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NECESSARY CONDITIONS AND SUFFICIENT CONDITIONS OF WEAK MINIMUM FOR SOLUTIONS WITH CORNER POINTS

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1. Introduction

In some textbooks on Calculus of Variation, the second Weierstrass-Erdmann condition is found to be incorrectly formulated. We have shown in [c], by means of a counter-example, that such condition is not valid in the ordinary case of weak minimum.

It is not easy to find the historical origin of the mistake. It seems that Weierstrass and Erdmann worked only with strong extrema, in which case they have obtained the called Weierstrass-Erdmann's second condition.

According to Bolza ([b], p. 69), the weak metric was introduced by Kneser (1900). Later on we found the same second condition related with weak extrema attached with the names of Weierstrass and Erdmann.

So we think that the source of the mistake was the inadequacy of the Kneser metric to the study of weak extrema in the case where the functions, with fixed end-points, exhibit corner points.

The aim of this paper is to introduce, for such functions, the adequate weak metric where the second Weierstrass-Erdmann's conditions holds.

We show the adequacy of our weak metric by making a systematic study about the necessary conditions and sufficient ones in the above piecewise case.

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We add, in this study the case of smooth curves with moving end-points, since it is closely related to the problem of curves with corner points and fixed end-points (or moving ones); we believe we have found an adequate language in order with all these cases collectively.

We add likewise, in the final considerations, some results concerning the study of sufficient conditions for strong minimum.

The functional we consider here are of the following type:

$$I(y) = \int_a^b F(x, y, y')dx,$$

where $F: D\times\mathbb{R} \to \mathbb{R}$, $F \in C^4$, with D an open connected set in $\mathbb{R}^2$.

We regard, as admissible, those real functions y, of a real variable, defined and continuous on the closed interval $[a,b]$, of class D^1 (piecewise C^1), whose graphs are contained in D.

2. Necessary conditions of weak minimum

2.1. The usual metric. The Weierstrass-Erdmann's conditions. A counter-example.

Let M be the set of admissible functions of class D^1, defined on an interval $[a,b]$ and whose graphs have in common the end-points $A = (a, a_1)$, $B = (b, b_1)$, that is,

$$M = \{y: [a,b]\to\mathbb{R} \mid y \in D^1,\ y(a)=a_1,\ y(b)=b_1,\ \text{graph } (y) \subset D\}.$$

In the calculus of variations, the following metrics are introduced in M:

Strong distance between two functions $y_1, y_2 \in M$ is the real number

$$d_0(y_1, y_2) = \sup_{a\le x\le b} |y_1(x) - y_2(x)|.$$

Weak distance between two such functions is the real number

$$d_1(y_1, y_2) = \sup_{a\le x\le b} |y_1(x)-y_2(x)| + \sup_{a\le x\le b} |y_1'(x)-y_2'(x)| \quad (*)$$

We shall denote by

$$V_0(\psi, \varepsilon) = \{y \in M \mid d_0(\psi, y) < \varepsilon\},$$

the strong neighbourhood of center $\psi \in M$ and radius $\varepsilon > 0$, and by

$$V_1(\psi, \varepsilon) = \{y \in M \mid d_1(\psi, y) < \varepsilon\},$$

the weak neighbourhood of center $\psi \in M$ and radius $\varepsilon > 0$.

From this point on, these distances and neighbourhoods will be called *vertical*, in order to distinguish them from those to be introduced later in 2.2. We shall denote them by

$$d_0^v(y_1, y_2),\ d_1^v(y_1, y_2),\ V_0^v(\psi, \varepsilon),\ V_1^v(\psi, \varepsilon).$$

The following concepts of local minimum are usual ones:

$\psi \in M$ provides the functional I with a strong vertical minimum if there exists a strong neighbourhood $V_0^v(\psi, \varepsilon)$ such that

$$y \in V_0^v(\psi, \varepsilon) \Longrightarrow I(\psi) \le I(y).$$

$\psi \in M$ provides the functional I with a weak vertical minimum if there exists a weak neighbourhood $V_1^v(\psi, \varepsilon)$ such that

$$y \in V_1^v(\psi, \varepsilon) \Longrightarrow I(\psi) \le I(y).$$

In the calculus of variations, the following is a well-known theorem

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(*) If $y \in D^1$, $\sup_{a\le x\le b} |y'(x)|$ should be understood in the closed interval $[a,b]$ without those points where $y'(x)$ does not exist.

Theorem T_1: *If* $\psi \in D^1$ *provides the functional* I *with a weak vertical minimum, then* ψ *will fulfill the following integral equation*

$$F_{y'}(x, y, y') = \int_a^x F_y(\xi, y, y')d\xi + C, \quad \textit{where } C \textit{ is a constant.}$$

From this theorem there results the so-called

1^{st} *Weierstrass-Erdmann condition: If* $\psi \in D^1$ *provides the functional* I *with a weak vertical minimum, then* $F_{y'}(x, \psi(x), \psi'(x))$ *is a continuous function of* x *at the corner points of* ψ.

Some authors such as *Akhieser* [a], *Pars* [p], also present following "theorem"

Theorem T_2: *If* $\psi \in D^1$ *provides the functional* I *with a weak vertical minimum, then* ψ *will fulfill the following integral equation*

$$F(x, y, y') - y'F_{y'}(x, y, y') = \int_a^x F_x(\xi, y, y')d\xi + C, \quad \textit{where } C \textit{ is a constant.}$$

From this "*theorem*" there results the so-called

2^{nd} *Weierstrass-Erdmann condition: If* $\psi \in D^1$ *provides the functional* I *with a weak vertical minimum, then* $F(x, \psi(x), \psi'(x)) - \psi'(x)F_{y'}(x, \psi(x), \psi'(x))$ *is a continuous function of* x *at the corner points of* ψ.

A counter-example. The following counter-example will show that "theorem" T_2 and the 2^{nd} Weierstrass-Erdmann condition that it implies are both *false*.

Consider the functional

$$I(y) = \int_0^2 (6y'^4 - 14y'^3 + 9y'^2)dx$$

where $D = \mathbb{R}^2$, $A = (0, 0)$, $B = (2, 1)$, and the function

$$\psi(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ x-1, & 1 \leq x \leq 2 \end{cases}.$$

The graph of F as a function of y', sketched in the figure, shows that F has an absolute minimum at the point $y' = 0$, a local maximum at the point $y' = \frac{3}{4}$ and a local minimum at the point $y' = 1$.

For each function $y \in D^1$ such that

$$y'(x) = \begin{cases} \text{anything}, & 0 \leq x < 1 \\ > \frac{3}{4}, & 1 < x \leq 2, \end{cases}$$

we have $I(y) \geq I(\psi) = 1$.

