^{\perp}(p)$.*

Observe that if $\langle\, ,\rangle^n_{p.t}$ is k-general, then X is k-perturbable in p (assuming that X was already $(k-1)$-perturbable).

Proposition (3.1). *For differentiable one-parameter families of inner products, like $\langle\, ,\rangle^n_{p,t}$, it is a generic property to be k-general. In fact, those which are not k-general have infinite co-dimension in a sense to be defined below.*

Proof. To simplify notation we consider a one-parameter family of inner products on $\mathbb{R}^m$, with matrix $g_t = (g_{ij}(t))^m_{i,j=1}$. An ℓ-jet of such a family is just the collection of ℓ-jets of $g_{ij}(t)$ at $t=0$. Let $g_t^{-1} = (g^{ij}(t))$ denote the inverse matrix of $(g_{ij}(t))$; the ℓ-jet of the inverse is determined by the ℓ-jets of $\{g_{ij}(t)\}$. Now $\operatorname{grad}_{g(t)} f = \Sigma \dfrac{\partial f}{\partial x_i} \cdot g^{ij}(t) \dfrac{\partial}{\partial x_j}$, so $\operatorname{grad}_{g(t)} f$ is bilinear in f and $g^{ij}(t)$.

Let $F_1, \ldots, F_N$ be a basis of the homogeneous polynomials of degree $(k+1)$ on $\mathbb{R}^m$.

We say that the ℓ-jet of $g(t)$ is *k-general* if

$$\left\{ \sum_{i,j} \frac{\partial F_h}{\partial x_i} , \frac{\partial^s g^{ij}(0)}{\partial t^s} \cdot \frac{\partial}{\partial x_j} \right\}_{h=1}^{N}\,{}_{s=0}^{\ell}$$

spans all homogeneous polynomial vector fields of degree k on $\mathbb{R}^m$. Now we have the following statements:

I: there is some ℓ-jet which is k-general if ℓ is sufficiently big;
II: the set of ℓ-jets which are not k-general is an algebraic set (because the condition is equivalent with a number of determinants being zero);
III: the co-dimension of the set of ℓ-jets which are not k-general goes to infinite for fixed k as ℓ goes to infinite;
IV: if for some ℓ, the ℓ-jet of $g(t)$ is k-general, then $g(t)$ is k-general.

Observe that the statements II and IV are evident. Statement I we prove below, while statement III follows from the same arguments. Finally proposition (3.1) follows from the above statements, while it follows from III and IV how to interprete infinite co-dimension.

Proof of statement I. $F_1, \ldots, F_N$ denotes again a basis of the homogeneous polynomials of degree $(k+1)$ on $\mathbb{R}^m$. We consider the vector space of all vector fields on $\mathbb{R}^m$, generated by vector fields of the form

$$\sum_{i,j} \frac{\partial F_\ell}{\partial x_i} A^{ij} \frac{\partial}{\partial x_j}$$

where A_{ij} is a symmetric $m \times m$ matrix. This vector space is invariant under the action induced by the linear transformations in $\mathbb{R}^m$.

Since this vector space contains:

$$x_1^k \frac{\partial}{\partial x_1}\left(F = \frac{1}{k+1}\cdot x_1^{k+1},\ A^{ij} = \delta_{ij}\right) \text{ and}$$

$$x_1^k\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\left(F = \frac{1}{k+1}\cdot x_1^{k+1},\ A^{ij} = \begin{pmatrix} 1 & 1 & 0 & \cdot \\ 1 & 2 & 0 & \cdot \\ 0 & 0 & 1 & \\ \cdot & \cdot & & 1 \end{pmatrix}\right),$$

it must contain all homogeneous polynomial vector fields degree k.

