BOL. SOC. BRAS. MAT., VOL. 16 Nº 1 (1985), 45-83

SIMPLE PROOFS OF LOCAL CONJUGACY THEOREMS FOR DIFFEOMORPHISMS OF THE CIRCLE WITH ALMOST EVERY ROTATION NUMBER

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Introduction

If $\alpha \in \mathbb{R}$ satisfies the diophantine condition, DC_β: $\exists \beta \geq 0, \exists \gamma > 0, \forall p/q \in \mathbb{Q}$ implies $|\alpha - p/q| \geq \gamma q^{-2-\beta}$ and if $0 \leq \beta \leq 1$, we propose to give simple proofs of some theorems of [2, Annexe] (i.e. the local conjugacy theorems of diffeomorphisms of the circle to rotations of rotation number α).

The main new idea is the use of the Schwarzian derivative of a diffeomorphism $f \in D^3(\mathbb{T}^1_d)$, $Sf \equiv S(f) = D^2\text{Log}\, Df - \frac{1}{2}(D\, \text{Log}\, Df)^2$. To solve $f \circ h = h \circ R_\alpha$, R_α being the translation of $\mathbb{T}^1$ or $\mathbb{R}$ $R_\alpha(\theta) = \theta + \alpha$, we take the Schwarzian derivatives: $S(f \circ h) = S(h \circ R_\alpha)$ which reduces to $((Sf) \circ h)(Dh)^2 = (Sh) \circ R_\alpha - Sh$; to obtain Sh we solve the linear difference equation and as $0 \leq \beta \leq 1$, $\alpha \in DC_\beta$ we only "lose" 2 derivatives for the function $((Sf) \circ h)(Dh)^2$. The use of the Schwarzian derivative was in [2,IX.2.1], implicitly, one of the main ingredients of the proof of the fundamental theorem of [2]. The fundamental theorem of [2] has recently been generalized by J.C. Yoccoz ([8] and [9]).

In the present paper we will only be interested in the local theorems (i.e. the existence of a "smooth" diffeomorphism h when f is a smooth enough perturbation of R_α). Our proofs are very similar to the case of rotation numbers of constant type we have already given in [3] and [4].

Research supported in part by NSF Grant MCS 8120790.

Recebido em 31/01/85. Cópia revista em 15/07/85.

In I, we study the Sobolev smoothness of h for $\alpha \in DC_\beta$, $0 \leq \beta \leq 1$, following the proof [3,IV.4.3] we have given for rotation numbers of constant type in the Hölder case (see also [3,VII.10] for the Sobolev case for rotation numbers of constant type). From I.13 to I.17, we study (rapidly) the Hölder case (that is already known by [2,A]).

In II, we study V. I. Arnold's local theorem ([1] and [2,A]). For that if $\alpha \in DC_\beta$, $0 \leq \beta \leq 1$, we also use the Schwarzian derivative trick and similar ideas to those of the proof we have given in [4,VIII] in the case of rotation numbers of constant type. We refer the reader to [4,VIII] for some details. The present proof, that only uses the standard implicit function theorem, yields also V. I. Arnold's result on analytic dependence on analytic parameters.

All the results we obtained in [4,VIII.7 to 14] stay true for $\alpha \in DC_\beta$, $0 \leq \beta \leq 1$, with the same proofs as the ones given in [4,VIII.7 to 14] (for the global conjugacy the reader will use [2], [8] and [9]).

In particular, by the same proof as in [4,VIII.12], one obtains Siegel's theorem for $\alpha \in DC_\beta$, $0 \leq \beta \leq 1$. By a similar proof as the one given for II.8 the reader can prove directly, as an exercise, Siegel's theorem, by introducing the appropriate Hardy Sobolev spaces on the unit disk $\{z \in \mathbb{C}, |z| < 1\}$.

Let D^0_δ be the set of mappings that are of the form identity $+\phi$, where ϕ is a $\mathbb{Z}$-periodic $\mathbb{C}$-analytic function on $B_\delta = \{z \in \mathbb{C}, |\mathrm{Im}\, z| < \delta\}$, continuous on $\bar{B}_\delta$. In III, following V.I. Arnold [1], we study the dependance on α of the functional equation

$$(+) \quad \lambda_\alpha + f\circ h_\alpha = h_\alpha \circ R_\alpha$$

where $\lambda_\alpha \in \mathbb{C}$, $f \in D^0_\delta$ and $h_\alpha \in D^0_{\delta/2}$. To do this we complexify α. We only allow α to belong to a closed set $\Omega \subset \mathbb{C}$ such that $\Omega \cap \mathbb{R} = C_\gamma = \{\alpha \in \mathbb{R} \mid \forall\, p/q \in \mathbb{Q}, |\alpha - (p/q)| \geq \gamma q^{-2-\beta}\}$ where $0 < \beta < 1$ is fixed and $\gamma > 0$ is small but fixed. The set Ω is defined in order to study non-tangential limits to C_γ (in contrast to V. I. Arnold [1] we define Ω directly, cf. [1, p. 244]).

