PRETZEL – FIBERED LINKS

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1. Introduction

In this paper a link in S^3 will allways be a tame link. In the early 60's Murasugi [Mu] and Stallings [St1] proved two striking results on fibered links. Murasugi gave a proof of the following theorem: An alternating link is fibered if and only if its reduced Alexander polynomial is monic. Stallings proved the following general result: A link $L \subset S^3$ is fibered if and only if $\pi_1(L)$ contains a finitely generated normal subgroup, whose quotient is Z. On one side Murasugi's work is constructive but with the restriction of asking for alternating links, on the other hand Stallings' result is quite general but it is usually hard to verify (see also [H]).

Goldsmith [Go] constructed a wide class of fibered links, what she called symmetric links, using cyclic branched coverings. Her results were extended by Birman [Bi] to include new examples.

Stallings [St2] proved that if a link can be represented as an homogeneous braid (i.e. on each column the braid has either all overcrossings or all undercrossings) then it is fibered (see also Birman - Williams [B-W]).

In this paper we will study another class of links, which under simple conditions imposed on a spanning surface are fibered links with that surface as fiber.

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Using this result and Stallings' theorem on homogeneous braids [St2], we can give explicity different fibrations of a fibered link. (see also [Ga], [Th], [Fr]). One example is the 6^3_3 link in Rolfsen's table [Ro]. Our results show that this link is fibered with fiber as shown in figure 0(a). Stallings' results apply to the homogeneous braid form of this link (shown in figura 0(b), obtained by flipping α in 0(a) as shown.

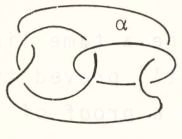


Figura 0(a)

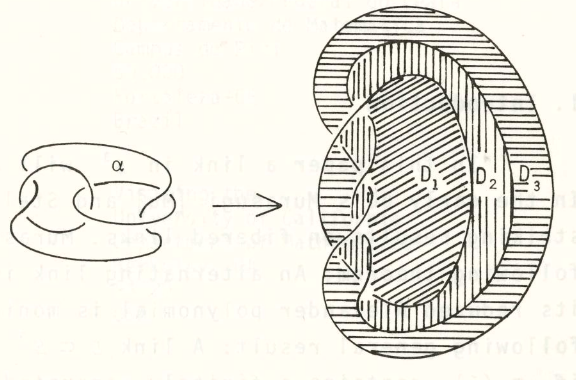


Figura 0(b)

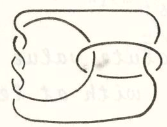
The Euler characteristic of the fiber for 0(a) is *-1* while that 0(b) is *-3* (it is obtained from *3* disks, D_1, D_2, D_3, and *6* strips along the crossings). Hence they are clearly distinct.

2. Definitions and Theorems

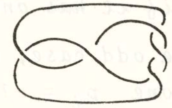
In this section we give some definitions essential for our work and state the main result.

Definition 2.1. *A pretzel link is a projected link as in figure 1, specified by* $n+1$ *numbers,* $p_1, \ldots, p_{n+1}$. *These numbers determine the number of crossings in each column in the following way: in the odd case, there are* $2p_i+1$ *crossings in each column, in the*

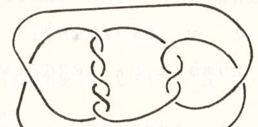
even case $2p_i$. *We will use the convention* $+1$ *and* -1 *for the crossings and respectively.*



1(a)
even case
$p_1=-2, p_2=1, p_3=-1.$



1(b)
odd case
$p_1=-1, p_2=-1, p_3=1.$



1(c)
odd case
$p_1=-1, p_2=-3, p_3=1, p_4=0.$

Figure 1

Definition 2.2. *A pretzel surface is the Seifert surface for a pretzel link given by two disks joined by* $n+1$ *strips connecting them, each strip having* $2p_i$ *or* $2p_i+1$ *twists (see figure 1 above). A pretzel-fibered link is one which is fibered with fiber a pretzel surface. If* L *is pretzel-fibered, the pretzel surface corresponding to the given presentation may not be the fiber of a fibration of* $S^3 - \mathring{N}(L)$.