From the above argument it follows that there is a set of vector fields of the form

$$\left\{\sum_{i,j} \frac{\partial F_\ell}{\partial x_i} A_s^{ij} \frac{\partial}{\partial x_j}\right\}_{\ell=1}^{N}{}_{s=1}^{S}$$

with A_s^{ij} symmetric in i, j, spanning all homogeneous vector fields of degree k. Now we take $g_t = (g_{ij}(t))$ so that $\frac{\partial^s g^{ij}}{\partial_t s}(0) = A_s^{ij}$. Then clearly the S-jet of g_t is k-general.

b. Mechanical systems. Let (M, g) be a Riemannian manifold; this time we assume that g has no curvature. For a potential function $V: M \to \mathbb{R}$, the corresponding mechanical system on $T^*(M)$ has the following explicit form in local coordinates: let $x_1, \dots, x_n$ be local coordinates on an open $U \subset M$ with respect to which g is the Euclidean inner product and let $x_1, \dots, x_n, p_1, \dots, p_n$ on $T^*(U)$ be the associated canonical coordinates in the sense that we identify x_i on U with $x_i\pi$ on $T^*(U)$ and that the co-vector $\Sigma\, \bar{p}_i \mathrm{d}x_i\,|_{(\bar{x}_1, \dots, \bar{x}_n)}$ has coordinates $\bar{x}_1, \dots, \bar{x}_n, \bar{p}_1, \dots, \bar{p}_n$. The canonical 2-form has, in these coordinates, the form $\omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}p_i$. The kinetic energy, restricted to $T^*(U)$, is $K(x, p) = \frac{1}{2}\sum_i p_i^2$. The Hamiltonian corresponding to V is $H_V = K + V \circ \pi$, which defines the Hamiltonian vector field X_V by the relation $\omega(X_V, -) = \mathrm{d}\,H_V$. In local coordinates we have that $X_V = \sum_i p_i \frac{\partial}{\partial x_i} - \Sigma \frac{\partial V}{\partial x_i}\frac{\partial}{\partial p_i}$.

We take a point $q \in T^*(M)$ which is not on the zero section and assume that X_V is $(k-1)$-pertubable in q (here we mean with "perturbable": "perturbable within the class of mechanical systems in the Hamiltonian sense"). We want to find sufficient conditions for X_V to be k-perturbable in q.

As before we denote the time t map of X_V by $\varphi_t: T^*(M) \to T^*(M)$ (which may be only partially defined). The symplectic structure on $T^*(M)$ defines a linear symplectic structure on each tangent space: the derivative $\mathrm{d}\varphi_t$ preserves these structures. φ_t also preserves the Hamiltonian function H_V. For each $q' \in T^*(M)$ there is a Lagrangian subespace $L_{q'} \subset T_{q'}(T^*(M))$, namely the tangent space of the fibre $\pi^{-1}(\pi(q'))$. We define $L_{q,t}$ as $(\mathrm{d}\varphi_t)^{-1}(L_{\varphi_t(q)})$. This variable Lagrangian subspace in $T_q(T^*(M))$ plays a role analogous to the variable inner product $\langle\,,\rangle_{p,t}^n$ in the previous discussion on gradient systems.

The perturbations which we consider are obtained by adding to V a function $\widetilde{V}$ whose k-jet is zero in each point of the projection of the X_V integral curve through q. This means that the $(k-1)$-jet of $X_{V+\widetilde{V}} - X_V = \widetilde{X}_{\widetilde{V}}$ is zero in all points of the integral curve through q.