In III.6, we show, that if f-Id is "small enough" and $\alpha \in \Omega$, then we can find solutions λ_α and h_α of (+), that depend continuously on $\alpha \in \Omega$ and holomorphically on $\alpha \in \mathrm{Int}\,\Omega$. We define the mapping G by $G(\alpha,f) = (\lambda_\alpha, h_\alpha) \in \mathbb{C} \times D^0_{\delta/2}$ and denote by J the inclusion of $\mathbb{C} \times D^0_{\delta/2}$ into $\mathbb{C} \times D^0_{\delta/4}$.

In III.13, we show that the mapping $J\circ G(z,f): z \in \Omega \to (\lambda_z, h_z) \in \mathbb{C}\times D^0_{\delta/4}$ is C^1-*holomorphic* (i.e. $z \to J\circ G(z,f)$ is C^1 in the sense of Whitney and satisfies $\bar{\partial}(J\circ G(z,)) = 0$). This solves a question left open by V.I. Arnold [1,p. 251]. The proof is completely elementary and reduces to the study (in III.9) of non-tangential limits of the derivatives of $\alpha \in \mathrm{Int}\,\Omega \to G(\alpha,f)$, with respect to α, when $\alpha \to C_\gamma$.

The method is general, simple and can be applied to other problems in small divisors (see, for example, a very similar question studied by Belokolos in [B]). For a related question, we refer the reader to J. Pöschel's work [5].

In III.16, we recall well known properties of C^1-holomorphic mappings on compact sets of $\mathbb{C}$. For reasons we explain in 16, we prefer the terminology C^1-*holomorphic* to *monogenic* functions (every C^1-holomorphic function defines a monogenic function in the sense of Emile Borel).

I would like to thank Faye Yeager for typing with great care the manuscript, Claudine Harmide and Marie-Jo Lécuyer for typing the corrections, and Jean-Christophe Yoccoz for helping me to improve the presentation of the paper.

This paper was written during my visit at the Mathematical Sciences Research Institute and I would like to acknowledge support from the N.S.F.

I. THE SOBOLEV CASE

1. Notations

Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be the circle and $d\theta$ the normalized Haar measure.

We denote, for $\alpha \in \mathbb{T}^1$ (resp. $\alpha \in \mathbb{R}$), by $R_\alpha : x \to x+\alpha$ the translations of $\mathbb{T}^1$ (resp. $\mathbb{R}$). On $\mathbb{T}^1$, R_α is also called a rotation and its rotation number is α.

We denote by $C^k(\mathbb{T}^1)$ the functions of class C^k ($k \in \mathbb{N}$) on the circle and we will also identify functions of $C^k(\mathbb{T}^1)$ with functions $\phi: \mathbb{R} \to \mathbb{R}$ of class C^k and $\mathbb{Z}$-periodic.

We define the norms:

$$\|\phi\|_{C^0} = \sup_{\theta \in \mathbb{T}^1} |\phi(\theta)|$$

$$\|\phi\|_{C^k} = \sup_{0 \leq j \leq k} \|D^j \phi\|_{C^0}$$

where $D\phi(\theta) = \dfrac{d\phi(\theta)}{d\theta}$.

2. Let $0 \leq \beta$ and DB be the set of the numbers satisfying a diophantine condition of exponent $2+\beta$:

$$DC_\beta = \left\{\alpha \in \mathbb{R} \;\Big|\; \exists \gamma > 0, \;\; \forall\, p/q \in \mathbb{Q}, \;\; \left|\alpha - \frac{p}{q}\right| \geq \frac{\gamma}{q^{2+\beta}}\right\}.$$

If $\beta > 0$, then $\mathbb{R}-DC_\beta$ is of Lebesgue measure 0.

We define the Sobolev spaces, $k \in \mathbb{N}$,

$$W^{k,2} \equiv W^{k,2}(\mathbb{T}^1) = \left\{\phi \in L^2(\mathbb{T}^1, d\theta, \mathbb{R}) \mid D^j\phi \in L^2, \; j=0,\ldots,k\right\},$$

D being the derivative in the sense of distributions (and $D^0\phi = \phi$).

For $k \geq 1$, we have $C^k(\mathbb{T}^1) \subset W^{k,2}(\mathbb{T}^1) \subset C^{k-(1/2)}(\mathbb{T}^1) \subset C^{k/2}(\mathbb{T}^1)$.

If $\phi \in W^{1,2}$, we have:

$$|\phi(x) - \phi(y)| \leq \|D\phi\|_{L^2} |x-y|^{1/2}$$

If, moreover $\int_0^1 \phi(\theta)\,d\theta = 0$, then

$$(*) \qquad \|\phi\|_{C^0} \leq \frac{1}{2(3)^{1/2}} \|D\phi\|_{L^2},$$

see [3,VII].

