LAX PAIRS

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In the pioneering paper [3], Lax stated a condition under which certain one parameter families of operators $\{L(t)\}$ are isospectral, i.e., all the $L(t)$ have the same spectrum. Lax actually asserted that his condition implied the unitary equivalence of all the $L(t)$. Unfortunately it is not clear from [3] exactly when this condition is applicable. Furthermore there seems to be no clear statement nor proof of this in the Literature [See [1], Chapter 3 where this is discussed]. The purpose of this note is to state and prove such a result and show as Lax affirmed that it can indeed be used to prove the unitary equivalence of Schrödinger operators whose potentials evolve according to the Korteweg-de Vries equation.

The main theorem depends heavily on Kato's paper [2]. We refer to it for background and notation.

Theorem. *Let H be a Hilbert space, $\{L(t)\}$, $\{A(t)\}$, $t \in [0,T]$ families of self-adjoint operators in H and Y, V, W Hilbert spaces which are densely and continuously imbedded in H and satisfy $Y \subseteq V$, $Y \subseteq W$. Suppose in addition:*

1. *For each $t \in [0,T]$ $D(L(t)) = V$ and $D(A(t)) = W$.*
2. *$t \longmapsto L(t)$ is strongly continuously differentiable (V,H).*
3. a) *For each $t \in [0,T]$, $A(t)$ maps Y continuously into V.*
   b) *$t \longmapsto A(t)$ is strongly continuous (Y,V).*
   c) *$t \longmapsto A(t)$ is continuous $[0,T] \to B(W,H)$ and strongly continuously differentiable (W,H).*
4. *There is a family of isomorphisms $\{S(t)\}$, $t \in [0,T]$ from Y onto H such that $t \longmapsto S(t)$ is strongly differentiable (Y,H) and*

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$$S(t)A(t)S(t)^{-1} = A(t) + B(t),$$

where $B(t) \in B(H)$ *and* $t \longmapsto B(t)$ *is strongly continuous* (H).

5. $L(t)$ *maps* Y *into* W *and, for each* $t \in [0,T]$ $\phi \in Y$,

$$\frac{d}{dt} L(t)\phi = -i[A(t),L(t)]\phi,$$

the derivative being taken in H, *which makes sense in virtue of 2. Also the commutator above makes sense given the mapping properties of* $L(t)$ *and* $A(t)$.

Then all the operators $L(t)$ *are unitarily equivalent.*

Remark. A pair of operator families $\{L(t)\}$, $\{A(t)\}$ satisfying the above condition is called a *Lax pair*.

In the following, $C_b^n(\mathbb{R})$ denotes the space of all C^n functions $u:\mathbb{R} \to \mathbb{R}$ such that $\|u\|_n = \sup_{\substack{j\le n\\ x\in\mathbb{R}}} |u^{(j)}(x)| < \infty$.

Corollary. *Suppose that for all* $t \in [0,T]$, $u(t) \in C_b^6(\mathbb{R})$, $t \longmapsto u(t)$ *is continuous* $[0,T] \to C_b^6(\mathbb{R})$ *and* $C^1[0,T] \to C_b^1(\mathbb{R})$. *If* u *satisfies the Korteweg-de Vries equation*

$$u_t + uu_x + u_{xxx} = 0$$

then the Schrödinger operators

$$L(t) = \partial_x^2 + u(t)/6 \quad (t \in [0,T])$$

are all unitarily equivalent operators in $H = L^2(\mathbb{R})$.

We remark that his corollary has been rigourously shown [see [1]] by other methods.

Proof of Theorem. We apply Theorem 6.1 of [2]. By 3c) and Remark 6.2 of [2] there is a family $\{U(t)\}$, $t \in [0,T]$ in $B(H)$ such that

(i) $U(t)$ is strongly continuous (H).

(ii) $U(t)\,W \subseteq W$ for each $t \in [0,T]$ and $\{U(t)\}$ is strongly continuous (W).

(iii) $\{U(t)\}$ is strongly continuously differentiable (W,H) and for all $\phi \in H$ $t \longmapsto U(t)\phi$ is a solution to the initial value problem

$$\begin{cases} \dfrac{d}{dt}\psi(t) = -i\,A(t)\psi(t), \\ \psi(0) = \phi. \end{cases} \qquad (*)$$

Each $U(t)$ is a unitary operator in H: By differentiation all solutions of (*) have constant H norm. Thus $U(t)$ is isometric. By reversing the initial value problem (i.e. taking as initial point t instead of 0 and solving backwards, which is clearly possible) we conclude $U(t)$ is surjective and thus unitary.

$U(t)$ leaves Y invariant. To see this apply Theorem 6.2 of [2] with hypotheses 3. and 4. Note $t \longmapsto A(t)$ is continuous $[0,T] \to B(Y,H)$ because the inclusion $Y \to W$ is continuous by the closed graph theorem. Thus if $\phi \in Y$ the initial value problem (*) has a strongly continuous solutions $\psi: [0,T] \to Y$. By uniqueness of solutions [which follows from linearity and the constancy of the norms of solutions to (*)] we conclude the invariance of Y.