Remark. We observe that in the odd case we have a knot if n is even (see figure 1(b)) and a link with two components if n is odd (see figure 1(c)). In the even case we always have a link

(1(a)). We remark also that a cyclic reordering of the p_i's gives the same link and same pretzel surface.

Theorem. *A pretzel link* $(p_1,\ldots,p_{n+1})$ *is pretzel-fibered if and only if it has one of the following forms:*

(A) *in the odd case, either each* p_i *has absolute value 1 with at least one* $p_i = -1$ *or each* $|p_i + 1| = 1$ *with at least one* $p_i = 0$, *or*

(B) *in the even case, each* p_i *has absolute value 1 for* $1 \leq i \leq n$ *and* p_{n+1} *is as follows:*

(1) $p_{n+1} = 0$,

(2) p_{n+1} *is arbitrary and* k, *the number of negative* p_i *for* $1 \leq i \leq n$, *is* $\frac{n}{2}$ *(hence* n *is even), or*

(3) $|p_{n+1}| = 2$ *and the number of negative* p_i*'s is* $\frac{n+1}{2}$.

This theorem for p_i odd and n even was known to R. Parris ([P]).

3. Proof of the theorem

The fundamental group of the pretzel surface S is a free group on n elements. Let us take generators $u_1,\ldots,u_n$ as shown in figure 2. Similarly $\pi_1(S^3-S)$ is a free group on n elements. We will take generators $x_1,\ldots,x_n$ as shown.

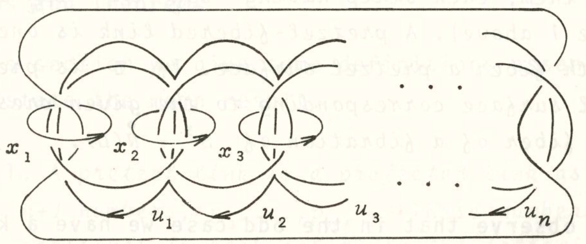


Figure 2

Take U to be an open neighborhood of interior S, with boundary the union of two surfaces S_1 and S_2 (as in [C-T]). We define maps j_1 and j_2 by the following diagram:

$$\begin{array}{ccccc} \pi_1(S_1) & \xrightarrow{i} & \pi_1(S^3-U) & \xrightarrow{\cong} & \pi_1(S^3-(U \cup K)). \\ & \searrow\cong & & & \swarrow\cong \\ & & \pi_1(S) & \xrightarrow{j_1} & \pi_1(S^3-S) \end{array}$$

(Similarly for j_2).

Stallings' fibration theorem gives the following:

Theorem. ([St], [N]). *A link is fibered with fiber* S *if and only if* j_1, j_2 *are isomorphisms.*

The following lemma ([C-T],[St]) will be the main tool in proving our theorem.

Lemma. (a) *If* c,d *is a pair of linearly independent elements of* $\pi_1(S^3-S)$ *and* u *is a nontrivial element of* $\pi_1(S)$ *and if* j *is an isomorphism, then* $(c,d;j(u) = 1)$ *is infinite cyclic.*

(b) *The group* $(c,d;\ c^m=d^n)$ *is infinite cyclic if and only if* $|m|=1$ *or* $|n|=1$.

We will treat the proof of the odd case and even case separately.

Proof of the odd case: We suppose that there exists an odd number of twists in each column given by $(2p_1+1,\ldots,2p_{n+1}+1)$. The map j_1 is given by:

$$j_1(u_1) = x_1^{p_1} x_2^{-(p_2+1)}$$

$$j_1(u_2) = x_2^{p_2} x_3^{-(p_3+1)}$$

$$\vdots$$

$$j_1(u_n) = x_n^{p_n}(x_n^{-1} \cdots x_1^{-1})^{-(p_{n+1}+1)}$$

$$j_1(u_n^{-1} \cdots u_1^{-1}) = (x_n^{-1} \cdots x_1^{-1})^{p_{n+1}} x_1^{-(p_n+1)}.$$