We define

$$W_0^{k,2} \equiv W_0^{k,2}(\mathbb{T}^1) = \left\{\phi \in W^{k,2} \;\Big|\; \int_0^1 \phi(\theta)\,d\theta = 0\right\}.$$

3. **Lemma**. *Let $0 \leq \beta \leq 1$ and $k \geq 2$, then for every $\alpha \in DC_\beta$ and $\phi \in W_0^{k,2}$, there exists a unique $\psi \in W_0^{k,2}$ such that*

$$\psi \circ R_\alpha - \psi = \phi$$

and furthermore

$$\|D^{k-2}\psi\|_{L^2} \leq C_1 \gamma^{-1} \|D^k \phi\|_{L^2}.$$

C_1 being a universal constant and

$$\gamma = \inf_{q \in \mathbb{N},\, q \geq 1,\, p \in \mathbb{Z}} q^{1+\beta} |q\alpha - p|.$$

Proof. It is enough to prove the lemma for $k = 2$. We can write

$$\phi = \sum_{n \neq 0} \frac{a_n}{n^2} e^{2\pi i n\theta}$$

with

$$\sum_{n \neq 0} |a_n|^2 = \frac{1}{(2\pi)^4} \|D^2\phi\|_{L^2}^2.$$

We have formally

$$\psi \sim \sum_{n \neq 0} \frac{a_n}{n^2(e^{2\pi i n\alpha} - 1)} e^{2\pi i n\theta}$$

and we deduce the result from

$$\sup_{n \neq 0} \frac{1}{n^2 |e^{2\pi i n\alpha} - 1|} \leq \frac{C^{te}}{\gamma}. \qquad \blacksquare$$

4. Let us consider the mapping:

$$\Psi: (\psi, \lambda) \in W_0^{1,2}(\mathbb{T}^1) \times \mathbb{R} \to \lambda + D\psi - \tfrac{1}{2}\psi^2 \in L^2.$$

As $W_0^{1,2}(\mathbb{T}^1) \subset C^0(\mathbb{T}^1)$ the mapping is well defined and of class C^∞. Let $\Psi(\psi, \lambda) = \phi$, then $\lambda = \int_0^1 (\tfrac{1}{2}(\psi(\theta))^2 + \phi(\theta))\,d\theta$.

We have $\Psi(0,0) = 0$ and $D\Psi(0,0)(\Delta\psi,\Delta\lambda) = D(\Delta\psi) + \Delta\lambda$, where $D\Psi(\psi,\lambda)$ is the derivative of map Ψ at the point (ψ,λ). Since $D\Psi(0,0)$ is an isomorphism, if follows by the implicit function theorem, that Ψ is a C^∞-diffeomorphism of a neighborhood of $(0,0) \in W_0^{1,2} \times \mathbb{R}$ onto a neighborhood of $0 \in L^2$.

5. For $k \in \mathbb{N}$, $k \geq 1$, let $D^k(\mathbb{T}^1) = \{f \in \mathrm{Diff}_+^1(\mathbb{R}) \mid f - \mathrm{Id} \in C^k(\mathbb{T}^1)\}$. The group $D^k(\mathbb{T}^1)$ is the universal covering space of the group of C^k orientation preserving diffeomorphisms of the circle with the C^k-topology. We denote by Id or $\mathrm{Id}|_X$ the identity mapping of a set X.

For $k \in \mathbb{N}$, $k \geq 2$, we define

$$D^{k,2}(\mathbb{T}^1) = \left\{f \in D^1(\mathbb{T}^1) \mid f - \mathrm{Id} \in W^{k,2}\right\};$$

$D^{k,2}(\mathbb{T}^1)$ is a topological group for the $W^{k,2}$-topology and we have, since $W^{k,2} \subset C^{k-1}$, $D^{k,2}(\mathbb{T}^1) \subset D^{k-1}(\mathbb{T}^1)$ (see [3,VI and VII]).

6. Schwarzian Derivatives

For $f \in D^{3,2}(\mathbb{T}^1)$ we define the Schwarzian derivative of f by:

$$(1) \qquad Sf \equiv S(f) = D^2 \mathrm{Log} Df - \frac{1}{2}(D \mathrm{Log} Df)^2$$
$$= \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2.$$

We have

$$(2) \qquad Sf = -2(Df)^{1/2} D^2 \left(\frac{1}{(Df)^{1/2}}\right)$$

If $f \in D^{3,2}(\mathbb{T}^1)$ satisfies $Sf = 0$, then $f = R_\lambda$ for some $\lambda \in \mathbb{R}$; indeed one has $D((Df)^{-1/2}) = c \in \mathbb{R}$ and $\int_0^1 D((Df)^{-1/2})d\theta = 0$, so $c = 0$ and Df is constant.