The k-jet of $\widetilde{X}_{\widetilde{V}}$ at some point $q_t = \varphi_t(q)$ can be uniquely represented as a homogeneous polynomial vector field $\overline{X}_t$ on $T_{q_t}(T^*(M))$ of degree k; the $(k+1)$-jet of $\widetilde{V} \circ \pi$ in q_t can in the same way be represented by a homogeneous polynomial $\overline{V}_t$ of degree $(k+1)$ on $T_{q_t}(T^*(M))$. If we denote the symplectic structure in $T_{q_t}(T^*(M))$ by ω_t, then $\omega_t(\overline{X}_t, -) = \mathrm{d}\overline{V}_t$. Due to the construction, $\overline{V}_t$ has the property that for any two points $w, w' \in T_{q_t}(T^*(M))$ with $(w - w') \in L_{q_t} \oplus \langle X_V(q_t)\rangle$, $\overline{V}_t(w) = \overline{V}_t(w')$. Note that this implies that $\overline{X}_t$ is tangent to the levels of $\mathrm{d}H$ (as a linear functon on $T_{q_t}(T^*(M))$) and that the projection of $\overline{X}_t$ on $T_{q_t}(T^*(M))/\langle X(q_t)\rangle$ is well defined. This last vector field is "parameter Hamiltonian" in the sense that it generates a parameter symplectic diffeomorphism. As in the case of gradient systems, X_V is perturbable in g if $\{L_{q,t}\}$ is k-general in the following sense:

Definition. *We say that $t \mapsto L_{q,t} \subset T_q(T^*(M))$ is k-general if, for each $s > 0$ there are $0 < t_1 \le t_2 \le \dots \le t_\ell < s$ and homogeneous polynomials f_i of degree $(k+1)$ on $T_q(T^*(M))/(L_{q,t_i} \oplus \langle X_V(q)\rangle)$ such that $\{f_i \circ p_{t_i}\}_{i=1}^{\ell}$ defines a basis of the polynomials of degree $(k+1)$ on $T_q(T^*(M))/\langle X_V(q)\rangle$. Here, p_{t_i} denotes the projection $T_q(T^*(M)) \to T_q(T^*(M))/L_{q,t_i} \oplus \langle X_V(q)\rangle$.*

Now we want to prove the analogue of proposition (3.1).

Proposition (3.2). *For differentiable one parameter families of Lagrangian subspaces $t \mapsto L_{q,t}$ it is a generic property to be k-general. In fact, those which are not k-general form a set of infinite co-dimension.*

Proof. We take a linear coordinate system $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ on $T_q(T^*(M))$ such that the symplectic form is $\sum_i \mathrm{d}\xi_1 \wedge \mathrm{d}\eta_i$, $X(q) = \frac{\partial}{\partial \xi_1}$ and such that $L_{q,0} = \{\xi_i = 0\}$.

Then, Lagrangian subspaces near $L_{q,0}$ can be given as $\{\xi_i = \sum_j A_{ij}\eta_j\}$ with $A_{ij} = A_{ji}$. So one-parameter families $t \mapsto L_{q,t}$ can be given by a t-dependent symmetric matrix $A_{ij}^{(t)}$ with $A_{ij}^{(0)} = 0$. Polynomials of the form $f \circ p_t$ can be written as polynomials of $(\xi_2 - \sum_j A_{2j}^{(t)}\eta_j), \ldots, (\xi_n - \sum_j A_{nj}^{(t)}\eta_j)$. Let F^{k+1} be the vector space of polynomials on $T_q(T^*(M))$ spanned by all polynomials of the degree $(k+1)$ of $(\xi_2 - \Sigma a_{2j}\eta_j), \ldots, (\xi_n - \Sigma a_{nj}\eta_j)$ for various symmetric a_{ij}. F^{k+1} is clearly invariant under the group of symplectic transformations mapping $\frac{\partial}{\partial \xi_1}$ to itself, F^{k+1} contains the polynomial ξ_2^{k+1}. All hyperplanes near $\{\xi_2 = 0\}$ containing $\frac{\partial}{\partial \xi_1}$ can be transformed into one another by symplectic transformations which do not move $\frac{\partial}{\partial \xi_1}$, so F^{k+1} contains all monomials of the form

$$(a_2\xi_2 + \ldots + a_n\xi_n + b_1\eta_1 + \ldots + b_n\eta_n)^{k+1}$$

with $a_2 >> a_i$, b_i. Hence F^{k+1} consists of all homogeneous polynomials of degree $(k+1)$ of $(\xi_2, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$.

