ON ISOMETRIC IMMERSIONS OF A TORUS INTO A SPACE FORM WITH THE SAME MEAN CURVATURE FUNCTION

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Blaine Lawson and Renato Tribuzy proved in [2] that, up to congruences, compact surfaces in the Euclidean space R^3 are, essentially, determined by the first fundamental form and only the trace of the second. The possible exception is the case of constant mean curvature which leads to a famous Hopf's conjecture that the surface is the round sphere. An explicit statement of their result is the following: "let M be a compact orientable surface with a Riemannian metric and let $h: M \to R$ be a smooth function. If h is not constant, then there exist at most two geometrically distinct isometric immersions of M into $M^3(c)$ with mean curvature h". Here $M^3(c)$ is a space form of dimension 3 and curvature c. Two isometric immersions are said to be geometrically distinct if they are not congruent.

In this paper we generalize this fact to isometric immersions of a torus into an n-dimensional space form $M^n(c)$ under the additional hypothesis of parallel normalized mean curvature vector $\frac{H}{|H|}$, where the mean curvature $|H|$ is non-constant and never vanishes. More precisely, we prove the following: *let T be a torus with a Riemannian metric and $h: T \to R$ be a positive non-constant smooth function. Then, there exist at most two geometrically distinct full isometric immersions of T into $M^n(c)$ with parallel normalized mean curvature vectors and mean curvature function h*, (cfr. Theorem 2).

Here $h = |H|$, where H is the mean curvature vector.

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The hypothesis of $\frac{H}{|H|}$ being parallel in the normal bundle of the immersions is essential in our proof because permits the existence of holomorphic quadratic differentials which in a torus are constant multiples of each other, by the Riemann-Roch's theorem.

In [4] is proved that given a non-constant function on a surface homeomorphic to the 2-sphere, there exists at most *one* isometric immersions of the surface in $M^3(c)$ having such a functions as mean curvature. This result also holds when the ambient space is an n-dimensional space form $M^n(c)$ under the additional hypothesis of parallel normalized mean curvature vector. In fact, by a theorem in [3] we have reduction of codimension and the immersion lies in a 3-dimensional space form, and so the above result can be applied.

We also prove that if a torus T admits two isometric immersions $x\colon T \to M^4(c)$ and $\tilde{x}\colon T \to M^3(c)$ with the same mean curvature function such that x is a full immersion and has parallel normalized mean curvature vector, then $\tilde{x}$ is one of two immersions determined by the coefficients of the second fundamental forms of x (cfr. Th. 1).

§1. Preliminaries. Let T be a torus equipped with a Riemannian metric and let e_1, e_2 be a global orthonormal frame on T associated to local isothermal parameters $z = x + iy$. That is, $e_1 = \frac{\partial}{\partial x}/\lambda$, $e_2 = \frac{\partial}{\partial y}/\lambda$, where $\lambda = |\frac{\partial}{\partial x}| = |\frac{\partial}{\partial y}|$. The metric on T is then given by

$$ds^2 = \lambda^2 |dz|^2. \tag{1.1}$$

Let ω_1, ω_2 the dual frame and

$$\omega_{12} = a_1\omega_1 + a_2\omega_2 \tag{1.2}$$

the connection 1-form on T. Then

$$d\omega_{12} = -K\, \omega_1 \wedge \omega_2 \tag{1.3}$$

where K is the Gaussian curvature of T.

Let $M^n(c)$ be an n-dimensional space form of curvature c and $x\colon T \to M^n(c)$ an isometric immersion. If $e_3,\dots,e_n$ is an orthonormal frame of normal vectors e_α and (h^α_{ij}) are the coefficients of the secont fundamental forms with respect to e_α relative to e_1, e_2, the mean curvature vector is given by

$$H = \frac{1}{n}\sum_{\alpha=3}^{n} (h^\alpha_{11} + h^\alpha_{22}).$$

Suppose $\frac{H}{|H|}$ is defined and parallel in the normal bundle of x. We assume that e_α have been chosen with $e_3 = \frac{H}{|H|}$. Then the globally defined form