We have if f and g belong to $D^{3,2}(\mathbb{T}^1)$

$$(3) \qquad S(f \circ g) = ((Sf) \circ g)(Dg)^2 + S(g)$$

and therefore

$$(4) \qquad S(f^{-1}) = -((Sf) \circ f^{-1})(Df^{-1})^2.$$

7. **Lemma**. *Let $f \in D^{3,2}(\mathbb{T}^1)$ such that $S(f) = a$ with $a \in C^0(\mathbb{T}^1)$ $a \geq 0$ or $a \leq 0$; then $a = 0$.*

Proof. We have by (2)

$$D^2((Df)^{-1/2}) = -\frac{a}{2(Df)^{1/2}}$$

and the lemma follows from the fact

$$\int_0^1 D^2((Df(\theta))^{-1/2})d\theta = 0. \qquad \blacksquare$$

8. For $k \geq 3$, we define the sets: $K^{k,2} = \{h \in D^{k,2}(\mathbb{T}^1) \mid h(0)=0, \|D^k h\|_{L^2} \leq 1\}$ ($\|D^k h\|_{L^2} < 1$ implies $\|Dh - 1\|_{C^0} < 1/12$, cf. (*)). For the induced weak topology from $W^{k,2}$ on $K^{k,2}$ (for the definition see [3,VII.6]), the set $K^{k,2}$ is compact, convex and metrizable. By [3,VII.6] the mapping $g \in K^{k,2} \to Sg$ is weakly continuous.

9. Let $\alpha \in DC_\beta$, $0 \leq \beta \leq 1$, and

$$\gamma = \inf_{q \in \mathbb{N}, q \geq 1, p \in \mathbb{Z}} q^{1+\beta}|q\alpha - p|.$$

Taking $q = 1$, one sees that $\gamma < 1/2$.

Theorem. *There exists constants $\varepsilon > 0$ and $C > 0$ such that, if α is as above and $f \in D^5(\mathbb{T}^1)$ has rotation number $\rho(f) = \alpha$ and satisfies*

$$\|f - R_\alpha\|_{C^5} \leq \varepsilon\gamma,$$

then there exists a unique $h \in D^{3,2}(\mathbb{T}^1)$, *such that:* $h(0) = 0$ *and*

$$f = h \circ R_\alpha \circ h^{-1}$$

Moreover h *satisfies the following inequality:*

$$\|D^3 h\|_{L^2} \leq \frac{c}{\gamma} \|f - R_\alpha\|_{C^5}.$$

Proof. Let us define a map

$$\Phi : K^{3,2} \to K^{3,2}$$

which has a fixed point h which is the diffeomorphism we are looking for.

If $f = h \circ R_\alpha \circ h^{-1}$ then $f \circ h = h \circ R_\alpha$ and therefore using (3):

$$((Sf) \circ h)(Dh)^2 = Sh \circ R_\alpha - Sh.$$

For $h \in K^{3,2}$, let

$$\mu(f,h) = -\int_0^1 ((Sf) \circ h)(Dh)^2 \, d\theta.$$

By 3, we can find $\psi \in L^2$, satisfying $\int_0^1 \psi(\theta)\, d\theta = 0$ and such that:

$$(5) \qquad \psi \circ R_\alpha - \psi = ((Sf) \circ h)(Dh)^2 + \mu(f,h)$$

(one has $((Sf) \circ h)(Dh)^2 \in W^{2,2}$, see [H,VI.5]). We have

$$\|\psi\|_{L^2} \leq \frac{c_1}{\gamma} \|D^2(((Sf) \circ h)(Dh)^2)\|_{L^2} \leq$$

$$\leq \frac{c_1}{\gamma} (\|D^2(Sf)\|_{C^0} \|(Dh)^4\|_{C^0} + 5\|D(Sf)\|_{C^0} \|Dh\|^2_{C^0} \|D^2 h\|_{C^0} +$$

$$+ 2\|Sf\|_{C^0} \|D^2 h\|^2_{C^0} + 2\|Sf\|_{C^0} \|Dh\|_{C^0} \|D^3 h\|_{L^2}).$$

Using that $h \in K^{3,2}$ (i.e. $\|D^3 h\|_{L^2} < 1$) and that, if $\phi \in W_0^{1,2}$, one has

$$(*) \qquad \|\phi\|_{C^0} \leq \frac{1}{2(3)^{1/2}} \|D\phi\|_{L^2} \quad \text{(cf. [H}_1\text{,VII]),}$$

we obtain that

$$(6) \qquad \|\psi\|_{L^2} \leq C_2 \varepsilon$$

$C_2 > 0$ being a constant.

By 4, we can write

$$(7) \qquad \psi = D\psi_1 - \tfrac{1}{2}(\psi_1)^2 + \lambda, \quad \lambda = \int_0^1 \tfrac{1}{2}(\psi_1)^2 \, d\theta$$

with $\psi_1 \in W_0^{1,2}$, $\lambda \in \mathbb{R}$, satisfying

$$(8) \qquad \begin{cases} \|D\psi_1\|_{L^2} \leq c_3 \varepsilon \\ |\lambda| \leq c_3 \varepsilon \end{cases}$$

where $c_3 > 0$ is a constant and (ψ_1, λ) is unique under the restrictions imposed by (8), if $\varepsilon > 0$ is small enough. We define $h_1 \in D^{3,2}(\mathbb{T}^1)$, h_1 being uniquely determined by the conditions:

$$h_1(0) = 0$$

and

$$D \operatorname{Log} Dh_1 = \psi_1$$

let $\psi_2 \in W_0^{2,2}$ such that $D\psi_2 = \psi_1$, then $\psi_3 = e^{\psi_2 + a}$, $a \in \mathbb{R}$ being chosen such that $\int_0^1 \psi_3(\theta) d\theta = 1$; finally $h_1(\theta) = \int_0^\theta \psi_3(t) dt$.