Using the lemma we have that if j_1 is an isomorphism, then

$$(x_i, x_{i+1};\ j_1(u_i) = x_i^{p_i} x_{i+1}^{-(p_{i+1}+1)} = 1)$$

and

$$(x_n^{-1} \cdots x_1^{-1}, x_1;\ j_1(u_n^{-1} \cdots u_1^{-1}) = 1)$$

are infinite cyclic. It is easy to see then that either every p_i is 1 in absolute value or every p_i+1 is, for $1 \le i \le n+1$.

The map induced on homology by j_1 is given by the following integer matrix.

$$\begin{pmatrix} p_1 & 0 & 0 & \cdots & 0 & p_{n+1}+1 \\ -(p_2+1) & p_2 & 0 & & 0 & p_{n+1}+1 \\ 0 & -(p_3+1) & p_3 & & \cdot & \cdot \\ \cdot & 0 & & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & p_{n-1} & p_{n+1}+1 \\ 0 & 0 & \cdot & \cdot & -(p_n+1) & p_n+p_{n+1}+1 \end{pmatrix}$$

If j_1 is to be an isomorphism, this matrix must be invertible. Hence the determinant is ± 1, i.e.,

$$p_1 p_2 \cdots p_n + p_1 p_2 \cdots p_{n-1}(p_{n+1}+1) + \cdots + p_1(p_3+1)\cdots(p_{n+1}+1) + (p_2+1)\cdots(p_{n+1}+1) = \pm 1.$$

If every p_i is 1, then the value of this determinant is

is $1+2+\cdots+2^n \neq \pm 1$. So in the case that every p_i has absolute value 1, we must have at least one $p_i = -1$. Similarly, in the case that every $|p_i+1| = 1$, at least one p_i must have value 0.

To summarize, if j_1 is an isomorphism, then either every p_i is 1 in absolute value with at least one -1 or every p_i+1 is 1 in absolute value with at least one $p_i = 0$.

We must now check that these two cases do indeed give isomorphisms. We will find the inverse.

In the first case, some $p_i = -1$. Assume, for convenience, that it is p_{n+1}, so $p_{n+1}+1 = 0$. Then $j_1(u_n) = x_n^{\pm 1}$ and $x_n = [j_1(u_n)]^{\pm 1}$. Hence

$$j_1(u_{n-1}) = x_{n-1}^{\pm 1} x_n^{-(p_n+1)} = x_{n-1}^{\pm 1}[j_1(u_n)]^{\pm(p_n+1)}$$

so

$$x_{n-1} = \{[j_1(u_{n-1})]^{\pm 1}[j_1(u_n)]^{\pm(p_n+1)}\}^{\pm 1}.$$

Continuing in this manner, we can write each x_i as a word in the u_j.

Similarly in the second case, assume it is p_1 that is 0. Then $j_1(u_1) = x_2^{\pm 1}$ so $x_2 = [j_1(u_1)]^{\pm 1}$. And $x_3 = [j_1(u_1)]^{\pm p_2}[j_1(u_2)]^{\pm 1}$, etc.

We must also check the map j_2. The argument is quite similar. The map j_2 is given by

$$j_2(u_1) = x_1^{p_1+1} x_2^{-p_2}$$

$$j_2(u_2) = x_2^{p_2+1} x_3^{-p_3}$$

$$\vdots$$

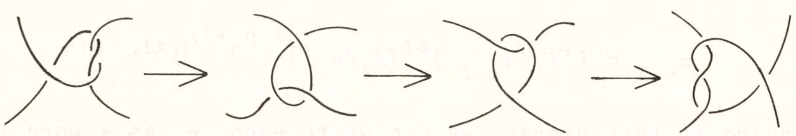
$$j_2(u_n) = x_n^{p_n+1}(x_n^{-1} \cdots x_1^{-1})^{-p_{n+1}}$$

$$j_2(u_n^{-1} \cdots u_1^{-1}) = (x_n^{-1} \cdots x_1^{-1})^{p_{n+1}+1} x_1^{-p_1}.$$

The lemma tells us that either every $|p_i+1|$ is 1 or every $|p_i|$ is 1. Again using the fact that the determinant of the map induced on homology is ± 1, we have that if j_2 is an isomorphism then either p_i is 1 in absolute value and at least one is -1, or every p_2+1 is 1 in absolute value and at least one is 0.