Next, if $y \in V_1^v(\psi, \frac{1}{4})$, and

so $\sup_{0<x<2} |y'(x) - \psi'(x)| < \frac{1}{4}$, there follows that $I(y) \geq I(\psi) = 1$.

This proves that ψ provides the given functional with a weak vertical minimum. However, the function $F(\psi'(x)) - \psi'(x)F_{y'}(\psi'(x))$ is not continuous at the corner point $x = 1$. Indeed,

$$F(\psi'(x)) - \psi'(x)F_{y'}(\psi'(x)) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 < x \leq 2. \end{cases}$$

2.1.1. Remarks

(a) Theorem T_2 and the consequent 2^{nd} *Weierstrass-Erdmann* condition will become valid if we replace the weak vertical minimum hypothesis by that of strong vertical minimum (we shall prove this in 2.4.1.).

(b) Analogously, theorem T_2 will become true if we require that all admissible functions of M be C^1, keeping the hypothesis of weak vertical minimum (we shall prove this in 2.4.1.).

2.2. The suitable metric

Let $\psi \in M$. What we are going to call a regular parametrization of ψ to be a mapping

$$P: [a, b] \to \mathbb{R}^2, \quad P(t) = [x(t), y(t)],$$

satisfying the following conditions:

$$x(a)=a, \ x(b)=b, \ x \in C^1, \ \dot{x}(t) > 0, \quad \forall\, t \in [a, b], \quad y(t) = \psi(x(t)).$$

Note that, although ψ admits infinitely many regular parametrizations, the functional I, when "calculated" in each one of them, takes the same value $I(\psi)$, that is,

$$I(P) = \int_a^b F(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)})\dot{x}(t)dt = \int_a^b F(x, \psi(x), \psi'(x))dx = I(\psi).$$

Any two regular parametrizations P_1 and P_2, of the same $\psi \in M$, will be called equivalent, and this will be indicated by $P_1 \equiv P_2$. It is clear that $I(P_1) = I(P_2) = I(\psi)$.

Let N be the set of regular parametrizations of the functions of M, that is

$$N=\{P: [a, b] \to \mathbb{R}^2 \mid x \in C^1, \ y \in D^1, \ P(a)=A, \ P(b)=B, \ \dot{x}(t)>0, \ \forall\, t \in [a, b], \ \text{im}\ P \subset D\}.$$

In the set N the following metrics will be introduced:

Strong distance between $P_1, P_2 \in N$ is the real number

$$d_0(P_1, P_2) = \sup_{a \le t \le b} \{|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|\}.$$

Weak distance between $P_1, P_2 \in N$ is the real number

$$d_1(P_1, P_2) = \sup_{a \le t \le b} \{|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|\} + \sup_{a \le t \le b} \left|\frac{\dot{y}_1(t)}{\dot{x}_1(t)} - \frac{\dot{y}_2(t)}{\dot{x}_2(t)}\right| .$$

We shall denote by

$$V_0(P_1, \varepsilon) = \{P \in N \mid d_0(P_1, P) < \varepsilon\},$$

the strong neighbourhood of center $P_1 \in N$ and radius $\varepsilon > 0$, and by

$$V_1(P_1, \varepsilon) = \{P \in N \mid d_1(P_1, P) < \varepsilon\},$$

the weak neighbourhood of center $P_1 \in N$ and radius $\varepsilon > 0$.

The following concepts of local minimum are natural:

$P_1 \in N$ provides the functional I with a strong minimum if there exists a strong neighbourhood $V_0(P_1, \varepsilon)$ such that $P \in V_0(P_1, \varepsilon) \Longrightarrow I(P_1) \le I(P)$.

$P_1 \in N$ provides the functional I with a *weak minimum* if there exists a weak neibhbourhood $V_1(P_1, \varepsilon)$ such that $P \in V_1(P, \varepsilon) \Longrightarrow I(P_1) \le I(P)$.

2.2.2. Remark

If ψ has a regular parametrization P_1 that minimizes I, any other regular parametrization P_2 of ψ still minimizes I; in other words:

If P_1 provides the functional I with a weak (strong) minimum and $P_2 \equiv P_1$, then P_2 still provides I with a weak (strong) minimum.

2.3. Fundamental theorem. The Weierstrass-Erdmann's conditions

With the weak metric introduced in 2.2., we are able to develop, in a satisfactory way, the theory of necessary conditions (and of sufficient conditions) of weak minimum for functions with corner points and fixed end-points. First we have the following fundamental theorem

Theorem T_3: *If a regular parametrization of $\psi \in M$ provides the functional I with a weak minimum, then ψ will fulfill the following integral equations*

$$F_{y'}(x, y, y') = \int_a^x F_y(\xi, y, y')d\xi + C_1$$

$$F(x, y, y') - y'F_{y'}(x, y, y') = \int_a^x F_x(\xi, y, y')d\xi + C_2,$$

where C_1 and C_2 are constants.

Let $P_1(t) = [t, \psi(t)]$ be the natural parametrization of ψ and P a regular parametrization defined by:

$$P(t) = [t + \alpha\lambda(t), \psi(t) + \beta\mu(t)], \alpha, \beta \in \mathbb{R}, \lambda \in C^1, \mu \in D^1,$$

$$\lambda(a) = \lambda(b) = \mu(a) = \mu(b) = 0.$$

For each pair of functions λ, μ, the mapping P defines a two-parameter family (with α and β as parameters) and the functional, evaluated at the family, becomes a function of the parameters, that is, $I(P) = J(\alpha, \beta)$.

It is possible to show that, given $\varepsilon > 0$, there exists $\sigma > 0$, for which:

$$|\alpha| < \sigma, |\beta| < \sigma \Longrightarrow P \in V_1(P_1, \varepsilon).$$

Therefore, the function J has a local minimum at the point $\alpha = \beta = 0$ and, since the partial derivatives $J_\alpha(0, 0)$, $J_\beta(0, 0)$ do exist, they are both zero.

By means of a well-known technique, one can establish the following equalities:

$$J_\beta(0, 0) = \int_a^b \{F_{y'}(t, \psi(t), \dot\psi(t)) - \int_a^t F_y(\tau, \psi(\tau), \dot\psi(\tau))d\tau\}\dot\mu(t)dt$$

$$J_\alpha(0, 0) = \int_a^b \{F(t, \psi(t), \dot\psi(t)) - \dot\psi(t)F_{y'}(t, \psi(t), \dot\psi(t)) - \int_a^t F_x(\tau, \psi(\tau), \dot\psi(\tau))d\tau\}\dot\lambda(t)dt.$$

Since the functions λ and μ are arbitrary, one concludes that

$$F_{y'}(x, \psi(x), \psi'(x)) = \int_a^x F_y(\xi, \psi(\xi), \psi'(\xi))d\xi + C_1$$

$$F(x, \psi(x), \psi'(x)) - \psi'(x)F_{y'}(x, \psi(x), \psi'(x)) = \int_a^x F_x(\xi, \psi(\xi), \psi'(\xi))d\xi + C_2,$$

and the theorem is proved.