By (5) and (7) we have:

$$(9) \qquad \mu(f,h) + ((Sf) \circ h)(Dh)^2 = Sh_1 \circ R_\alpha - Sh_1.$$

Using (8) and (*), if $\varepsilon > 0$ small enough, we have $h_1 \in K^{3,2}$, $D \operatorname{Log} Dh_1$ satisfies (8) and therefore we have defined a mapping:

$$\Phi : K^{3,2} \to K^{3,2}$$
$$h \to h_1$$

10. Lemma. *The mapping* Φ *is continuous for the weak topology on* $K^{3,2}$.

Proof. Let $(h_i)_{i\geq 0}$ be a sequence of $K^{3,2}$ converging to h, and $\Phi(h_i) = g_i$. By (9), we have:

(10) $\mu_i + ((Sf)\circ h_i)(Dh_i)^2 = (Sg_i)\circ R_\alpha - Sg_i$.

Let $(g_{i_n})_{n\geq 0}$ be a converging subsequence of $(g_i)_{i\geq 0}$, converging to g (this is possible since $K^{3,2}$ is compact for the weak topology).

As $g_{i_n} \to Sg_{i_n}$ is continuous for the weak topology on $K^{3,2}$, Sg satisfies (8).

Since the maps $h_i \to \mu(f,h_i) = -\int_0^1 ((Sf)\circ h_i)(Dh_i)^2 d\theta$, $h_i \to ((Sf)\circ h_i)(Dh_i)^2$ are continuous for the weak topology on $K^{3,2}$, passing to the limit in (10), we obtain:

$$\mu + ((Sf)\circ h)(Dh)^2 = (Sg)\circ R_\alpha - Sg$$

and since g satisfies (8), by uniqueness, $g = \Phi(h)$ and the lemma follows. ■

11. End of the Proof of the Theorem

By Schauder-Tychonov's fixed point theorem there exists $h \in K^{3,2}$ and $\mu \in \mathbb{R}$ such that

$$\mu + ((Sf)\circ h)(Dh)^2 = (Sh)\circ R_\alpha - Sh.$$

From (3) and (4) we obtain:

$$\mu + S(f\circ h\circ R_{-\alpha}) = -(S(h^{-1})\circ h)(Dh)^2$$

hence

$$\frac{\mu}{(Dh)^2}\circ h^{-1} + (S(f\circ h\circ R_{-\alpha}))\circ h^{-1}(D(h^{-1}))^2 + S(h^{-1}) = 0.$$

It follows that

$$S(f\circ h\circ R_{-\alpha}\circ h^{-1}) = \mu(Dh^{-1})^2.$$

By 7, one obtains:

$$S(f\circ h\circ R_{-\alpha}\circ h^{-1}) = 0$$

and therefore

$$f = R_\lambda\circ h\circ R_\alpha\circ h^{-1}.$$

As $\rho(f) = \alpha \in \mathbb{R}-\mathbb{Q}$, we conclude, using [2,III.4], that $\lambda = 0$ and we have obtained

$$f = h\circ R_\alpha\circ h^{-1}.$$

The inequality in the theorem follows from (8). ■

12. Remarks.

1. By the same proof, for $k \geq 5$, there exists $\varepsilon_k > 0$ and $C_k > 0$ such that, if $f \in D^k(\mathbb{T}^1)$, $\rho(f) = \alpha$ and $\|f-R_\alpha\|_{C^k} \leq \varepsilon_k\gamma$, then there exists $h \in D^{k-2,2}(\mathbb{T}^1)$, $h(0) = 0$, such that $f = h\circ R_\alpha\circ h^{-1}$ and with the inequality:

$$\|D^{k-2}h\|_{L^2} \leq \frac{C_k}{\gamma}\|f-R_\alpha\|_{C^k}.$$

2. If $0 \leq \beta < 1$, the $h \in K^{3,2}$ we constructed is smoother; in fact $h\text{-Id} \in W^{4-\beta,2}(\mathbb{T}^1)$ (and even better $h\text{-Id} \in C^{4-\beta}(\mathbb{T}^1)$, if $0 < \beta < 1$, cf. [3,VI] and 14).

13. Hölder Spaces

For $k \in \mathbb{N}$ and $0 < a < 1$, let $C^{k+a}(\mathbb{T}^1)$ be the space of functions $\phi \in D^k(\mathbb{T}^1)$ such that $D^k\phi$ is Hölder of exponent a. We define the norm

$$\|\phi\|_{C^{k+a}} = \sup(\|\phi\|_{C^k}, |D^k\phi|_{C^a})$$

with

$$|\phi|_{C^a} = \sup_{x\neq y}\frac{|\phi(x)-\phi(y)|}{|x-y|^a}$$

14. If $p > 1$ and $\alpha \in DC_\beta$, then by the same proof as for [3,IV.3.7] (see also [6]) we have the inequality:

$$\left(\sum_{0<|n|\leq N}\frac{1}{\|n\alpha\|^p}\right)^{1/p} \leq \frac{C}{\gamma}N^{1+\beta}$$