$$\sum_{\alpha=4}^{n} (h^\alpha_{11} - ih^\alpha_{12})^2(\omega_1 + i\omega_2)^4 \tag{1.4}$$

is homorphic on T because each matrix (h^α_{ij}), $\alpha \geq 4$, has trace zero. If $n = 4$, also the quadratic form

$$(h^4_{11} - ih^4_{12})(\omega_1 + i\omega_2)^2 \tag{1.5}$$

is holomorphic on T. Hence we may assume that e_1, e_2 diagonalize (h^4_{ij}), hence also (h^3_{ij}), because the normal bundle of x is flat (by hypothesis e_3 is parallel, hence also e_4).

If $x\colon T \to M^4(c)$ is an isometric immersion, the Gauss equation is

$$\det(h^3_{ij}) + \det(h^4_{ij}) = K - c \tag{1.6}$$

and the Codazzi equations can be written

$$-e_j(h^\alpha_{ii}) + e_i(h^\alpha_{ij}) + (-1)^j a_i(h^\alpha_{ii} - h^\alpha_{jj}) + 2(-1)^j a_j h^\alpha_{ij} = 0, \tag{1.7}$$

where $\omega_{ij} = (-1)^j\{a_i\omega_i + a_j\omega_j\}$, $i = 1,2$, $i \neq j$, $\alpha = 3,4$ and a_i is given in (1.2).

§2. Isometric immersions of a torus with the same mean curvature function in $M^3(c)$ and $M^4(c)$. Let $x\colon T \to M^4(c)$ be an isometric immersion into a 4-dimensional space form of curvature c. We say that x is a full immersion into $M^4(c)$ if there is no totally geodesic submanifold $M^3(c)$ of $M^4(c)$ such that

$x(T) \subset M^3(c)$. We say that x has parallel normalized mean curvature vector if $\frac{H}{|H|}$ is parallel in the normal bundle of x. Immersions with parallel normalized mean curvature vectors were studied in [1] and [3].

Lemma 1. Let $x: T \to M^4(c)$ be a full isometric immersion with parallel normalized mean curvature vector $\frac{H}{|H|}$. Let e_1, e_2 be a global frame on T associated to local isothermal parameters diagonalizing the matrices (h^3_{ij}) and (h^4_{ij}) of the second fundamental forms of x relative to the orthonormal normal frame e_3, e_4, with $e_3 = \frac{H}{|H|}$. Then, there exist two local isometric immersion of T into $M^3(c)$ with mean curvature $|H|$, whose second fundamental forms are given by

$$\begin{pmatrix} h^3_{11} & h^4_{11} \\ h^4_{11} & h^3_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} h^3_{11} & -h^4_{11} \\ -h^4_{11} & h^3_{22} \end{pmatrix}$$

Proof: We will use the Codazzi equations for x, which can be written, for $\alpha = 3,4$,

$$-e_2(h^\alpha_{11}) + e_1(h^\alpha_{12}) + a_1(h^\alpha_{11}-h^\alpha_{22}) + 2a_2 h^\alpha_{12} = 0$$

$$-e_1(h^\alpha_{22}) + e_2(h^\alpha_{12}) + a_2(h^\alpha_{11}-h^\alpha_{22}) - 2a_1 h^\alpha_{12} = 0.$$

Since e_1, e_2 diagonalize both (h^3_{ij}) and (h^4_{ij}) we have

$$-e_2(h^3_{11}) + a_1(h^3_{11}-h^3_{22}) = 0 \tag{2.1}$$

$$-e_1(h^3_{22}) + a_2(h^3_{11}-h^3_{22}) = 0 \tag{2.2}$$

$$-e_2(h^4_{11}) + 2a_1 h^4_{11} = 0 \tag{2.3}$$

$$e_1(h^4_{11}) + 2a_2 h^4_{11} = 0. \tag{2.4}$$

If we change h^4_{11} for $-h^4_{11}$ in (2.3) and (2.4) we obtain

$$-e_2(-h^4_{11}) + 2a_1(-h^4_{11}) = 0 \tag{2.3)'$$

and

$$e_1(-h^4_{11}) + 2a_2(-h^4_{11}) = 0. \tag{2.4)'$$

On the other hand, the Gauss equation for x is

$$h^3_{11}h^3_{22} - (h^4_{11})^2 = K-c, \tag{2.5}$$

where K is the Gaussian curvature of T.