From this theorem and the continuity of the integral function there result

The Weierstrass-Erdmann's conditions: If a regular parametrization of $\psi \in M$ provides the functional with a weak minimum, then the functions $F_{y'}(x, \psi(x), \psi'(x))$ and $F(x, \psi(x), \psi'(x)) - \psi'(x)F_{y'}(x, \psi(x), \psi'(x))$ are continuous at the corner points of ψ.

2.4. Comparison between minima

The following implications are immediate:

I_1: strong vertical minimum $\Longrightarrow$ weak vertical minimum

I_2: strong minimum $\Longrightarrow$ weak minimum

In that follows we are going to establish other comparisons between minima which are fundamental in this study. First we antecipate, in a short way, the subsequent relationships:

I_3: strong minimum $\Longleftrightarrow$ strong vertical minimum

I_4: weak minimum $\Longrightarrow$ weak vertical minimum

I_5: weak vertical minimum $\not\Longrightarrow$ weak minimum

In order to justify these relationships, let us consider a function $\psi \in M$ together with any of its regular parametrizations, say, $P_1 \in N$.

Regarding Proposition I_3, we shall prove the theorem below

Theorem T_4: *A necessary and sufficient condition for ψ to provide a strong vertical minimum is that P_1 provides the functional with a strong minimum.*

We shall prove the necessity, since the sufficiency is immediate.

Let $P_1(t) = [t, \psi(t)]$ be the natural parametrization of ψ, $P \in N$ a regular parametrization of $y \in M$, with $P(t) = [t + \lambda(t), \psi(t) + \mu(t)]$, and $P_2 \equiv P_1$ defined by $P_2(t) \doteq [t + \lambda(t), \psi(t + \lambda(t))]$.

From the hypothesis we know that there exists $\varepsilon > 0$ for which:

$$d_0^v(\psi, y) < \varepsilon \Longrightarrow I(\psi) \leq I(y).$$

Since $d_0^v(\psi, y) = d_0(P_2, P)$, $I(\psi) = I(P_1)$, $I(y) = I(P)$, we have:

$$d_0(P_2, P) < \varepsilon \Longrightarrow I(P_1) \leq I(P).$$

Thus, it suffices to show that there exists $\delta > 0$, corresponding to $\varepsilon > 0$, such that:

$$d_0(P_1, P) < \delta \Longrightarrow d_0(P_2, P) < \varepsilon.$$

By the triangular property,

$$d_0(P_2, P) \leq d_0(P_1, P_2) + d_0(P_1, P),$$

we see that it is sufficient to show that there exists $0 < \delta < \frac{\varepsilon}{2}$ for which:

$$d_0(P_1, P) < \delta \Longrightarrow d_0(P_1, P_2) < \frac{\varepsilon}{2}.$$

Once we have

$$d_0(P_1, P_2) = \sup_{a \leq t \leq b} \{|\lambda(t)| + |\psi(t+\lambda(t)) - \psi(t)|\},$$

and since the function $\psi(x)$ is uniformly continuous on $[a, b]$, there exists $0 < \delta < \frac{\varepsilon}{4}$ such that, $\forall\, t \in [a, b]$,

$$|\lambda(t)| < \delta \Longrightarrow |\psi(t + \lambda(t)) - \psi(t)| < \frac{\varepsilon}{4}.$$

Thus, taking $d_0(P_1, P) = \sup_{a \leq t \leq b} \{|\lambda(t)| + |\mu(t)|\} < \delta$, and so $\sup_{a \leq t \leq b} |\lambda(t)| < \delta$, there results:

$$d_0(P_1, P_2) \leq \sup_{a \leq t \leq b} |\lambda(t)| + \sup_{a \leq t \leq b} |\psi(t+\lambda(t)) - \psi(t)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Regarding proposition I_4, the following theorem holds:

Theorem T_5: *If P_1 provides a weak minimum, then ψ provides the functional I with a weak vertical minimum.*

Its proof is immediate.

Explicitly, proposition I_5 means that:

If $\psi \in D^1$ provides a weak vertical minimum, one cannot conclude that P_1 provides the functional with a weak minimum.

In fact, the contradiction of this proposition implies, by the fundamental theorem T_3, the truth of the 2^{nd} *Weierstrass-Erdmann* condition, which is false by virtue of the counter-example of item 2.1.

We would like to point out that proposition I_5 contains, in our opinion, the fundamental idea of this work, and justifies the introduction of the weak metric in the set N of regular parametrizations, by means of which this theory is being developed.

We would like to remark that proposition I_5, in the case where the admissible functions are of class C^1, may be replaced by the following theorem.

Theorem T_6: *If $\psi \in C^1$ provides a weak vertical minimum, then P_1 provides the functional with a weak minimum.*

The proof is analogous to the one of theorem T_4.

2.4.1. Remark

Finally, we are now able to justify observations (a) and (b) of 2.1.1. The former is a consequence of propositions I_3, I_2, together with the fundamental theorem T_3. The latter is due to theorem T_6, together with the fundamental theorem T_3.

2.5. Smooth curves with moving end-points

Until now we have always considered admissible functions of class D^1 (or C^1), all of them defined on the same closed interval $[a, b]$, and taking the same values at the end-points. In this section we shall admit that these functions are C^1, but not necessarily defined in the same interval.

Let $\tilde{M}$ be the set,

$$\tilde{M} = \{y : [x_1, x_2] \to \mathbb{R} \mid y \in C^1, \text{ graph } (y) \subset D\},$$

where $[x_1, x_2]$ is any closed interval on the real line.

Clearly, in this set we cannot introduce those "vertical" metrics of item 2.1 and so we will use the metrics of 2.2.

Let $\tilde{N}$ be the set of regular parametrizations of the functions of $\tilde{M}$, all of them defined on the same closed interval $[a, b]$, that is to say,

$$\tilde{N} = \{P\colon [a, b] \to \mathbb{R}^2 \mid P \in C^1, \quad \dot{x}(t) > 0, \quad \forall\, t \in [a, b], \text{ im } P \subset D\}.$$

In this set we shall define the following metrics:

Strong distance between $P_1, P_2 \in \tilde{N}$ is the real number

$$d_0(P_1, P_2) = \sup_{a \le t \le b} \{|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|\}.$$

Weak distance between $P_1, P_2 \in \tilde{N}$ is the real number

$$d_1(P_1, P_2) = \sup_{a \le t \le b} \{|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|\} + \sup_{a \le t \le b} \left| \frac{\dot{y}_1(t)}{\dot{x}_1(t)} - \frac{\dot{y}_2(t)}{\dot{x}_2(t)} \right|.$$

The concepts of strong and weak minima are introduced in the same way as the corresponding concepts of item 2.2., and the remark 2.2.1. remains valid: If $\psi\colon [a, b] \to \mathbb{R}$ has a parametrization that minimizes I, any other parametrization of ψ still minimizes I.