Let $F_1, \ldots, F_N$ be a basis of the homogeneous polynomials in $\xi_2, \ldots, \xi_n$; $F_i^{(t)}$ is defined as $F_i(\xi_2 - \Sigma A_{2j}^{(t)}\eta_j, \ldots, \xi_n - \Sigma A_{nj}^{(t)}\eta_j)$. It is clearly a generic property for one-parameter families $t \mapsto L_{q,t}$ as above that for the corresponding $F_i^{(t)}$,

$$\left\{ \frac{\partial^m F_i^{(t)}}{\partial t^m} \bigg|_{t=0} \right\}_{i=1}^{N} \ {}_{m=1}^{S}$$

spans F^{k+1} for S sufficiently big. The argument of infinite co-dimension is the same as in the case of gradient systems.

Remark. It turns out that the actual one-parameter families $L_{q,t}$ of Lagrangian subspaces occuring for mechanical systems is so that $A_{ij}^{(t)}$ satisfies $A_{ij}^{(t)} = -t \cdot \delta_{ij} + 0(t^3)$. Also for such one-parameter families, the above arguments remain valid.

4. The proofs of theorems A and B.

Lemma (4.1). *Let (M, g) be a Riemannian manifold, $f: M \to \mathbb{R}$ a smooth function, and $p \in M$ so that $df(p) \neq 0$. Let $\langle\,,\rangle_{p,t}^n$ be the one-parameter family of inner products on $(\text{grad } f)^\perp(p)$ as defined in the previous section. Let $d_n^2\, f(p)$ be the second derivative of f normal to $(\text{grad } f)(p)$, i.e., $d_n^2\, f(p)$ is the second order derivative of $f \circ (Exp_p|(\text{grad } f)^\perp(p))$, where $Exp_p: T_p(M) \to M$ is the exponential map determined by the metric g. We interpret this second derivative as a bilinear form on $(\text{grad } f)^\perp(p)$. With this notation, we have*

$$\frac{\partial}{\partial t}\langle\,,\rangle_{p,t}^n\Big|_{t=0} = 2 \cdot (d_n^2\, f(p)).$$

Proof. We choose local coordinates $x_1, \ldots, x_n$ so that:

$$p = (0, 0, \ldots, 0);$$
$$(\text{grad } f)(p) = (\|df\|, 0, \ldots, 0);$$
$$g_{ij}(0) = \delta_{ij};$$
$$\partial_k g_{ij}(0) = 0.$$

This means that $x_1, \ldots, x_n$, as far as the 2-jet, are exponential or normal coordinates. Hence the matrix of $(d_n^2 f)(p)$ is just $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j=2}^n$.

Next we have to calculate $\frac{\partial}{\partial t}\langle\,,\rangle_{p,0}^n$. In the following calculation we work in local coordinates and "add" points as in $\mathbb{R}^n$:

$$\varphi_t(x) = x + t \cdot X(x) + 0(t^2),$$
where $X = \text{grad } f$,
$$d\varphi_t(x) = \text{id} + t \cdot dX(x) + 0(t^2).$$

Take $\frac{\partial}{\partial x_i} \in (\text{grad } f)^\perp(p)$ (i.e., $i \geq 2$) and calculate $(d\varphi_t(p))\frac{\partial}{\partial x_i}$:

$$(d\varphi_t(p))\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} + t \cdot \Sigma \frac{\partial X_j}{\partial x_i}(p)\frac{\partial}{\partial x_j} + 0(t^2) = \frac{\partial}{\partial x_i} + $$
$$+ t \cdot \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial}{\partial x_j} + 0(t^2)$$

(due to the conditions imposed on g, expressed in the present coordinates).