These cases can again be shown to give isomorphisms, the argument being essentially the same as before.

Hence those pretzel links with an odd number of twists in each column that are fibered are precisely those that either have 3 or -1 twists in each column, with at least one -1, or have -3 or 1 twists in each column with at least one 1. Notice that these pretzel links are independent of the order of the columns since one can exchange an adjacent -1 and 3 (or 1 and -3) as shown below.



$p_i=-1 \quad p_{i+1}=3$ $\qquad\qquad$ $p_i=3 \quad p_{i+1}=-1$

Figure 3

Proof of the even case: We will suppose that there exists an even number of twists in each strip given by $(2p_1,\dots,2p_{n+1})$. The map is given by

$$j_1(u_1) = x_1^{p_1}x_2^{-p_2}$$

$$j_1(u_2) = x_2^{p_2}x_3^{-p_3}$$

$$\vdots$$

$$j_1(u_n) = x_n^{p_n}(x_n^{-1}\dots x_1^{-1})^{-p_{n+1}}$$

$$j_1[u_n^{-1}\dots u_1^{-1}] = j_1(u_{n+1}) = (x_n^{-1}\dots x_1^{-1})^{p_{n+1}}x_1^{-p_1}$$

Let $v_{ij} = j_1(u_i)\dots j_1(u_j) = x_i^{p_i}x_j^{-p_j}$ for $1 \le i < j \le n+1$. Using the lemma, if j_1 is an isomorphism, then $(x_i,x_j;\ v_{ij} = x_i^{p_i}x_j^{-p_j} = 1)$ is infinite cyclic and therefore $|p_i|$ or $|p_j|$ is 1 for every $i \neq j$. If $|p_1| \neq 1$ then $|p_2|=|p_3|=\dots=|p_{n+1}|=1$. So all but at most one p_i has absolute value 1. Assume $|p_1| = |p_2| = \dots = |p_n| = 1$.

On homology, j_1 induces the following matrix.

$$\begin{bmatrix} p_1 & 0 & 0 & \dots & 0 & p_{n+1} \\ -p_2 & p_2 & 0 & \dots & 0 & p_{n+1} \\ 0 & -p_3 & 0 & \dots & 0 & p_{n+1} \\ 0 & 0 & 0 & \dots & p_{n-1} & p_{n+1} \\ 0 & 0 & 0 & \dots & -p_n & p_n+p_{n+1} \end{bmatrix}$$

As before, if j_1 is to be an isomorphism, this determinant must have value ± 1, i.e.

$$\sum_{i=1}^{n+1} p_1p_2\dots\hat{p}_i\dots p_{n+1} = p_1\dots p_n + p_{n+1}\left(\sum_{i=1}^{n} p_1\dots\hat{p}_i\dots p_n\right) = \pm 1.$$

Let k be the number of -1's in $p_1,\dots,p_n$. Then the value of this determinant is $(-1)^k + p_{n+1}(n-2k)$ and hence $p_{n+1}(n-2k)=0$ or -2. We have several cases.

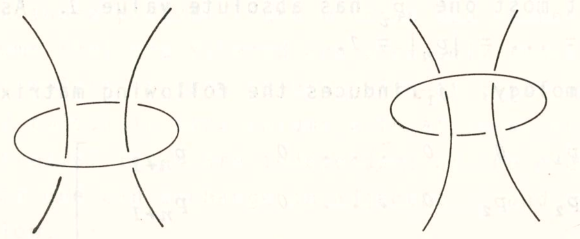
First, if $p_{n+1}(n-2k) = 0$, either (a) $p_{n+1} = 0$ or (b) $n = 2k$. In case (a), we have $j_1(u_{n+1}) = x_1^{\pm 1}$ and we can use $j_1(u_1) = x_1^{\pm 1}x_2^{\pm 1}$ to get x_2, etc. So we have an isomorphism.