Now, by adding the equations (2.1) to (2.4) and subtracting (2.3) from (2.2), we get

$$-e_2(h^3_{11}) + a_1(h^3_{11}-h^3_{22}) + e_1(h^4_{11}) + 2a_2 h^4_{11} = 0 \tag{2.6}$$

and

$$-e_1(h^3_{22}) + a_2(h^3_{11}-h^3_{22}) + e_2(h^4_{11}) - 2a_1 h^4_{11} = 0. \tag{2.7}$$

Observe that (2.5), (2.6) and (2.7) are the Gauss and the Codazzi equations for a local isometric immersions $x_1: T \to M^3(c)$ whose second fundamental form has matrix

$$\begin{pmatrix} h^3_{11} & h^4_{11} \\ h^4_{11} & h^3_{22} \end{pmatrix},$$

with respect to e_1, e_2.

Similarly, by adding (2.1) and (2.4)', and then, subtracting (2.3)' from (2.2), we get

$$-e_2(h^3_{11}) + a_1(h^3_{11}-h^3_{22}) + e_1(-h^4_{11}) + 2a_2(-h^4_{11}) = 0 \tag{2.6)'$$

and

$$-e_1(h^3_{22}) + a_2(h^3_{11}-h^3_{22}) + e_2(-h^4_{11}) - 2a_1(-h^4_{11}) = 0. \tag{2.7)'$$

Again, (2.5), (2.6)' and (2.7)' are the Gauss and Codazzi equations for a local isometric immersion $x_2: T \to M^3(c)$ whose matrix of the second fundamental form is

$$\begin{pmatrix} h^3_{11} & -h^4_{11} \\ -h^4_{11} & h^3_{22} \end{pmatrix}$$

relative to e_1, e_2. Since both x_1 and x_2 have mean curvature function $|H|$, Lemma 1 is proved.

Let T be a torus equipped with a Riemannian metric and let $x,\tilde{x}: T \to M^4(c)$ two isometric immersions and H and $\tilde{H}$ the mean curvature vectors such that $|H| = |\tilde{H}|$ and both normalized mean curvature vectors $\frac{H}{|H|}$ and $\frac{\tilde{H}}{|\tilde{H}|}$ are parallel in the normal bundle of x and $\tilde{x}$, respectively. Choose normal frames $e_3 = \frac{H}{|H|}$, $e_4 \perp e_3$ and $\tilde{e}_3 = \frac{\tilde{H}}{|\tilde{H}|}$, $\tilde{e}_4 \perp \tilde{e}_3$. Denote by (h^α_{ij}) and $(\tilde{h}_{ij})$ the matrices of the second fundamental forms relative to e_α and $\tilde{e}_\alpha$, $\alpha = 3,4$, taken with respect to an orthonormal tangent frame e_1, e_2 associated to local isothermal parameters z on T. Let

$$H^3 = (h^3_{11}-h^3_{22}-2ih^3_{12}) \quad \text{and} \quad H^4 = (h^4_{11}-h^4_{22}-2ih^4_{12}), \qquad (2.8)$$

and define $\tilde{H}^3$ and $\tilde{H}^4$ similarly.

Lemma 2. Let $x,\tilde{x}: T \to M^4(c)$ be isometric immersions with the same mean curvature functions and both having parallel normalized mean curvature vectors $e_3 = \frac{H}{|H|}$ and $\tilde{e}_3 = \frac{\tilde{H}}{|\tilde{H}|}$. Suppose $H^4 \neq 0$ and $\tilde{H}^4 \neq 0$. Then, either

$$H^3 = \tilde{H}^3 \qquad (2.9)$$

or the mean curvature is constant.