We close this chapter stating the following theorem

Theorem T$_7$: *Let* $P_1 \in \tilde{N}$ *be a regular parametrization of function* $\psi\colon [a, b] \to \mathbb{R}$, $\psi \in \tilde{M}$.

If P_1 *provides the functional with a weak minimum, then:*

(a) ψ *will fulfill the differential equations below*

$$\frac{d}{dx} F_{y'} = F_y, \quad \frac{d}{dx}(F - y'F_{y'}) = F_x$$

(b) $F_{y'}(a, \psi(a), \psi'(a)) = F_{y'}(b, \psi(b), \psi'(b)) = 0$

$F(a, \psi(a), \psi'(a)) = F(b, \psi(b), \psi'(b)) = 0.$

3. Sufficient conditions of weak minimum

3.1. A H_σ-family. Fundamental lemma

Let $\psi\colon [a, b] \to \mathbb{R}$, $\psi \in C^1$, be a solution of *Euler's* equation $\frac{d}{dx} F_{y'} = F_y$. We say that ψ is normal if $F_{y'y'}(x, \psi(x), \psi'(x)) \neq 0$, $\forall\, x \in [a, b]$.

By arguments of continuity and compactness there exists a neighbourhood $V \subset D \times \mathbb{R}$, of the image of the mapping $\Gamma\colon [a, b] \to \mathbb{R}^3$, $\Gamma(x) = (x, \psi(x), \psi'(x))$, where $F_{y'y'}(x, y, y') \neq 0$.

In the neighbourhood V, *Euler's* equation can be normalized, that is, expressed in the form $y'' = f(x, y, y')$, $f\colon V \to \mathbb{R}$, $f \in C^2$,

$$f = \frac{F_y(x, y, y') - F_{y'x}(x, y, y') - y'F_{y'y}(x, y, y')}{F_{y'y'}(x, y, y')}$$

This equation admits a unique maximal solution relatively to any triple of initial values $(\xi, \eta, \eta') \in V$, which is of class C^2.

By a theorem of *Hilbert* [a], this equation and *Euler's* equation are equivalent in V.

Since the solution ψ is defined on a compact $[a, b]$, it is not maximal and therefore admits an extension to a new compact $[\tilde{a}, \tilde{b}]$, $\tilde{a} < a < b < \tilde{b}$.

One we have fixed $\xi \in [\tilde{a}, \tilde{b}]$, the set of solutions of *Euler's* equation relatively to the initial values $(\xi, \eta, \eta') \in V$ define a two-parameter family.

$$y = \psi(x; \eta, \eta'); \quad \psi(\xi; \eta, \eta') = \eta, \quad \psi'(\xi; \eta, \eta') = \eta'.$$

$\psi\colon [\tilde{a}, \tilde{b}] \to \mathbb{R}$ is the member of the family corresponding to the initial values $(\xi; \eta_0, \eta_0')$. It will be denoted by: $\psi(x) = \psi(x; \eta_0, \eta_0')$.

By well-known theorems about continuity and differentiability of solutions with respect to the initial values, it is possible to determine a number $\sigma > 0$, such that:

(a) Every solution $\psi(x; \eta, \eta')$ becomes defined on the interval $[\tilde{a}, \tilde{b}]$, if $|\eta - \eta_0| < \sigma$, $|\eta' - \eta_0'| < \sigma$.

(b) $\psi(x;\, \eta,\, \eta')$ is of class C^2 in the set: $\tilde{a} < x < \tilde{b}$, $|\eta - \eta_0| < \sigma$, $|\eta' - \eta_0'| < \sigma$.

A family of solutions of *Euler's* equation, that performs both conditions (a) and (b), will be denoted by H_σ. We will indicate by H_δ, $\delta < \sigma$, the subfamily of H_σ such that: $|\eta - \eta_0| < \delta$, $|\eta' - \eta_0'| < \delta$.

The following *fundamental lemma* is basic for the study of sufficient conditions of weak (or strong) minimum:

Lemma T_8: *Let $\psi: [a, b] \to \mathbb{R}$ be a normal solution of Euler's equation, fulfilling the following condition: Does not exist, relatively to ψ, a conjugate point of a in the interval $]a, b]$. Therefore it is possible to determine a H_δ - family (of solutions of Euler's equation) and two neighbourhoods $V_A \subset \mathbb{R}^2$, $V_B \subset \mathbb{R}^2$,* of points $A = [a, \psi(a)]$, $B = [b, \psi(b)]$, *such that:*

(a) *Whatever the points $Q_1 = [x_1, y_1] \in V_A$ and $Q_2 = [x_2, y_2] \in V_B$, through these points passes one and only one member of the family H_δ.*

(b) *Corresponding to such member, there does not exist a conjugate point of x_1 in the interval $]x_1, x_2]$.*

Proof. From the above considerations, the solution ψ may be extended to an interval $[\tilde{a}, \tilde{b}]$, $\tilde{a} < a < b < \tilde{b}$.

Relatively to ψ, a conjugate point of a in the interval $]a, b]$ does not exist. So, it is possible to find a point ξ, $\tilde{a} < \xi < a$, sufficiently close to a, such that: there is no conjugate point of ξ, relatively to ψ, in the interval $]\xi, d]$, where d is a point sufficiently close to b, $b < d < \tilde{b}$. Such a conclusion is a consequence of separation theorem of *Sturm*.

Let H_σ be the family of solutions of *Euler's* equation

$$y = \psi(x;\, \eta,\, \eta');\quad \psi(\xi;\, \eta,\, \eta') = \eta,\quad \psi'(\xi;\, \eta,\, \eta') = \eta'.$$

By a well-known property, the functions $\psi_\eta(x;\, \eta_0,\, \eta_0')$ and $\psi_{\eta'}(x;\, \eta_0,\, \eta_0')$, $\tilde{a} \le x \le \tilde{b}$, $\eta_0 = \psi(\xi)$, $\eta_0' = \psi'(\xi)$, are both solutions of *Jacobi's* equation (equation of variations), constructed with the function ψ. Besides, they are linearly independent, since:

$$\psi_\eta(\xi;\, \eta_0,\, \eta_0') = 1,\quad \psi_{\eta'}(\xi;\, \eta_0,\, \eta_0') = 0,\quad \psi'_{\eta'}(\xi;\, \eta_0,\, \eta_0') = 1$$

Therefore, the function χ, $\chi: [\tilde{a}, \tilde{b}] \to \mathbb{R}$, given by

$$\chi(x) = \psi_{\eta'}(a;\, \eta_0,\, \eta_0')\, \psi_\eta(x;\, \eta_0,\, \eta_0') - \psi_\eta(a;\, \eta_0,\, \eta_0')\, \psi_{\eta'}(x;\, \eta_0,\, \eta_0')$$

is likewise a solution of *Jacobi's* equation, satisfying the conditions $\chi(a) = 0$, $\chi'(a) \neq 0$.

