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EXAMPLE OF A COMPLETE MINIMAL IMMERSION IN $\mathbb{R}^3$ OF GENUS ONE AND THREE EMBEDDED ENDS

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1. Introduction

In this work we will construct an example of a complete minimal immersion of the torus punctured at three points in $\mathbb{R}^3$ with embedded ends. The total curvature of such an immersion is -12π. The result is a consequence of the application to minimal surfaces of the theory of elliptic functions of the complex plane $\mathbb{C}$ through the Weierstrass representation.

Let M_γ be a compact surface of genus γ, let $Q_1,\dots,Q_N$ be points of M_γ, and let $x: M = M_\gamma - \{Q_1,\dots,Q_N\} \to \mathbb{R}^n$ be a complete minimal immersion. If $D_j \subset M_\gamma$ is a topological disk centered at Q_j, $j = 1,\dots,N$, $Q_i \notin D_j$, $i \neq j$, then $F_j = x(D_j \cap M)$ is an *end* of the immersion x, and we will say that x is a *complete minimal immersion in* R^n, *of genus* γ *and with* N *ends*. We will prove the following theorem:

Theorem. *There exists a complete minimal immersion in* $\mathbb{R}^3$, *of genus one, with three ends and the following properties:*

a) *The total curvature is* -12π

b) *The ends are embedded.*

Among the complete minimal immersions in $\mathbb{R}^3$, of genus one and three ends, the above immersion has greatest total curvature.

To prove the theorem we will consider the complex plane $\mathbb{C}$ with coordinate $z = u+iv$, the lattice $L = L(1,i) = \{m+ni \in \mathbb{C};\ m,n \in \mathbb{Z}\}$,

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the points $w_1 = 1/2$, $w_2 = (1+i)/2$, $w_3 = i/2$ and the quotient torus $T = \mathbb{C}/L$ with complex structure induced by the canonical projection $\pi: \mathbb{C} \to \mathbb{C}/L$. Through π we will identify the meromorphic functions and the meromorphic differentials of T with the elliptic functions and elliptic differentials of L, respectively. Then we will define

$$g = \frac{a}{P'} \quad \text{and} \quad w = P dz,$$

where P is the Weierstrass P-function of the lattice L and $a \in \mathbb{R}-\{0\}$ will be chosen conveniently. Finally we will show that the couple (g,w) is the Weierstrass representation of a complete minimal immersion,

$$x: M = T-\{Q_1,Q_2,Q_3\} \to \mathbb{R}^3, \quad Q_1 = \pi(w_1), \quad Q_2 = \pi(0) \text{ and } Q_3 = \pi(w_3),$$

which has the required properties.

In order to prove this (see [7], Lemma 8.1 and [4], §3), it is sufficient that (g,w) satisfy the following conditions:

(c_1) *w is a holomorphic function in M. $Q \in M$ is a pole of order m of g if and only if Q is a zero of order $2m$ of w.*

(c_2) *If δ is a closed path in M then*

$$Re \int_\delta gw = 0 \quad \text{and} \quad \overline{\int_\delta w} = \int_\delta g^2 w.$$

(c_3) *Every divergent path ℓ in M has infinite length.*

This paper contains parts of my Doctoral Dissertation at IMPA [2], under the orientation of M. do Carmo. I was recently informed that the surface described in the theorem was proved to be embedded by D. Hoffman.

2. Elliptic Functions

We will use the following notation: Let $L = L(\lambda,\lambda') = \{m\lambda+n\lambda' \in \mathbb{C},\ m,n \in Z\}$ be a lattice where $\lambda,\lambda' \in \mathbb{C}$ and $\operatorname{Im} \lambda/\lambda' > 0$, let P be the P-function of Weierstrass of L, and let $\alpha,\beta: [0,1] \to \mathbb{C}$ be the paths

(2.1) $$\alpha(t) = \frac{\lambda'}{3} + t\lambda, \qquad \beta(t) = \frac{\lambda}{3} + t\lambda'.$$

We define the complex numbers:

(2.2) $$\eta = -\int_\alpha P dz, \qquad \eta' = -\int_\beta P dz.$$

η and η' are invariants of L associated to the non-trivial homology classes of the torus $T = \mathbb{C}/L$.