Proof. We assume that the global orthonormal frame e_1, e_2 on T is locally associated to isothermal parameters. Denote by ω_1, ω_2 the dual frame. Since (h^4_{ij}) has trace zero, the quadratic differential $H^4(\omega_1+i\omega_2)^2$ is holomorphic. By Lemma 2.18 in [4], the quadratic differential $(H^3-\tilde{H}^3)(\omega_1+\omega_2)^2$ is holomorphic, because (h^3_{ij}) and $(\tilde{h}^3_{ij})$ have the same trace (by the hypothesis of equal mean curvature functions), hence the difference has trace zero. On the other hand, by the choice of e_3, e_4 both are parallel. Hence the second fundamental forms relative to e_3 and e_4 are simultaneously diagonalizable. Therefore, we may assume that e_1, e_2 diagonalize both (h^3_{ij}) and (h^4_{ij}).

By the Riemann-Roch's theorem the two holomorphic quadratic differentials $H^4(\omega_1+i\omega_2)^2$ and $(H^3-\tilde{H}^3)(\omega_1+i\omega_2)^2$ are constant multiples of each other, because the surface is a torus. Thus, we may write

$$H^3 - \tilde{H}^3 = bH^4, \quad b = \text{constant} \qquad (2.10)$$

Similarly, $\tilde{H}^4 = aH^4$, $a = \text{constant}$.

Now, because $H^3 = h^3_{11}-h^3_{22}$ and $\tilde{H}^3 = \tilde{h}^3_{11}-\tilde{h}^3_{11}-2i\tilde{h}^3_{12}$, we have

$$H^3-\tilde{H}^3 = (h^3_{11}-h^3_{22}) - (\tilde{h}^3_{11}-\tilde{h}^3_{22}-2i\tilde{h}^3_{12}) = h^3_{11} - (2|H|-h^3_{11}) - \tilde{h}^3_{11} + (2|H|-\tilde{h}^3_{11}) - 2i\tilde{h}^3_{12} = 2(h^3_{11}-\tilde{h}^3_{11}) - 2i\tilde{h}^3_{12},$$

where $|H|$ is the mean curvature function. On the other hand, $H^4 = 2h^4_{11}$. Therefore, writing $b = b_1+ib_2$, $h^3_{11}-\tilde{h}^3_{11}-i\tilde{h}_{12} = bh^4_{11}$, which gives that

$$h^3_{11} - \tilde{h}^3_{11} = b_1h^4_{11}$$

and

$$\tilde{h}^3_{12} = -b_2h^4_{11}. \qquad (2.11)$$

From (2.10), writing $a = a_1+ia_2$, we also have

$$\tilde{h}^4_{11} = a_1h^4_{11}$$

$$\tilde{h}^4_{12} = a_2h^4_{11}. \qquad (2.12)$$

1st Case. Suppose $H^3 - \tilde{H}^3$ is not pure imaginary. The Gauss equations of the two immersions give that

$$\det(h^3_{ij}) + \det(h^4_{ij}) = \det(\tilde{h}^3_{ij}) + \det(\tilde{h}^4_{ij}).$$

Hence,

$$h_{11}(2|H|-h^3_{11}) - (h^4_{11})^2 = \tilde{h}^3_{11}(2|H|-\tilde{h}^3_{11}) - (\tilde{h}^3_{12})^2 - (\tilde{h}^4_{11})^2 - (\tilde{h}^4_{12})^2.$$

By using (2.11) and (2.12) we get

$$2(h^3_{11}-\tilde{h}^3_{11})|H| - ((h^3_{11})^2-(\tilde{h}^3_{11})^2) - (h^4_{11})^2 + b^2_2(h^4_{11})^2 + a^2_1(h^4_{11})^2 + a^2_2(h^4_{11})^2 =$$

$$= 2(h^3_{11}-\tilde{h}^3_{11})|H| - ((h^3_{11})^2-(\tilde{h}^3_{11})^2) - (1-b^2_2-a^2_1-a^2_2)(h^4_{11})^2 =$$

$$= 2b_1|H|h^4_{11} - b_1h^4_{11}(h^3_{11}+\tilde{h}^3_{11}) - (1-b^2_2-a^2_1-a^2_2)(h^4_{11})^2 = 0.$$