Since there is no conjugate of a in the interval $]a, d]$, we have: $\chi(x) \neq 0$, $\forall\, x$, $a < x \le d$.

To complete the proof, let us consider the following system of implicit equations in the unknowns η and η',

$$\begin{cases} \psi(x_1;\, \eta,\, \eta') = y_1 \\ \psi(x_2;\, \eta,\, \eta') = y_2, \end{cases}$$

where $Q_1 = [x_1, y_1]$ and $Q_2 = [x_2, y_2]$ are points close to $A = [a, \psi(a)]$ and $B = [b, \psi(b)]$, respectively. Since

$$\begin{cases} \psi(a;\, \eta_0,\, \eta_0') = \psi(a) \\ \psi(b;\, \eta_0,\, \eta_0') = \psi(b) \end{cases} \quad \text{and the Jacobian} \quad \begin{vmatrix} \psi_\eta(a;\, \eta_0,\, \eta_0') & \psi_{\eta'}(a;\, \eta_0,\, \eta_0') \\ \psi_\eta(b;\, \eta_0,\, \eta_0') & \psi_{\eta'}(b;\, \eta_0,\, \eta_0') \end{vmatrix} \neq 0$$

(once we have $\chi(b) \neq 0$), we may assert, by the implicit function theorem, the existence and uniqueness of a pair of C^2 - functions

$$\begin{cases} \eta = \Psi_1(x_1,\, y_1,\, x_2,\, y_2) \\ \eta' = \Psi_2(x_1,\, y_1,\, x_2,\, y_2) \end{cases},\quad \text{defined on } V_A \times V_B,$$

where $V_A \subset \mathbb{R}^2$, $V_B \subset \mathbb{R}^2$, are suitable disjoint neighbourhoods of points A, B, and taking values in a suitable neighbourhood of point $[\eta_0, \eta'_0]$: $|\eta - \eta_0| < \rho < \sigma$, $|\eta' - \eta'_0| < \rho < \sigma$. This proves part (a) of the theorem.

With respect to part (b), let us consider the solution $\psi_{\eta'}(x;\ \eta_0,\ \eta'_0)$ of *Jacobi's* equation. Since there is no conjugate of point ξ in the interval $]\xi,\ d]$, we have:

$$\psi_{\eta'}(x;\ \eta_0,\ \eta'_0) \neq 0, \quad \forall\ x, \quad \xi < x \leq d, \quad \text{or}$$

$$\psi_{\eta'}(x;\ \eta_0,\ \eta'_0) \neq 0, \quad \forall\ x, \quad a_0 \leq x \leq d,$$

where a_0, $\xi < a_0 < a$, is a point sufficiently close to ξ. Therefore, by arguments of continuity and compactness, there exists a number δ, $0 < \delta < \rho$, such that:

$$\psi_{\eta'}(x;\ \eta,\ \eta') \neq 0, \quad \forall x, \quad a_0 \leq x \leq d, \quad |\eta - \eta_0| < \delta, \quad |\eta' - \eta'_0| < \delta.$$

Noting that $\psi_{\eta'}(x;\ \eta,\ \eta')$ is a solution of *Jacobi's* equation, constructed with the function $\psi(x;\ \eta,\ \eta')$, and performing both conditions $\psi_{\eta'}(\xi;\ \eta,\ \eta') = 0$, $\psi'_{\eta'}(\xi;\ \eta,\ \eta') = 1$, we may conclude that: if one restricts neighbourhoods V_A and V_B in such a way that $|\eta - \eta_0| < \delta$, $|\eta' - \eta'_0| < \delta$, then there does not exist, relatively to $\psi(x;\ \eta,\ \eta')$, any conjugate point of x_1 in the interval $]x_1,\ x_2]$, since this would imply that $\psi_{\eta'}(x;\ \eta,\ \eta')$ is zero at some interior point of that interval (*Sturm's* separation theorem). The theorem is proved.

3.1.1. Uniformity lemma

In the theory of sufficient conditions of weak minimum for curves with fixed end-points, the following is a well-known theorem

Theorem T_9: *Let* $\psi: [a,\ b] \to I\!R$, $\psi \in C^1$, *be a solution of Euler's equation performing the following conditions:*

(a) ψ *is positive normal;*

(b) *there does not exist, relatively to* ψ, *a conjugate point of* a *in the interval* $]a,\ b]$.

Then, ψ *provides the functional with a weak vertical minimum, in the set* M *of functions of class* D^1.

This theorem has been proved, such as in *Gelfand, Fomin* [g], in the set of functions of class C^1. However, by the rounding argument, we may conclude that the theorem is still valid in the set of functions of class D^1.

From the fundamental lemma T_8 and the theorem above, we conclude that every member $\psi(x;\ Q_1,\ Q_2)$ for the family H_δ, having as end-points $Q_1 = [x_1,\ y_1] \in V_A$ and $Q_2 = [x_2,\ y_2] \in V_B$, provides the functional with a weak vertical minimum; in other words, for each pair $Q_1 \in V_A$, $Q_2 \in V_B$, there is a positive number $\varepsilon(Q_1,\ Q_2)$, such that:

$$\forall\ y \in D^1,\ y(x_1) = y_1,\ y(x_2) = y_2,$$

$$y \in V_1^v(\psi(x;\ Q_1,\ Q_2),\ \varepsilon(Q_1,\ Q_2)) \Longrightarrow I(\psi(x;\ Q_1,\ Q_2)) \leq I(y).$$

Moreover, if one properly reduces neighbourhood V_A and V_B then the following result holds:

Uniformity lemma: The choice of $\varepsilon(Q_1,\ Q_2)$ *in the above implication may be done in order to exist* $\varepsilon > 0$ *such that, if* $Q_1 \in V_A$ *and* $Q_2 \in V_B$, *then* $\varepsilon \leq \varepsilon(Q_1,\ Q_2)$.

Proof. Let $\psi(x;\ Q_1,\ Q_2)$ be a member of the family H_δ, and let $K \subset V$ be a compact, x - convex set such that:

$$x \in [\tilde{a},\ \tilde{b}],\ Q_1 \in V_A,\ Q_2 \in V_B \Longrightarrow (x,\ \psi(x;\ Q_1,\ Q_2),\ \psi'(x;\ Q_1,\ Q_2)) \in K.$$

Furthermore, let $m > 0$ be the minimum of $F_{y'y'}(x,y,y')$, $(x,y,y') \in K$.