We also define the complex numbers

(2.3) $$w_1 = \lambda/2, \qquad w_2 = \frac{\lambda+\lambda'}{2}, \qquad w_3 = \lambda'/2$$

and

(2.4) $$e_j = P(w_j), \quad j=1,2,3, \quad g_3 = e_1e_2e_3, \quad g_2 = \sum_{i<j} e_ie_j.$$

We need the following lemma:

Lemma. *Let $L = L(1,i)$ be a lattice. Then, with the notation* (2.1), (2.2), (2.3) *and* (2.4), *we have:*

a) $\eta = \pi$, $\eta' = -\pi i$,

b) $e_j \in \mathbb{R}$, $j=1,2,3$, $e_2 = 0$, $e_1 = -e_3 > 0$, $g_3 = 0$ and $g_2 = 4e_1^2$,

c) $P'(w_j) = 0$, $P''(w_j) \in \mathbb{R}$, $j = 1,2,3$.

d) $$\frac{1}{P-e_1} = \frac{1}{2e_1^2}[P(z-w_1)-e_1], \qquad \frac{1}{P-e_3} = \frac{1}{2e_1^2}[P(z-w_3)-e_3].$$

Proof. Choose paths α, β and invariants η, η' as in (2.1) and (2.2). In the lattice $\tilde{L} = L(i,-1)$, obtained from L by a rotation of an angle $\pi/2$, we have the paths $\tilde{\alpha}$, $\tilde{\beta}$ and the invariants $\tilde{\eta}$ and $\tilde{\eta}'$.

From the expression that appears in [6], page 24, we see that the Weierstrass function P is the same for both lattices. Then,

$$\tilde{\eta} = -\int_{\tilde{\alpha}} Pdz = -\int_{\beta} Pdz = \eta'.$$

On the other hand, the development in series that appears in [3], page 445, shows that

$$\tilde{\eta} = -i\eta.$$

From the equations above and from the relation of Legendre ([6], page 38) we obtain

$$\eta' = i\eta - 2\pi i = -i\eta.$$

Thus we have

$$\eta = \pi \quad \text{and} \quad \eta' = -\pi i,$$

and part (a) of the lemma is proved.

To prove (b) observe that the expressions in series for g_3 and $\tilde{g}_3$ that appear in [3], page 446, show that $g_3 = -\tilde{g}_3$. But, since the P-functions agree in the lattices L and $\tilde{L}$, $g_3 = \tilde{g}_3$. Thus, $g_3 = 0$. Since the lattice is axial (see [6], page 162), we have $e_j \in I\!R$, $j = 1,2,3$ and $0 < e_1 > e_2 > e_3 < 0$. On the other hand, from [6], page 47, we have $e_1 + e_2 + e_3 = 0$. Since $g_3 = 0$, we conclude that $e_1 = -e_3$, $e_2 = 0$, and part (b) of lemma is proved.

Part (c) of the lemma follows from [6], page 27, item (b), and the equation

$$P^2 = \frac{1}{6} P'' + \frac{1}{12} g_2,$$

that appears in [6], page 47.

To prove part (d) we observe that the quotient of the elliptic function $\frac{1}{P-e_j}$ and $P(z-w_j)-e_j$, $j=1,3$ is constant. In order to find these constants, it is sufficient to evaluate them on the point w_2.

3. Proof of the Theorem

Let $L = L(1,i)$ be a lattice and let $T = \mathbb{C}/L$ be the torus with complex structure induced by the canonical projection $\pi\colon \mathbb{C} \to \mathbb{C}/L$. Let $Q_1, Q_2, Q_3 \in T$ be given by

$$Q_1 = \pi(w_1), \quad Q_2 = \pi(0) \quad \text{and} \quad Q_3 = (w_3).$$

We will show that the couple (g,w),

$$g = \frac{a}{P'}, \qquad w = Pdz$$

where a is to be chosen conveniently, is a Weierstrass representation of a complete minimal immersion $x\colon M{=}T-\{Q_1,Q_2,Q_3\} \to I\!R^3$ with the properties expressed in the theorem, that is, we will show that the couple (g,w) satisfies the conditions (c_1), (c_2) and (c_3).

Proof of (c_1)

Clearly w is holomorphic in M. From the lemma, $e_2 = P(w_2) = P'(w_2) = 0$. Therefore, since P is an elliptic function of order 2, $Q = \pi(w_2) \in M$ is the only zero of w. On the other hand, P' is an elliptic function of order 3, hence, from the lemma, w_1, w_2, w_3 are simple zeros of P'. Thus, the points $\pi(w_j)$, $j=1,2,3$ are simple poles of g. Since $Q_1 = \pi(w_1)$ and $Q_3 = \pi(w_3)$ do not belong to M, condition (c_1) is satisfied.