Dividing by h^4_{11} we obtain

$$2b_1|H| - b_1(h^3_{11}+\tilde{h}^3_{11}) - (1-b^2_2-a^2_1-a^2_2)h^4_{11} = 0.$$

But $\tilde{h}^3_{11} = h^3_{11} - b_1h^4_{11}$. Substituting we get

$$2b_1|H| - 2b_1h^3_{11} + b^2_1h^4_{11} - (1-b^2_2-a^2_1-a^2_2)h^4_{11} = 0.$$

By hypothesis, either $H^3 - \tilde{H}^3 \equiv 0$ or $b_1 \neq 0$. In the last case dividing by b_1 we get

$$2|H| - 2h^3_{11} - \frac{1}{b_1}(1-b^2_2-a^2_1-a^2_2-b^2_1)h^4_{11} = 0.$$

But $2|H| - 2h^3_{11} = 2|H| - h^3_{11} - h^3_{11} = h^3_{22} - h^3_{11} = -H^3$. Hence

$$H^3 = -\frac{1}{2b_1}(1-b^2_2-a^2_1-a^2_2-b^2_1)H^4$$

proving that $H^3(w_1+iw_2)$ is holomorphic. By Lemma 2.18 in [4] the mean curvature is constant. Therefore, either $b = 0$, and so

$$H^3 = \tilde{H}^3,$$

or the mean curvature is constant, finishing the proof of 1st Case. Note that, in this case, we do not need the hypothesis of that $\tilde{H}^4 \neq 0$.

2nd Case. Suppose $H^3 - \tilde{H}^3$ is pure imaginary.

Because $(\tilde{h}^3_{ij})$ and $(\tilde{h}^4_{ij})$ commute (because the normal bundle of $\tilde{x}$ is flat) we have that

$$(\tilde{h}^3_{ij})(\tilde{h}^4_{ij}) = (\tilde{h}^4_{ij})(\tilde{h}^3_{ij}). \tag{2.13}$$

Since $\tilde{h}^4_{11} = -\tilde{h}^4_{22}$, (2.13) yields

$$\tilde{h}^4_{12}(\tilde{h}^3_{11}-\tilde{h}^3_{22}) = 2\tilde{h}^4_{11}\tilde{h}^3_{12}. \tag{2.14}$$

Now, either

$$\tilde{h}^4_{12} \equiv 0 \tag{2.15}$$

or

$$\tilde{h}^4_{12} \not\equiv 0. \tag{2.16}$$

Suppose (2.15) holds. By (2.14) either

$$\tilde{h}^4_{11} \equiv 0 \tag{2.17}$$

or

$$\tilde{h}^3_{12} \equiv 0. \tag{2.18}$$

In the first case, (2.17), we have that $\tilde{H}^4 \equiv 0$ (since $\tilde{h}^4_{12} \equiv 0$), which contradicts the hypothesis of the lemma. In the last case, (2.18) implies that

$$H^3 - \tilde{H}^3 \equiv 0, \tag{2.19}$$

since e_1, e_2 has been chosen to diagonalize (h^3_{ij}) and (h^4_{ij}) and by hypothesis $H^3 - \tilde{H}^3$ is pure imaginary. Therefore,

$$H_3 = \tilde{H}_3,$$

proving the lemma under the hypothesis (2.15).