$C > 0$ being a constant and if $x \in \mathbb{R}$

$$\|x\| = \inf_{\ell \in \mathbb{Z}} |x+\ell| .$$

15. By the same proof as in [3,IV.3.8.1] one proves, using 14, the following lemma:

Lemma. *Let* $a \in DC_\beta$, $k \in \mathbb{N}$, $0 < a < 1$, *with* $k+a > 1+\beta$ *and* $k+a-1-\beta \notin \mathbb{N}$ *then for every* $\phi \in C^{k+a}(\mathbb{T}^1)$ *with* $\int_0^1 \phi(\theta)d\theta = 0$, *there exists a unique* $\psi \in C^{k+a-1-\beta}(\mathbb{T}^1)$ *satisfying:*

$$\int_0^1 \psi(\theta)d\theta = 0,$$

$$\psi \circ R_a - \psi = \phi$$

and furthermore

$$\|\phi\|_{C^{k+a-1-\beta}} \leq \frac{C_1}{\gamma} |D^k\phi|_{C^a}$$

$C_1 > 0$ *being a constant.*

If $\ell = k+a-1-\beta \in \mathbb{N}$, $\ell \geq 1$, then one has to replace $\psi \in C^\ell$ by $\psi \in C^{\ell *}$ (i.e. $D^{\ell-1}\phi$ is "Zygmund smooth," see [3,IV2 and 3]).

16. Using the same ideas and strategy as in the proof of 9 (i.e. the use of Schwarzian derivative) and using the same Hölder techniques as in [3,VI.4.3] one proves the following theorem:

Theorem. *Let* $k > 5$; *then there exists* $\varepsilon_k > 0$ *and* $C_k > 0$ *such that if* $a \in DC_\beta$, $0 \leq \beta \leq 1$, *and* $f \in D^k(\mathbb{T}^1)$ *has rotation number* $\rho(f) = a$ *and satisfies*

$$\|f - R_a\|_{C^k} \leq \varepsilon_k \gamma$$

then there exists a unique $h \in D^{[k-1-\beta]}(\mathbb{T}^1)$, $h(0) = 0$, *such that*

$$f = h \circ R_a \circ h^{-1}$$

and h *satisfies*

$$\|h - \mathrm{Id}\|_{C^{[k-1-\beta]}} \leq \frac{C_k}{\gamma} \|f - R_a\|_{C^k} .$$

(If $x \in \mathbb{R}$, $[x]$ denotes the biggest integer $\leq x$).

17. Remark

One can even show that $h \in C^{k-1-\beta}$, if $k-1-\beta \notin \mathbb{N}^*$ (and if $k-1-\beta \in \mathbb{N}$, $h \in C^{(k-1-\beta)*}$). One can even introduce Besov spaces, cf. [3,VI].

II. THE ANALYTIC CASE

1. Notations

For $\delta > 0$, let

$$B_\delta = \{z \in \mathbb{C} \mid |\mathrm{Im} z| \leq \delta\}.$$

We again denote by R_a the translation

$$z \to R_a(z) = z + a \quad (a \in \mathbb{C}).$$

If $a \in \mathbb{R}$ then $R_a(B_\delta) = B_\delta$. Let $k \in \mathbb{N} \cup \{\infty\}$; for $\delta > 0$, O_δ^k denotes the space of functions $\phi: B_\delta \to \mathbb{C}$ of class C^k on B_δ, holomorphic on Int B_δ and $\mathbb{Z}$-periodic (i.e. $\phi(z+1) = \phi(z)$). We define the sup norm by

$$\|\phi\|_{C_\delta^0} = \sup_{z \in B_\delta} |\phi(z)| .$$

If $0 < \delta' < \delta$ and if $\phi \in O_\delta^0$, then $\phi \in O_\delta^\infty$, and one has Cauchy's inequality:

$$\|D\phi\|_{C_{\delta'}^0} \leq \|\phi\|_{C_\delta^0} / (\delta - \delta')$$

where $D\phi(z) = \frac{d\phi}{dz}(z)$ denotes the $\mathbb{C}$-derivative of ϕ at the point z.

2. **Hardy Sobolev Spaces**

For $k \in \mathbb{N}$, let

$$O_\delta^{k,2} = \left\{\phi(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi inz} \;\middle|\; a_n \in \mathbb{C},\; n \in \mathbb{Z},\; \|D^k\phi\|_{O_\delta^{0,2}} < +\infty\right\}$$

with

$$D\phi = \sum_{n\neq 0} 2\pi i n a_n e^{2\pi i nz}$$

and

$$\|\phi\|_{O_\delta^{0,2}} = \left(\sum_{n\in\mathbb{Z}} |a_n|^2 e^{4\pi|n|\delta}\right)^{1/2}.$$

With the norm

$$\|\phi\|_{O_\delta^{k,2}} = \left(|a_0|^2 + \|D^k\phi\|^2_{O_\delta^{0,2}}\right)^{1/2}$$

the space $O_\delta^{k,2}$ is a Hilbert space.