Notice that this argument shows that $\{U(t)\}$ is strongly continuous (Y). We now show that $\{U(t)\}$ is continuously strongly differentiable (Y,V). Indeed, if $\phi \in Y$

$$U(t)\phi - \phi = -i\int_0^t A(\theta)U(\theta)\phi\,d\theta,$$

the integral being a priori an H valued integral. But by 3a) it follows easily that $\theta \longmapsto A(\theta)U(\theta)\phi$ is continuous $[0,T] \to V$ and so by the fundamental theorem of calculus $t \longmapsto U(t)\phi$ is differentiable (V).

To complete the proof of the theorem we prove that if $\phi \in Y$, $U^*(t)L(t)U(t)\phi$ is constant. To do this it clearly suffices to prove that $\langle L(t)U(t)\phi,\ U(t)\psi\rangle$ is constant for all $\psi \in Y$. A straight forward argument shows $t \longmapsto L(t)U(t)\phi$ is strongly continuously differentiable (H) and

$$\frac{d}{dt} L(t)U(t)\phi = \left[\frac{d}{dt} L(t)\right]U(t)\phi + L(t)\ \frac{d}{dt}\ U(t)\phi.$$

Thus $t \longmapsto \langle L(t)U(t)\phi,\ U(t)\psi\rangle$ is a differentiable function and

$$\begin{aligned}\frac{d}{dt} \langle L(t)U(t)\phi,\ U(t)\psi\rangle &= \\ &= \langle \left[\frac{d}{dt} L(t)\right]U(t)\phi + L(t)\ \frac{d}{dt}\ U(t)\phi,\ U(t)\psi\rangle + \\ &+ \langle L(t)U(t)\phi,\ \frac{d}{dt}\ U(t)\psi\rangle = \langle\left[\frac{d}{dt}\ L(t)\right]U(t)\phi - \\ &- i\ L(t)A(t)U(t)\phi + i\ A(t)L(t)U(t)\phi,\ U(t)\psi\rangle = \\ &= 0\end{aligned}$$

by assumption 5. This completes the proof.

Proof of the Corollary. Let $Y = H^5(\mathbb{R})$, $W = H^3(\mathbb{R})$, $V = H^2(\mathbb{R})$

$$S(t) = \left[\partial_x^5 + I\right] \quad \text{(independently of } t\text{)}$$

$$A(t) = i\left[24\partial_x^3 + 3u(t)\partial_x + 3\partial_x u(t)\right] = i\left[24\partial_x^3 + 6u(t)\partial_x + 3u'(t)\right].$$

We show conditions 1 - 5 of the theorem hold:

1. Follows from the Kato-Rellich Theorem [Theorem X.12 [4]] and 3a. follows from elementary mapping properties of differential operators. To check 2. it suffices to observe that $u(t)$ [considered as a multiplication operator in $L^2(R)$] is strongly continuously differentiable (H) as a function of t. To verify 3b. note that, since $t \longmapsto u(t)$ is continuous with values in $C_b^3(\mathbb{R})$, $t \longmapsto 3u(t)\partial_x + 3\partial_x u(t)$ is continuous $B(H^3(\mathbb{R}),\ H^2(\mathbb{R}))$. From this it follows that $t \longmapsto A(t)$ is continuous $B(H^5(\mathbb{R}),\ H^2(\mathbb{R}))$. A similar argument proves 3c.

To prove 4 we begin with some domain considerations: Obviously

$$D(S(t)A(t)S(t)^{-1}) = \{\phi \in L^2(\mathbb{R})\ :\ A(t)S(t)^{-1}\phi \in H^5(\mathbb{R})\}.$$

If $w \in H^5(\mathbb{R})$ and $A(t)w \in H^5(\mathbb{R})$ then

$$24\partial_x^3 w = -3u(t)\partial_x w - 3\partial_x u(t)w - iA(t)w \in H^4(\mathbb{R}).$$

By elliptic regularity, $w \in H^7(\mathbb{R})$; Since $u(t) \in C_b^6(\mathbb{R})$ for each t the previous equation and elliptic regularity again imply $w \in H^8(\mathbb{R})$. From this follows immediately that

$$D(S(t)A(t)S(t)^{-1}) = H^3(\mathbb{R}).$$

By commutation properties of differential operators

$$[\partial^5 + I]A(t)[\partial^5 + I]^{-1} = A(t) + T(t)[\partial^5 + I]^{-1},$$

where $T(t)$ is a differential operator of order 5 with at least continuous coefficients. Thus $T(t)$ maps $H^5(\mathbb{R})$ continuously into $L^2(\mathbb{R})$. Furthermore since the coefficients of $T(t)$ are x-derivatives of u of order ≤ 6 it follows $t \longmapsto T(t)$ is continuous $[0,T] \to B(H^5(\mathbb{R}),\ L^2(\mathbb{R}))$.

Finally property 5. is precisely the commutation relation shown in Lax's paper.

Bibliography

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