Suppose (2.16) holds. Dividing (2.14) by $\tilde{h}^4_{12}$ we get

$$\tilde{h}_{11}-\tilde{h}_{22} = \frac{2\tilde{h}^4_{11}\tilde{h}^3_{12}}{\tilde{h}^4_{12}} = -\frac{2a_1h^4_{11}b_2h^4_{11}}{a_2h_{11}} = -\frac{2a_1b_2}{a_2}h^4_{11}, \tag{2.20}$$

by (2.11) and (2.12). Therefore, (2.20) and (2.11) yield

$$\tilde{H}^3 = -\frac{2a_1b_2}{a_2}h^4_{11} + ib_2h^4_{11} = (-\frac{2a_1b_2}{a_2} + ib_2)h^4_{11}.$$

This gives that

$$\tilde{H}^3(\omega_1+i\omega_2)^2 = (-\frac{a_1 b_2}{a_2} + i\frac{b_2}{2})H^4(\omega_1+i\omega_2)^2,$$

proving that $\tilde{H}^3(\omega_1+i\omega_2)^2$ is holomorphic, and so by Lemma 2.18 in [4], the mean curvature is constant. This finishes the proof of the lemma.

Theorem 1. Let $x: T \to M^4(c)$ be a full isometric immersion with parallel normalized mean curvature vector $\frac{H}{|H|}$. Suppose there exists an isometric immersion $\tilde{x}: T \to M^3(c)$ with the same mean curvature function $|H|$. If $|H|$ is not constant, then $\tilde{x}$ is given by one of the two immersion of Lemma 1.

Proof: In the previous notation, we may assume that (h^3_{ij}) and (h^4_{ij}) are diagonalized. We will show that

$$H^3 - \tilde{H}^3 = \text{pure imaginary}. \tag{2.21}$$

Suppose $H^3 - \tilde{H}^3$ is not pure imaginary. We may apply the argument in the 1st Case of Lemma 2 to conclude that

$$H^3 = \tilde{H}^3. \tag{2.22}$$

In fact, by the observation at the end of the 1st Case of Lemma 2, the argument applies even when $\tilde{x}$ is an isometric immersion of T into $M^3(c)$. From the Gauss equations for x and $\tilde{x}$, we obtain

$$\det(h^3_{ij}) + \det(h^4_{ij}) = \det(\tilde{h}^3_{ij}),$$

hence, by (2.22)

$$\det(h^4_{ij}) = 0. \tag{2.23}$$

Since (H^4_{ij}) is diagonalized, (2.23) gives that $h^4_{ij} \equiv 0$, contradicting the hypothesis of that x is a full immersion. Therefore (2.21) holds and we have

$$h^3_{11} = \tilde{h}^3_{11} \quad (\text{hence } h^3_{22} = \tilde{h}^3_{22}).$$

Again, by comparing the Gauss equations for x and $\tilde{x}$, we get

$$\tilde{h}^3_{12} = \pm h^4_{11}.$$

Thus, $(\tilde{h}_{ij})$ is

$$\begin{pmatrix} h^3_{11} & h^4_{11} \\ h^4_{11} & h^3_{22} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} h^3_{11} & -h^4_{11} \\ -h^4_{11} & h^3_{22} \end{pmatrix}$$

which are the matrices of the second fundamental forms of the two local immersions of T into $M^3(c)$ given by Lemma 1.

§3. Isometric immersions of a torus in $M^n(c)$ with the same mean curvature function. Let T be a torus with a Riemannian metric, e_1, e_2 a global orthonormal tangent frame associated to local isothermal parameters on T. Let ω_1, ω_2 be the dual frame.

Theorem 2. Let T be a torus with a Riemannian metric and $h: T \to R$ be a positive non-constant smooth function. Then, there exists at most two full isometric immersions of T in $M^n(c)$ with parallel normalized mean curvature vectors and the same mean curvature function h.

Proof. We separate the proof in two steps.