For each triplet $(Q_1,\ Q_2,\ \alpha)$, $0 < \alpha < m$, let us consider the quadratic functional that associates, to the function $h: [x_1,\ x_2] \to I\!R$, $h(x_1) = h(x_2) = 0$, the number

$$\int_{x_1}^{x_2} (Rh'^2 + Sh^2)dx - \alpha \int_{x_1}^{x_2} h'^2 dx, \qquad (1)$$

where

$$R = \frac{1}{2} F_{y'y'}(x, \psi(x; Q_1, Q_2), \psi'(x; Q_1, Q_2)),$$

$$S = \frac{1}{2}\left[F_{yy}(x, \psi(x; Q_1, Q_2), \psi'(x;Q_1,Q_2)) - \frac{d}{dx}F_{yy'}(x, \psi(x; Q_1, Q_2), \psi'(x; Q_1,Q_2))\right].$$

The corresponding *Euler's* equation,

$$-\frac{d}{dx}\left[(R-\alpha)h'\right] + Sh = 0, \qquad (2)$$

becomes reduced, for $\alpha = 0$, to the *Jacobi's* equation (equation of variations) of *Euler's* equation $\frac{d}{dx} F_{y'} = F_y$, relatively to $\psi(x; Q_1, Q_2)$.

Let $h(x; Q_1, Q_2, \alpha)$ be the solution of (2) such that $h(\xi; Q_1, Q_2, \alpha) = 0$, $h'(\xi; Q_1, Q_2, \alpha) = 1$.

We know that there does not exist, relatively to $\psi(x; A, B)$, any conjugate of ξ in the interval $]\xi, d]$, that is, $h(x; A, B, 0) \neq 0$, $\xi < x \leq d$. Therefore, by the continuity of the solution $h(x; Q_1, Q_2, \alpha)$, there exist $\tilde{V}_A \subset V_A$, $\tilde{V}_B \subset V_B$, $\nu > 0$, such that:

$$Q_1 \in \tilde{V}_A,\ Q_2 \in \tilde{V}_B,\ 0 < \alpha < \nu \Longrightarrow h(x; Q_1, Q_2, \alpha) \neq 0, \quad \forall x,\ \xi < x \leq d.$$

By a well-known property, $\forall Q_1 \in \tilde{V}_A$, $\forall Q_2 \in \tilde{V}_B$, $\forall \alpha$, $0 < \alpha < \nu$, the quadratic functional (1) becomes positive definite [c]. So, for a fixed c, $0 < c < \nu$, we have:

$$\int_{x_2}^{x_1} (Rh'^2 + Sh^2)dx > c\int_{x_2}^{x_1} h'^2dx, \quad \forall Q_1 \in \tilde{V}_A,\ \forall Q_2 \in \tilde{V}_B.$$

On the other hand, if $(x, \psi(x; Q_1,Q_2)+h(x), \psi'(x;Q_1,Q_2)+h'(x)) \in K$, $x \in [\tilde{a}, \tilde{b}]$, we have

$$I(\psi+h) - I(\psi) = \int_{x_1}^{x_2} (Rh'^2 + Sh^2)dx + \int_{x_1}^{x_2} (Mh^2 + Nh'^2)dx,$$

where

$$\psi=\psi(x; Q_1, Q_2),\ M = \frac{1}{3!}(\bar{F}_{yyy}h+3\bar{F}_{yyy'}h'),\ N = \frac{1}{3!}(3\bar{F}_{yy'y'}h+\bar{F}_{y'y'y'}h'),$$

with

$$\bar{F}_{rst} = F_{rst}(x, \psi(x; Q_1, Q_2) + \theta h(x), \psi'(x; Q_1, Q_2) + \theta h'(x))$$

and θ, $0 < \theta < 1$, given by *Taylor's* formula.

Corresponding to a given number $\delta > 0$, there exists $\varepsilon > 0$ such that

$$d_1^{\nu}(h, 0) < \varepsilon \Longrightarrow |M| < \delta,\ |N| < \delta.$$

Besides, by using Schwarz's inequality, we have

$$h^2(x) = \left(\int_{x_1}^{x} h'dx\right)^2 \leq (x-x_1)\int_{x_1}^{x} h'^2dx \leq (x-x_1)\int_{x_1}^{x_2} h'^2dx, \text{ that is}$$

$$\int_{x_1}^{x_2} h^2dx \leq \frac{(x_2-x_1)^2}{2}\int_{x_1}^{x_2} h'^2dx \leq \frac{(d-\xi)^2}{2}\int_{x_1}^{x_2} h'^2dx.$$

So, if $|M| < \delta$ and $|N| < \delta$, we have

$$\left|\int_{x_1}^{x_2} (Mh^2 + Nh'^2)dx\right| \leq \delta\left(1 + \frac{(d-\xi)^2}{2}\right)\int_{x_1}^{x_2} h'^2dx \text{ and}$$

$$I(\psi+h) - I(\psi) > \left[c - \delta\left(1 + \frac{(d-\xi)^2}{2}\right)\right]\int_{x_1}^{x_2} h'^2dx.$$

Therefore, once we have fixed δ, $0 < \delta < \dfrac{c}{1 + \frac{(d-\xi)^2}{2}}$, there exists $\varepsilon > 0$ for which:

$$d_1^{\nu}(h, 0) < \varepsilon \Longrightarrow I(\psi+h) - I(\psi) \geq 0, \quad \forall Q_1 \in \tilde{V}_A,\ \forall Q_2 \in \tilde{V}_B.$$

Being so, the uniformity lemma is proved.

3.2. Smooth curves with moving end-points

3.2.1. Theorem

Functional I, when calculated in the member $\psi(x; Q_1, Q_2)$ of family H_δ, $Q_1 \in V_A$, $Q_2 \in V_B$, defines a function $J: V_A \times V_B \to \mathbb{R}$, $J \in C^2$,

$$J(Q_1, Q_2) = \int_{x_1}^{x_2} F(x, \psi(x; Q_1, Q_2), \psi'(x; Q_1, Q_2))dx.$$

The existence of a minimum of the function J at point $[A, B]$, with $A = [a, \psi(a)]$, $B = [b, \psi(b)]$, is directly connected with the problem of determination of weak (or strong) minimum for smooth curves with moving end-points. This is what we shall make clear by means of the following theorem

Theorem T_{10}: *Let* $\psi: [a, b] \to \mathbb{R}$, $\psi \in C^1$, *be a solution of Euler's equation performing the following conditions:*

(a) ψ *is a positive normal solution;*

(b) *there does not exist, relatively to* ψ, *conjugate point of* a *in the interval* $]a, b]$;

(c) $I(\psi) = J(A, B) \leq J(Q_1, Q_2)$, $\forall\, Q_1 \in V_A$, $\forall\, Q_2 \in V_B$.