Instead of $(d\varphi_t(p))\frac{\partial}{\partial x_i}$ we are rather interested in $(dP_t)\frac{\partial}{\partial x_i}$. It is not hard to see that we can multiply X with a function g, identically equal to one along the integral curve through p, such that the time t map of $g \cdot X$ transforms $X(p)^\perp$ to $X(\varphi_t(p))^\perp$ and hence induces the derivative of the Poincaré map (dP_t). In this way we easily see that

$$(dP_t)\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} + t\left(\sum_j\left(\frac{\partial X_j}{\partial x_i}(p) + \frac{\partial g}{\partial x_i}(p) \cdot X_j(p)\right)\frac{\partial}{\partial x_j} + 0(t^2).$$

Finally we calculate

$$\left\langle (\mathrm{d}P_t)\frac{\partial}{\partial x_i}, (\mathrm{d}P_t)\frac{\partial}{\partial x_j} \right\rangle = \delta_{ij} + 2t\frac{\partial^2 f}{\partial x_i \partial x_j} + 0(t^2).$$

This proves the lemma.

Proof of theorem A. (M, g) is, as before, a Riemannian manifold. It follows easily from the above lemma that the map assigning, to each $(k+1)$-jet of a function $f: M \to \mathbb{R}$ at p, the corresponding k-jet of $\langle\, , \rangle^n_{p,t}$ has maximal rank (restricting to those $(k+1)$-jets of which the representative has no critical point in p). This implies that for $\varepsilon > 0$ and $k \in \mathbb{N}$, there is for big S, depending on $k \in \mathbb{N}$ and $\dim(M)$, see proposition (3,1), an open and dense subset of the C^S-functions on M, consisting of those functions f such that for each $p \in M$, grad X is k-general in p or within distance ε from p, f has a critical point. This implies the theorem.

Lemma (4.2). *Let (M, g) be a Riemannian manifold with zero curvature. Let $x_1, \dots, x_n$ be local coordinates on M with respect to which g is the Euclidean inner product; $x_1, \dots, x_n, p_1, \dots, p_n$ are corresponding canonical coordinates on the co-tangent bundle. Let $V: M \to \mathbb{R}$ be a potential function, $q = (x, p) \in T^*(M)$ a point with $p \neq 0$ and let $L_{q,t}$ be the corresponding one-parameter family of Lagrangian subspaces in $T_q(T^*(M))$ as defined in section 3; $A^{(t)}_{ij}$ is the corresponding one-parameter family of symmetric matrices. Then*

$$A^{(t)}_{ij} = -t\delta_{ij} - \frac{t^3}{3}\frac{\partial^2 V}{\partial x_i \partial x_j}(x) + 0(t^4).$$

Proof. We have to show that the x-components of

$$\left(\mathrm{d}\varphi_t(x,p)\left(\frac{\partial}{\partial p_i}\right) + \sum_j A^{(t)}_{ij}(x)\frac{\partial}{\partial x_j}\right),$$

with $A^{(t)}_{ij}(x) = |-t\delta_{ij} - \frac{t^3}{3}\frac{\partial^2 V}{\partial x_i \partial x_j}(x) + 0(t^4)$, are zero mod (t^4). In order to calculate $\mathrm{d}\varphi_t(x,p)$ we use the formula

$$\frac{\partial^k}{\partial t^k}\varphi_t(x,p)\,|_{t=0} = \frac{\partial^{k-1}}{\partial t^{k-1}} X(\varphi_t(x,p))\,|_{t=0},$$

where X is the Hamiltonian vectorfield determined by V. The righthand side can be calculated without knowing $\varphi_t(x,p)$:

$\frac{\partial^k}{\partial t^k} X(\varphi_t(x,p))\,|_{t=0}$ can be obtained as the derivate of $\frac{\partial^{k-1}}{\partial t^{k-1}} X(\varphi_t(x,p))\,|_{t=0}$ in the direction of $X(x,p)$. In the following calculations we shall denote parts of formulas which drop out later, by *.