(1) We have, for $k \geq 1$,

$$O_\delta^k \subset O_\delta^{k,2} \subset O_\delta^{k-1}.$$

If $\phi \in O_\delta^{1,2}$ and $\int_0^1 \phi(\theta)d\theta = 0$, then

(*) $$\|\phi\|_{C_\delta^0} \leq \frac{1}{2(3)^{1/2}} \|D\phi\|_{O_\delta^{0,2}}.$$

We use the following facts and we refer the reader to [4,VIII]:

(2) For $k \geq 1$, $O_\delta^{k,2}$ is a Banach algebra.

(3) For $k \geq 1$, if $g \in O_\delta^{k,2}$ and ϕ is holomorphic on a neighborhood of $g(B_\delta)$ then $\phi\circ g \in O_\delta^{k,2}$ and on a small enough neighborhood V of g in $O_\delta^{k,2}$ the map

$$\psi \in V \to \phi\circ\psi \in O_\delta^{k,2}$$

is holomorphic.

3. **The Spaces $D_\delta^{k,2}$ and $K_\delta^{k,2}$, $k \geq 1$**

Let

$$D_\delta^{k,2} = \left\{\mathrm{Id} + \phi \;\middle|\; \phi \in O_\delta^{k,2}\right\}$$

and

$$K_\delta^{k,2} = \left\{h = \mathrm{Id} + \phi \;\middle|\; \phi \in O_\delta^{k,2},\ \phi(0) = 0,\ \|\phi\|_{C_\delta^0} < \delta/2 \text{ and } \|D^k\phi\|_{O_\delta^{0,2}} < 1\right\}.$$

If $h \in K_\delta^{k,2}$, then

$$h(B_\delta) \subset B_{\frac{3}{2}\delta} \subset \mathrm{Int}(B_{2\delta}).$$

We use the following fact (see [4,VIII]): If $k \geq 1$, $\phi \in O_{2\delta}^0$ and $h \in K_\delta^{k,2}$, then $\phi\circ h \in O_\delta^{k,2}$ and the mapping $(\phi,h) \in O_{2\delta}^0 \times K_\delta^{k,2} \longrightarrow \phi\circ h \in O_\delta^{k,2}$ is holomorphic.

4. Let $\alpha \in DC_\beta$, $0\leq\beta\leq 1$ we define the number γ as we did in I.9.

Lemma. *There exists $C > 0$ such that, if $0 \leq \beta \leq 1$, $k \geq 2$ and $\alpha \in DC_\beta$ then for every $\phi \in O_\delta^{k,2}$ with $\int_0^1 \phi(t)dt = 0$, there exists a unique $\psi \in O_\delta^{k-2,2}$ with $\int_0^1 \psi(t)dt = 0$ and*

$$\psi\circ R_\alpha - \psi = \phi.$$

Furthermore we have

$$\|D^{k-2}\psi\|_{O_\delta^{0,2}} \leq \frac{C}{\gamma} \|D^k\phi\|_{O_\delta^{0,2}}.$$

The proof is the same as in I.3.

5. **Schwarzian Derivatives**

Let $f \in D_\delta^{3,2}$ with $|Df - 1| < \frac{1}{2}$ on B_δ then we define

$$Sf \equiv S(f) = D^2\mathrm{Log}\, Df - \tfrac{1}{2}(D\mathrm{Log}Df)^2$$

and we have the same formulas we had in I.6. By a similar proof, $Sf = 0$ implies $f = R_\alpha$ for some $\alpha \in \mathbb{C}$.

6. **Lemma**. *There exists* $0 < C < \frac{1}{2}$ *such that if* $a \in O_\delta^\circ$ *and* $f \in D_\delta^{3,2}$ *satisfy:*

$$|a(z) - 1| < C$$

$$|Df(z) - 1| < C$$

for $z \in B_\delta$, *and if*

$$Sf = \mu a .$$

for some $\mu \in \mathbb{C}$, *then it follows that* $\mu = 0$.

Proof. One has (cf. I.(2))

$$D^2((Df)^{-1/2}) = -\tfrac{1}{2}\mu a (Df)^{-1/2},$$

hence

$$0 = \int_0^1 D^2((Df)^{-1/2})(u)du = -\tfrac{1}{2}\mu\int_0^1 a(t)(Df(t))^{-1/2}dt.$$

If $C > 0$ is small enough, $\int_0^1 a(t)(Df(t))^{-1/2}dt \neq 0$, hence $\mu = 0$. ■

7. Let $O^{1,2}_{\delta,0} = \{\phi \in O^{1,2}_\delta, \int_0^1 \phi(u)du = 0\}$ and

$$\Psi: O^{1,2}_{\delta,0} \times \mathbb{C} \to O^{0,2}_\delta$$
$$(\psi,\lambda) \to D\psi - \tfrac{1}{2}(\psi)^2 + \lambda.$$

One has $\Psi(0,0) = 0$, and using as in I the implicit function theorem, we conclude that Ψ is a biholomorphic diffeomorphism of a neighborhood of $(0,0)$ in $O^{1,2}_{\delta,0} \times \mathbb{C}$ onto a neighborhood of 0 in $O^{0,2}_\delta$.