In case (b) where $n = 2k$, p_{n+1} can take on any value. First consider the case where $p_1 = p_3 = \dots = p_{n-1} = 1$ and $p_2 = \dots = p_n = -1$. We can write

$$x_n^{-1} = j_1(u_n)[j_1(u_1)j_1(u_3)\dots j_1(u_{n-3})j_1(u_{n-1})]^{-p_{n+1}}$$

and use x_n to obtain $x_{n-1}, x_{n-2},\dots,x_1$ in terms of the u_j's.

Note that a pretzel link where all but one p_i is 1 in absolute value is independent of the order of the p_i's. By a cyclic re-ordering, one may assume $|p_1| = \dots = |p_n| = 1$. Then one can exchange an adjacent $+1$ and -1 as shown in figure 4 below.



$p_i=1$ $p_{i+1}=-1$ $p_i=-1$ $p_{i+1}=1$

Figure 4

Hence we may choose any order we like.

In the second case, where $p_{n+1}(n-2k) = -2$, then $p_{n+1} = \frac{-2}{n-2k}$ must be an integer. So again there are two cases:

(a) if n is odd, $n-2k = \pm 1$ so $k = \frac{n\pm 1}{2}$ and

(b) if n is even, $n-2k = \pm 2$ so $k = \frac{n\pm 2}{2}$. We will show that in either case j_1 is an isomorphism.

For (a), if $k = \frac{n-1}{2}$, $p_{n+1} = -2$. Consider the following order: $p_1 = p_3 = \dots = p_n = 1$, $p_2 = p_4 = \dots = p_{n-1} = -1$. We can write $j_1(u_n) = x_n(x_n^{-1} \dots x_1^{-1})^2 = [x_1 \dots x_{n-1}]^{-1} x_n^{-1} [x_1 \dots x_{n-1}]^{-1} = [j_1(u_1)j_1(u_3)\dots j_1(u_{n-4})j_1(u_{n-2})]^{-1} x_n^{-1} [j(u_1)j_1(u_3) \dots j_1(u_{n-2})]^{-1}$.
Then

$$x_n^{-1} = [j_1(u_1)j_1(u_3) \dots j_1(u_{n-2})] j_1(u_n) [j_1(u_1)j_1(u_3) \dots j_1(u_{n-2})]$$

and from the equations left we can get $x_{n-1}, \dots, x_1$ in terms of $j_1(u_1), \dots, j_1(u_n)$.

If $k = \frac{n+1}{2}$, $p_{n+1} = 2$, we consider the following ordering: $p_1 = \dots = p_{n-1} = -1$, $p_2 = \dots = p_n = 1$. We can write $[j_1(u_n)]^{-1} \dots [j_1(u_1)]^{-1} = (x_n^{-1} \dots x_1^{-1})^2 x_1 = [(x_2x_3)(x_4x_5)\dots(x_{n-1}x_n)]x_1^{-1}[(x_2x_3) \dots (x_{n-1}x_n)]^{-1}$.

So:

$x_1^{-1} = [j_1(u_2)j_1(u_4)\dots j_1(u_{n-1})][j_1(u_1)\dots j_1(u_n)]^{-1}[j_1(u_2)j_1(u_4) \dots j_1(u_{n-1})]$ and we can proceed to get $x_2, \dots, x_n$, giving us an isomorphism.

For (b), when n is even, we have two cases once more: if $k = \frac{n-2}{2}$ and $p_{n+1} = -1$ or if $k = \frac{n+2}{2}$ and $p_{n+1} = 1$. With an appropriate re-ordering, these can be considered special cases of (b) in the first case (when $n = 2k$).

The map j_2, in the even case, is the same as j_1, so we have nothing else to prove.

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