$X(x,p) = \Sigma\, p_i \frac{\partial}{\partial x_i} - \Sigma \frac{\partial V}{\partial x_i}(x)\frac{\partial}{\partial p_i}$, which we also write as (first the x- and then the p- component):

$$X = (p, -\mathrm{d}V), \text{ so}$$

$$\partial_t X(\varphi_t(x,p))\,|_{t=0} = (-\mathrm{d}V(x), *)$$

$$\partial_{tt} X(\varphi_t(x,p))\,|_{t=0} = (-\mathrm{d}^2V(x)(p, -), *).$$

So $\varphi_t(x,p) = \left(x + t\cdot p - \frac{t^2}{2}\mathrm{d}V(x) - \frac{t^3}{3!}\mathrm{d}^2V(x)(p,-), *\right) + 0(t^4)$ and

$$\mathrm{d}\varphi_t(x,p) = \left(\begin{array}{c:c} Id - \frac{t^2}{2}\mathrm{d}^2V(x) + *\cdot t^3 & t\cdot \mathrm{Id} - \frac{t^3}{3!}\mathrm{d}^2V(x) \\ \hdashline * & * \end{array}\right) + 0(t^4).$$

Finally we have to apply this derivative to the vectors $\frac{\partial}{\partial p_i} + \sum_j A^{(t)}_{ij}\frac{\partial}{\partial x_j}$ and show that they have x-components equal to zero mod (t^4). This means that we have to apply $\mathrm{d}\varphi_t(x,p)$ to

$$\begin{pmatrix} A^{(t)}_{ij} \\ \mathrm{Id} \end{pmatrix} = \begin{pmatrix} -t\cdot\mathrm{Id} - \frac{t^3}{3}\mathrm{d}^2V \\ \mathrm{Id} \end{pmatrix} + 0(t^4).$$

We conclude that indeed the upper $n \times n$ block of the product, the x-components, is zero mod (t^4).

Proof of theorem B. Let (M, g) be again a Riemannian manifold with zero curvature and let $q = (x,p) \in T^*(M)$ be as above with $p \neq 0$. It follows easily from the above lemma that the map, which assigns to each $(k-1)$-jet of $V: M \to \mathbb{R}$ in $x = \pi(q)$ the k-jet of $(L_{q,t})$, has maximal rank (if we take into account the restrictions on the 2-jet of such families).

Combining this with proposition (3.2) we see that, in order to avoid $q = (x,p)$ to be k-general, the S-jet of V at x has to avoid a certain set A of co-dimension $d_{S,k}$. The set is the diffeomorphic image of an algebraic set and hence union of smooth manifolds. In fact, we have to avoid all $A_{S,k,x,p}$ for fixed S and k, and variable x and $p \neq 0$. This means that (for fixed x) the co-dimension of the set which is to be avoided is lowered by n. Since however $d_{S,k} \to \infty$ for $S \to \infty$, it is clear that for S sufficiently big, it is a C^S-generic property for potential functions $V: M \to \mathbb{R}$ that the $(S+1)$-jet of each $(L_{q,t})$ with $q=(x,p)$, $p \neq 0$, is k-general. This proves the first part of the theorem.

The second statement (involving the compact subset $K \subset T^*(M)$ avoiding the zero-section) follows from the fact that the S-jet in x, for which $(L_{q,t})$, $q=(x,p)$, $p \neq 0$, is not k-general, is closed.

5. Applications.

I. As we noted in the introduction, in any situation where one can produce local perturbations of sufficiently general type, one can prove that the Kupla-Smale theorem holds. For gradient vector fields (with fixed metric) it is generic that all singularities are hyperbolic (this is equivalent with the function having only nondegenerate singularities). There are no closed orbits, and by our theorem A one can make stable and unstable manifolds transverse. Hence:

Corollary (5.1). *On any Riemannian manifold (M, g), generic gradient vector fields are Kupka-Smale.*