1st step - The ambient space is $M^4(c)$. Let $x, \tilde{x}, \tilde{\tilde{x}}: T \to M^4(c)$ be full isometric immersions of the torus with the same mean curvature function $h = |H| = |\tilde{H}| = |\tilde{\tilde{H}}|$ and having

$$\frac{H}{|H|}, \quad \frac{\tilde{H}}{|\tilde{H}|} \quad \text{and} \quad \frac{\tilde{\tilde{H}}}{|\tilde{\tilde{H}}|}$$

parallel in the respective normal bundles. Consider the normal orthonormal frame $e_3 = \frac{H}{|H|}$ and $e_4 \perp e_3$ for the immersion x. Take the quadratic differentials $H^3(\omega_1+i\omega_2)^2$ and $H^4(\omega_1+i\omega_2)^2$, obtained from the coefficients of the second fundamental forms (h^3_{ij}) and (h^4_{ij}), with H^3 and H^4 as in (2.8). Similarly, we define $\tilde{H}^3$, $\tilde{H}^4$, $\tilde{\tilde{H}}^3$ and $\tilde{\tilde{H}}^4$ relative to the immersions $\tilde{x}$ and $\tilde{\tilde{x}}$ and consider analogous quadratic differentials.

Observe that, none of H^4, $\tilde{H}^4$ and $\tilde{\tilde{H}}^4$ can vanish since by hypothesis, x, $\tilde{x}$ and $\tilde{\tilde{x}}$ are full isometric immersions. Then, we may apply Lemma 2 to conclude that

$$H^3 = \tilde{H}^3 \quad \text{and} \quad H^3 = \tilde{\tilde{H}}^3, \tag{3.1}$$

with e_1, e_2 chosen to diagonalize (h^4_{ij}), hence also (h^3_{ij}). Therefore, by (3.1) also $(\tilde{h}^3_{ij})$ and $(\tilde{h}^4_{ij})$ are also diagonalized by e_1, e_2 because $(\tilde{h}^3_{ij})$ commutes with $(\tilde{h}^4_{ij})$. Now, the Gauss equations for x and $\tilde{x}$ give that

$$h^3_{11}h^3_{22} - (h^4_{11})^2 = \tilde{h}^3_{11}\tilde{h}^3_{22} - (\tilde{h}^4_{11})^2.$$

Hence, by (3.1),

$$h^4_{11} = \pm\tilde{h}^4_{11}. \tag{3.2}$$

Similarly, one proves that

$$h^4_{11} = \pm\tilde{\tilde{h}}^4_{11}. \tag{3.3}$$

Therefore, (3.2) and (3.3) gives that either $H^4 = \tilde{H}^4$ or $H^4 = \tilde{\tilde{H}}^4$. This, together with (3.1) yield

$$H^3 = \tilde{H}^3 = \tilde{\tilde{H}}^3 \tag{3.4}$$

and

$$\text{either } H^4 = \tilde{H}^4 \text{ or } H^4 = \tilde{\tilde{H}}^4. \tag{3.5}$$

Because x, $\tilde{x}$ and $\tilde{\tilde{x}}$ have the same mean curvature function, (3.4) implies that

$$(h^3_{ij}) = (\tilde{h}^3_{ij}) = (\tilde{\tilde{h}}^3_{ij}).$$

On the other hand, (3.5) yields

$$\text{either } (h^4_{ij}) = (\tilde{h}^4_{ij}) \text{ or } (h^4_{ij}) = (\tilde{\tilde{h}}^4_{ij}).$$

Therefore, either $x = \tilde{x}$ or $x = \tilde{\tilde{x}}$ (since the normal bundles are flat), up to congruences. This finishes the 1st step of the proof.

2nd step - The ambient space is $M^n(c)$. Consider three immersions $x, \tilde{x}, \tilde{\tilde{x}}: T \to M^n(c)$, with the same mean curvature function and all

having parallel normalized mean curvature vectors. Choose a normal orthonormal frame $e_3 = \frac{H}{|H|}$, $e_4, \ldots, e_n$, relative to x. We work with a global orthonormal frame associated to local isothermal parameter $z = x+iy$, with $e_1 = \frac{\partial}{\partial x}/\lambda$ and $e_2 = \frac{\partial}{\partial y}/\lambda$, where $\lambda = |\frac{\partial}{\partial x}| = |\frac{\partial}{\partial y}|$. Denote by (h^α_{ij}) the coefficients of the second fundamental forms relative to e_α, $\alpha = 3,\ldots,n$ with respect to e_1, e_2. Define

$$\phi = \sum_{\alpha=4}^{n} (h^\alpha_{11} - h^\alpha_{22} - 2ih^\alpha_{12})^2 (w_1 + iw_2)^4,$$

where w_1, w_2 is the dual frame. It is known that ϕ is holomorphic and globally defined on T, because each matrix (h_{ij}) has trace zero, for $\alpha = 4,\ldots,n$.