Therefore, if P_1 *is a regular parametrization of* ψ, P_1 *provides the functional with a weak minimum, in the set of regular parametrizations of admissible functions of class* D^1.

Let $P_1(t) = [t, \psi(t)]$, be the natural parametrization of ψ, P a regular parametrization of $y \in D^1$ defined by $P(t) = [t + \lambda(t), \psi(t) + \mu(t)]$, $P(a) = Q_1 \in V_A$, $P(b) = Q_2 \in V_B$ and let $P_2(t) = [t + \lambda(t), \psi(t + \lambda(t); Q_1, Q_2)]$. Both parametrized curves P_2 and P have in common the end-points Q_1 and Q_2; so, by theorem T_9 and the uniformity lemma, there exists $\varepsilon > 0$, independent from $Q_1 \in V_A$, $Q_2 \in V_B$, such that:

$$d_1^v(\psi(x; Q_1, Q_2), y) = d_1(P_2, P) < \varepsilon \Longrightarrow I(P_2) \leq I(P).$$

By hypothesis (c) of the theorem, we have $I(P_1) \leq I(P_2)$. Thus,

$$d_1(P_2, P) < \varepsilon \Longrightarrow I(P_1) \leq I(P).$$

Hence, it is sufficient to show that there exists $\delta > 0$, corresponding to $\varepsilon > 0$, such that:

$$d_1(P_1, P) < \delta \Longrightarrow d_1(P_2, P) < \varepsilon.$$

By the triangular property,

$$d_1(P_2, P) \leq d_1(P_1, P_2) + d_1(P_1, P),$$

we see to be sufficient to show that there exists $0 < \delta < \frac{\varepsilon}{2}$, such that:

$$d_1(P_1, P) < \delta \Longrightarrow d_1(P_1, P_2) < \frac{\varepsilon}{2}.$$

Being

$$d_1(P_1, P_2) = \sup_{a \leq t \leq b} \{|\lambda(t)| + |\psi(t+\lambda(t); Q_1, Q_2) - \psi(t)|\} + \sup_{a \leq t \leq b} |\psi'(t+\lambda(t); Q_1, Q_2) - \dot{\psi}(t)|$$

and since functions $\psi(x; Q_1, Q_2)$ and $\psi'(x; Q_1, Q_2)$ are uniformly continuous in a compact $x \in [a - r, b + r]$, $|x_1 - a| \leq r$, $|y_1 - \psi(a)| \leq r$, $|x_2 - b| \leq r$, $|y_2 - \psi(b)| \leq r$, $r > 0$ sufficiently small, there exists $0 < \delta < \frac{\varepsilon}{6}$, $\delta < r$, such that: $\forall\, t \in [a, b]$,

$$\begin{cases} |\lambda(t)| < \delta \\ |\mu(a)| < \delta \\ |\mu(b)| < \delta \end{cases} \Longrightarrow \begin{cases} |\psi(t + \lambda(t); Q_1, Q_2) - \psi(t)| < \frac{\varepsilon}{6} \\ |\psi'(t + \lambda(t); Q_1, Q_2) - \dot{\psi}(t)| < \frac{\varepsilon}{6} \end{cases}$$

Hence, taking

$$d_1(P_1, P) = \sup_{a \leq t \leq b} \{|\lambda(t)| + |\mu(t)|\} + \sup_{a \leq t \leq b} \left| \frac{\dot{\psi}(t) + \dot{\mu}(t)}{1 + \dot{\lambda}(t)} - \dot{\psi}(t) \right| < \delta$$

so that $\sup\limits_{a \leq t \leq b} |\lambda(t)| < \delta$, $|\mu(a)| < \delta$, $|\mu(b)| < \delta$, there results:

$$d_1(P_1, P_2) \leq \sup_{a \leq t \leq b} |\lambda(t)| + \sup_{a \leq t \leq b} |\psi(t+\lambda(t); Q_1, Q_2) - \psi(t)| + \sup_{a \leq t \leq b} |\psi'(t+\lambda(t); Q_1, Q_2) - \dot{\psi}(t)|$$

$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}$$

3.2.2. Remark

An important variational problem, regarding smooth curves with moving end-points, is the one in which the end-points belong to two given curves, Γ_1 and Γ_2. By virtue of the considerations we have presented here, it is clear that this problem is attached to the determination of the minimum of a function J restricted to curves Γ_1 and Γ_2, and a theorem, analogous to theorem T_{10}, could be stated and proved without difficulties.

3.3. Curves with corner points

3.3.1. Curves with one corner point and fixed end-points

Given a point $Q = [X, Y] \in D$, $a < X < b$, let $M(Q) = \{y \in M \mid y(X) = Y\}$, M being the set defined in 2.1.

We shall denote by $y_1 = y|_{[a,X]}$, $y_2 = y|_{[X,b]}$, the restrictions of $y \in M(Q)$ to the closed intervals $[a, X]$ and $[X, b]$.

Let yet $\psi \in M(C)$, $C = [c, \psi(c)]$, with $\psi_1 \in C^1$, $\psi_2 \in C^1$.

We shall admit the following hypothesis concerning function ψ:

(a) *ψ_1 and ψ_2 are positive normal solutions of Euler's equation;*

(b) *relatively to ψ_1, there is no conjugate point of a in the interval $]a, c]$;*

(c) *relatively to ψ_2, there is no conjugate point of c in the interval $]c, b]$.*

Under such conditions, if we apply the fundamental lemma T_8 to the functions ψ_1 and ψ_2, we may assert the existence of two families H_{δ_1}, H_{δ_2} (of solutions of *Euler's* equation) and of a neighbourhood V_c of point C, such that:
$\forall\ Q = [X, Y] \in V_c$,

(a) through A and Q, there passes one and only member of family H_{δ_1}, which will be indicated by $\psi_1(x; Q)$. One has: $\psi_1(a, Q) = \psi(a)$, $\psi_1(X; Q) = Y$;

(b) relatively to $\psi_1(x; Q)$, there is no conjugate point of a in the interval $]a, X]$;

(c) through Q and B, there passes one and only member of family H_{δ_2}, which will be indicated by $\psi_2(x; Q)$. One has: $\psi_2(X, Q) = Y$, $\psi_2(b, Q) = \psi(b)$;

(d) relatively to $\psi_2(x; Q)$, there is no conjugate point of X in the interval $]X, b]$.