II. In [11] there is a study of generic one-parameter families of gradient vector fields. That however was generic in the sense that both the function *and the metric* were perturbed. From theorem A it follows that only perturbing the function one can already make the necessary approximations. (Since one works here with one-parameter families, say on M, and interprete them as vector fields on $M \times \mathbb{R}$ with 0-component in the R direction, we have to restrict, like in the Hamiltonian case, the type of Poincaré map which comes into consideration. The only restriction on the Poincaré map is that it preserves the $\mathbb{R}$-coordinate.) In this way we obtain:

Corollary (5.2). *On any compact Riemannian manifold (M, g) generic one-parameter families of gradient vector fields are structurally stable.*

III. In the case of mechanical systems we cannot conclude that Kupka-Smale systems are generic. In principle there are two problems. namely

(a) the perturbations are not local: the support of any perturbation is a union of fobres of $T^*(M)$ and (b) the Poincaré maps are parameter symplectic. For general Hamiltonian systems Poincaré maps are also parameter symplectic; the consequences of this for closed orbits in the generic case were discussed in [6]. The conclusions of [6] also hold for generic mechanical systems since, if a mechanical system on M has a closed orbit $\gamma: S^1 \to T^*(M)$, there is some $r \in S^1$ and neighbourhood U of $\pi \cdot \gamma(r)$ in M such that

$$\gamma \,|\, (\pi \cdot \gamma)^{-1}(U) : (\pi \cdot \gamma)^{-1}(U) \to U$$

is an embedding. Then the perturbations, constructed in this paper, can be considered as local as far as we are concerned with perturbing γ and its Poincaré map. So:

Corollary (5.3). *The results in [6] on Poincaré maps of closed orbits in generic Hamiltonian systems apply also to generic mechanical systems.*

IV. Also in mechanical systems we can make stable and unstable manifolds transverse. So, for mechanical systems, we get the same results as in [4] for Hamiltonian systems. The non-local character of the perturbations is compensated by the fact that, also for an orbit belonging to the intersection of a stable and an unstable manifold in $T^*(M)$, the projection on M is "somewhere an embedding" like in the case of a periodic orbit. This corollary is a partial solution of a problem which was stated in [12].

References

[1] I. Kupka; *Contributions à la théorie des champs génériques*, Contributions to differential equations 2 (1963).

[2] S. Smale; *Stable manifolds for differential equations and diffeomorphisms*, Annali della Scuola Normale Superiore de Pisa, XVII, 97-116 (1963).

[3] L. Markus; *Generic properties of differential equations*, Nonlinear differential equations and nonlinear mechanics, ed. La Salle, Lefschetz, Acad. Press, 1963.

[4] R. C. Robinson; *Generic properties of conservative systems I, II*, American Jour. Math. 92 (1970), 562-603 and 897-906.

[5] S. Shahshahani; *Second order ordinary differential equations on differentiable manifolds*, Global Analysis, ed. SS. Chern and S. Smale, Proc. Symp. Pure Math., 14 (1970), 265-272., A.M.S.

[6] F. Takens; *Hamiltonian systems, generic properties of closed orbits and local perturbations*, Math. Ann. 188 (1970), 304-312.

[7] W. Klingenberg, F. Takens; *Generic properties of geodesic flows*, Math. Ann. 197 (1972), 323-334.

[8] H. G. Bothe; *Generic properties of ordinary differential equations of higher order*, Qualitative theory of differential equations, ed. M. Farkas, Coll. Mathematica Societatis János Bolyai 30 (1981), North Holland.
[9] F. Klok; *Generic singularities of the exponential map of Riemannian manifolds*, preprint ZW-8022, Groningen.
[10] Ph. Hartman; *Ordinary differential equations*, John Wiley, New York, 1964.
[11] J. Palis, F. Takens; *Stability of parametrized families of gradient vector fields*, to appear.
[12] L. Markus, K. R. Meyer; *Periodic orbits and solenoids in generic Hamiltonian dynamical systems*, Amer. J. Math. 102 (1980) 25-92.

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