We use "$\sim$" and "$\approx$" to distinguish entities relative to the immersions $\tilde{x}$ and $\tilde{\tilde{x}}$, analogous of those relative to the immersion x. Since ϕ, $\tilde{\phi}$ and $\tilde{\tilde{\phi}}$ are holomorphic on the torus, by the Riemann-Roch's theorem two of these are constant complex multiples of each other.

Since T is connected and $|H|$ is not constant, there exists a point $p \in T$ such that $dH(p) = 0$.

We claim that the curvature tensors $R^\perp$, $\tilde{R}^\perp$ and $\tilde{\tilde{R}}^\perp$ of the immersions x, $\tilde{x}$ and $\tilde{\tilde{x}}$, respectively, vanish in a neighborhood of p. Suppose this is not true. Then, there exists a sequence of points p_k converging to p such that, for each k either

$$R^\perp(p_k) \neq 0 \quad \text{or} \quad \tilde{R}^\perp(p_k) \neq 0 \quad \text{or} \quad \tilde{\tilde{R}}^\perp(p_k) \neq 0.$$

In this case, for a theorem in [1], each p_k has a neighborhood minimally immersed in an umbilical hypersurface of $M^n(c)$ and so $|H|$ is constant in such a neighborhood; therefore, $dH(p_k) = 0$ and then, by continuity, also $dH(p) = 0$, a contradiction.

Thus, there exists a neighborhood V of p such that $R^\perp \equiv 0$, $\tilde{R}^\perp \equiv 0$ and $\tilde{\tilde{R}}^\perp \equiv 0$ in V and this implies that the codimension of each of the three immersions x, $\tilde{x}$ and $\tilde{\tilde{x}}$ can be reduced to two. Therefore, in V, ϕ, $\tilde{\phi}$ and $\tilde{\tilde{\phi}}$ are reduced to the holomorphic quadratic forms

$$H^4(\omega_1+i\omega_2)^2, \quad \tilde{H}^4(\omega_1+i\omega_2)^2 \quad \text{and} \quad \tilde{\tilde{H}}^4(\omega_1+i\omega_2)^2.$$

Moreover, in V, both $\tilde{H}^4$ and $\tilde{\tilde{H}}^4$ are constant multiples of H^4.

We remark that the same computation in the first case of Lemma 2 can be applied to show that $H^3 = \tilde{H}^3$. For, we need first to prove that when (h^4_{ij}) is diagonalized in V, then both $H^3 - \tilde{H}^3$ and $H^3 - \tilde{\tilde{H}}^3$ are not pure imaginary. But this is a consequence of the following fact: there exist a point q in the boundary of V and a sequence $q_k \notin V$ such that q_k converges to q and $\tilde{R}^{\perp}(q_k) \neq 0$ (because $n > 4$), and so, by the mentioned theorem in [1],

$$\tilde{H}^3(q_k) = 0, \quad \text{hence, by continuity,} \quad \tilde{H}^3(q) = 0.$$

The same holds for $\tilde{\tilde{H}}^3$. Now let e_1, e_2 be an orthonormal frame associated to isothermal parameters that makes real the coefficients of ϕ. In V such a frame makes H^4 real and then (h^4_{ij}) is diagonalized. Hence, also (h^3_{ij}) is diagonalized and H^3 is real in V. By continuity, H^3 is real at q. Thus, $(H^3 - \tilde{H}^3)(q)$ is real and therefore $H^3 - \tilde{H}^3$ is real in V. The same holds for $H^3 - \tilde{\tilde{H}}^3$.

Thus, in V, we have that $H^3 = \tilde{H}^3$ and $H^3 = \tilde{\tilde{H}}^3$. The conclusion now follows by using the same argument of the first step of the proof.

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