We shall denote by $\psi(x; Q) \in M(Q)$ the function defined by

$$\psi(x; Q) = \begin{cases} \psi_1(x; Q), & a \le x \le X \\ \psi_2(x; Q), & X \le x \le b \end{cases}$$

By theorem T_9 and the uniformity lemma, we may assert, for each $Q \in V_c$, the existence of two vertical neighbourhoods $V_1^v(\psi_1(x; Q), \varepsilon)$, $V_2^v(\psi_2(x; Q), \varepsilon)$, ε independent from Q, such that:

(a) $\forall\ y_1 \in D^1$, $y_1(a) = \psi(a)$, $y_1(X) = Y$,
$y_1 \in V_1^v(\psi_1(x; Q), \varepsilon) \Longrightarrow I(\psi_1(x; Q)) \le I(y_1)$.

(b) $\forall\ y_2 \in D^1$, $y_2(X) = Y$, $y_2(b) = \psi(b)$,
$y_2 \in V_2^v(\psi_2(x; Q), \varepsilon) \Longrightarrow I(\psi_2(x; Q)) \le I(y_2)$.

The neighbourhood of center $\psi(x; Q)$ and radius $\varepsilon > 0$, in $M(Q)$ is simply the set

$$V_1^v(\psi(x; Q), \varepsilon) = \{y \in M(Q) \mid d^v(y, \psi(x; Q)) < \varepsilon\}.$$

Functional I, calculated in $\psi_1(x; Q)$, defines a function J_1,

$$J_1(Q) = \int_a^X F(x, \psi_1(x; Q), \psi_1'(x; Q))dx$$

and, calculated in $\psi_2(x; Q)$, defines another function J_2,

$$J_2(Q) = \int_X^b F(x; \psi_2(x; Q), \psi_2'(x; Q))dx.$$

Let:

$$J(Q) = J_1(Q) + J_2(Q) = \int_a^b F(x, \psi(x; Q), \psi'(x; Q))dx.$$

Under hypothesis (a), (b), (c) we may then assert, for each point $Q \in V_c$, the existence of a neighbourhood $V_1^v(\psi(x; Q), \varepsilon)$, ε independent from Q, such that: $\forall\ y \in D^1$,

$$y \in V_1^v(\psi(x; Q), \varepsilon) \Longrightarrow J(Q) \le I(y).$$

The existence of a minimum of the function J at point

$C = (c, \psi(c))$ is directly connected with the problem of determination of a weak (or strong) minimum, for curves with one corner point and fixed end-points. This is what we shall make clear by means of the next theorem

Theorem T_{11}: *Let $\psi \in M(C)$, $C = [c, \psi(c)]$, with $\psi_1 \in C^1$, $\psi_2 \in C^1$. Suppose that ψ and J fulfill the following conditions:*

(a) *ψ_1 and ψ_2 are positive normal solutions of Euler's equation;*

(b) *relatively to ψ_1, there is no conjugate point of a in the interval $]a, c]$;*

(c) *relatively to ψ_2, there is no conjugate point of c in the interval $]c, b]$;*

(d) $I(\psi) = J(C) \leq J(Q)$, $\forall\, Q \in V_c$.

Then, if P_1 is a regular parametrization of ψ, P_1 provides the functional with a weak minimum in the set N of regular parametrizations of the functions of M.

Proof. The proof of this theorem is just an appropriated adaptation of the proof of theorem T_{10}.

3.3.2. Remark

An important variational problem, regarding curves with one corner point and fixed end-points, is the one in which the corner-point belongs to a given curve Γ.

From our considerations, it is clear that this problem is attached to the determination of the minimum of function J conditioned to the curve Γ and a theorem, analogous to theorem T_{11}, could be stated and proved without difficulties.

3.3.3. Remaining cases

The case of n corner points and fixed (or moving) end-points may be dealt with in an analogous way, what reduces the problem to the study of the minimum of a function J having a suitable number of variables.

3.4. Final considerations

Completing this work, some results regarding sufficient conditions of strong minimum will be stated.

In problems involving such conditions, it is fundamental to consider the so-called *Weierstrass' function* E,

$$E: D\times\mathbb{R}^2 \to \mathbb{R},\quad E(x, y, y', w) = F(x, y, w) - F(x, y, y') - (w-y')F_{y'}(x, y, y')$$

or

$$E(x, y, y', w) = \frac{1}{2}(w-y')^2 F_{y'y'}(x, y, y' + \theta(w-y')), \qquad 0 < \theta < 1.$$

We say that a solution of *Euler's* equation $\psi: [a, b] \to \mathbb{R}$, $\psi \in C^1$ has positive excess if there exists a neighbourhood $V \subset D \times \mathbb{R}$, of the image of curve $\Gamma: [a, b] \to \mathbb{R}^3$, $\Gamma(x) = [x, \psi(x), \psi'(x)]$, such that:

$$(x, y, y') \in V, \quad w \neq y' \Longrightarrow E(x, y, y', w) > 0.$$

Regarding the study of sufficient conditions of strong minimum, in the case of smooth curves with moving end-points, one has the following theorem

Theorem T_{12}: *Let $\psi: [a, b] \to \mathbb{R}$, $\psi \in C^1$, be a solution of Euler's equation satisfying the following conditions:*

(a) *ψ is positive normal;*

(b) *there does not exist, relatively to ψ, conjugate point of a in the interval $]a, b]$;*

(c) *ψ has positive excess;*

(d) $I(\psi) = J(A, B) \leq J(Q_1, Q_2)$, $\forall Q_1 \in V_A$, $Q_2 \in V_B$.

Then, if P_1 *is a regular parametrization of* ψ, P_1 *provides the functional with a strong minimum, in the set of regular parametrization of admissible functions of class* D^1.

Regarding sufficient conditions of strong minimum, in the case of curves having one corner point and fixed end-points, one has the following theorem

Theorem T_{13}: *Suppose that* ψ *and* J, *defined in 3.3.1, satisfy the following conditions:*

(a) ψ_1 *and* ψ_2 *are positive normal solutions of Euler's equation;*

(b) *relatively to* ψ_1, *there is no conjugate point of* a *in the interval* $]a, c]$;

(c) *relatively to* ψ_2, *there is no conjugate point of* c *in the interval* $]c, b]$;

(d) ψ_1 *and* ψ_2 *have positive excess;*

(e) $I(\psi) = J(C) \leq J(Q)$, $\forall Q \in V_C$.

Then, ψ *provides a strong vertical minimum in the set* M.

References

[a] Akhiezer, N.I., *The Calculus of Variations*, translated by A.H. Frink Publishing Co., New York (1962).

[b] Bolza, O., *Lectures on the Calculus of Variations*, Dover Publications, Inc., New York (1961).

[c] Cesar, M.O., *Reformulation of the Second Weierstrass-Erdmann Condition*, Boletim da Sociedade Brasileira de Matemática, Vol. 13, Nº 1 (1982).

[g] Gelfand, I.M., Fomin, S.V., *Calculus of Variations*, translated by Richard A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963).

[p] Pars, L.A., *An Introduction to the Calculus of Variations*, Heinemann, London, Melbourne, Toronto (1962).

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