Proof of (c_2)

From the equation $(P')^2 = 4\prod_{j=1}^{3}(P-e_j)$ (see [6], page 46) and the lemma, we see that

$$gw = \frac{aP}{P'}\,dz = \frac{aPP'}{4\prod_{j=1}^{3}(P-e_j)}\,dz = \frac{a}{8e_1}\left(\frac{P'}{P-e_1} - \frac{P'}{P-e_3}\right)dz. \tag{3.1}$$

Now let α, β be nontrivial closed generators of the homology of T. Then, since $e_1 \in I\!R$ and $(P-e_j)$ are elliptic functions, we have

$$\mathrm{Re}\int_{\gamma} gw = \frac{a}{8e_1}\,\mathrm{Re}\left[\log(P-e_1)-\log(P-e_3)\right]_{\gamma} = 0, \quad \gamma \in \{\alpha,\beta\}.$$

On the other hand, by using the lemma and (3.1) we obtain

(3.2) $$g^2 w = \frac{a}{16e_1^3}\left[P(z-w_1)-P(z-w_3)-2e_1\right]dz.$$

And, again by the lemma,

$$\int_\alpha w = -\pi, \quad \int_\beta w = \pi i, \quad \int_\alpha g^2 w = \frac{-a^2}{8e_1^2} \text{ and } \int_\beta g^2 w = \frac{-a^2}{8e_1^2}.$$

Thus, if we choose $a = 2e_1\sqrt{2\pi}$, condition (c_2) is satisfied for α and β.

To complete the proof of (c_2) we need to show that

$$\operatorname{Res}_{Q_j} gw \in \mathbb{R} \text{ and } \overline{\operatorname{Res}_{Q_j} w} = -\operatorname{Res}_{Q_j} g^2 w, \quad j = 1,2,3.$$

From (3.1), it follows that

$$\operatorname{Res}_{Q_2} gw = 0.$$

By the lemma, we find that at the point Q_1

$$\operatorname{Res}_{Q_1} gw = \frac{a}{8e_1}\operatorname{Res}_{Q_1}\frac{P'}{P-e_1} = \frac{a}{16e_1^3}P'.P(z-w_1).$$

Finally, by using the local expression for $P(z-w_1)$, that appears in [6], page 356, and item (c) of the lemma, we find

that $\operatorname{Res}_{Q_1} gw \in \mathbb{R}$. Furthermore, since $\sum_{j=1}^{3}\operatorname{Res}_{Q_j} gw = 0$,

$\operatorname{Res}_{Q_3} gw \in \mathbb{R}$. Thus,

$$\operatorname{Res}_{Q_j} gw \in \mathbb{R}, \quad j=1,2,3.$$

On the other hand, from the local expressions for the function P (see [6], page 356) and from (3.2), we find that

$$\operatorname{Res}_{Q_j} w = \operatorname{Res}_{Q_j} g^2 w = 0,$$

whence condition (c_2) holds.

Proof of (c_3)

We need to show, for a divergent path ℓ in M, that

$$\int_\ell (1+|g|^2)|w| = \infty.$$

From the lemma, it follows that at the points $Q_1 = \pi(w_1)$ and $Q_3 = \pi(w_3)$ we have $P(w_1) = e_1 > 0$ and $P(w_3) = e_3 < 0$. Thus w is holomorphic and not zero in neighbourhoods of Q_1 and Q_3. Since $P'(w_1) = P'(w_3) = 0$, g has poles in Q_1 and Q_3. At the point $Q_2 = \pi(0)$, w has a pole of order 2. This is enough to prove condition (c_3).

We have shown that the couple (g,w) is a Weierstrass representation of a complete minimal immersion in $\mathbb{R}^3$, of genus one, with three ends and finite total curvature. Since g is of third order,

$$c = \int_M KdM = -12\pi.$$

Since the Euler's characteristic $\chi(M)$ of M is -3,

$$c = -12\pi = 2\pi[\chi(M)-3].$$

It follows from [5], Theorem 4, that the ends of the immersion are embedded. This concludes the proof of the theorem.

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