8. Let $\alpha \in DC_\beta$, $0 \le \beta \le 1$, and we define as in I.9 the number $0 < \gamma < \frac{1}{2}$. For $\delta > 0$, and $\gamma > 0$ we define:

$$V_{\varepsilon,\delta} = \left\{f = \mathrm{Id} + \phi \;\middle|\; \phi \in O^\circ_\delta, \|\phi\|_{C^\circ_\delta} < \varepsilon\gamma\right\}.$$

Theorem. *Given* $\delta > 0$ *there exists* $\varepsilon > 0$, $\varepsilon < \frac{1}{2}$, *and a holomorphic map*

$$F: V_{\varepsilon,\delta} \to \mathbb{C}\times K^{3,2}_{\delta/2}$$

such that, for $f \in V_{\varepsilon,\delta}$, $F(f) = (\lambda,h)$, *one has:*

$$(+) \qquad (f + \lambda)\circ h = h \circ R_\alpha$$

on $B_{\delta/2}$. *Furthermore, we have*

$$F(\mathrm{Id}|_{B_\delta}) = (\alpha, \mathrm{Id}|_{B_{\delta/2}})$$

and there exists $\eta > 0$ *such that, if* $(\lambda_1, h_1) \in \mathbb{C}\times K^{3,2}_{\delta/2}$ *satisfies* $\|h_1 - \mathrm{Id}\|_{O^{3,2}_{\delta/2}} < \eta$ *and* $(\lambda_1 + f)\circ h_1 = h_1 \circ R_\alpha$, *then* $F(f) = (\lambda_1, h_1)$.

Proof. We want to solve (+), so taking Schwarzian derivatives we obtain

$$((Sf)\circ h)(Dh)^2 = (Sh)\circ R_\alpha - Sh.$$

One has $h(B_{\delta/2}) \subset B_{3\delta/4} \subset B_{4\delta/5}$. If $\varepsilon > 0$ is small enough, by Cauchy's inequalities we have:

$$\|D^2(Sf)\|_{C^0_{4\delta/5}} \le \frac{C\varepsilon\gamma}{\delta^5}$$

$C > 0$ being a constant. If $\varepsilon > 0$ is small enough, let us define a holomorphic map

$$\Phi: V_{\varepsilon,\delta} \times K^{3,2}_{\delta/2} \to K^{3,2}_{\delta/2},$$

such that the map $h_0 \to \Phi(f,h_0)$ has a fixed point h.

For $h \in K^{3,2}_{\delta/2}$, let $\mu(f,h) = -\int_0^1 (((Sf)\circ h)(Dh)^2)(u)du \in \mathbb{C}$, the mapping $(f,h) \to \mu(f,h)$ is holomorphic. By 4, we can find a unique $\psi \in O^{0,2}_{\delta/2}$ with $\int_0^1 \psi(t)dt = 0$ satisfying

$$\psi\circ R_\alpha - \psi = ((Sf)\circ h)(Dh)^2 + \mu(f,h)$$

and since $\|D^3 h\|_{O^{0,2}_{\delta/2}} < 1$, we have

$$\|\psi\|_{O^{0,2}_{\delta/2}} \le \text{constant } \varepsilon\,\delta^{-5}$$

(to see that $((Sf)_0 h)(Dh)^2 \in O^{2,2}_{\delta/2}$ we use 2 and 3). The mapping $(f,h)\to\psi$ is holomorphic. If $\varepsilon>0$ is sufficiently small, we can define $\Psi^{-1}(\psi) = (\psi_1,\lambda)$, $(\psi_1,\lambda)\in O^{1,2}_{\delta/2,0}\times\mathbb{C}$, $\Psi^{-1}(0)=(0,0)$ and satisfying:

$$\psi = D\psi_1 - \tfrac{1}{2}(\psi_1)^2 + \lambda;$$

hence

$$\lambda = \int_0^1 (\psi + \tfrac{1}{2}(\psi_1)^2)(u)du.$$

It follows that the mapping $(f,h)\to(\psi_1,\lambda)$ is holomorphic.

Let $\psi_2\in O^{2,2}_{\delta/2}$, $\int_0^1\psi_2(u)du=0$ such that $D\psi_2=\psi_1$. Then let $\psi_3 = e^{\psi_2+c}$ with $e^c = 1/a$, $a=\int_0^1 e^{\psi_2(u)}du$; if $\varepsilon>0$ is sufficiently small, $a\neq 0$ and ψ_3 is well defined. Finally we define, if $\varepsilon>0$ is small enough,

$$h_1(z) = \int_0^z \psi_3(u)du.$$

One has, if $\varepsilon>0$ is small enough,

$$h_1\in K^{3,2}_{\delta/2} \quad (\text{since } |h_1(z)-z| \le (\tfrac{1}{4}+\delta^2)^{1/2}\|Dh_1-1\|_{C^0_{\delta/2}})$$

and we have therefore defined a holomorphic mapping

$$\Phi: V_{\varepsilon,\delta}\times K^{3,2}_{\delta/2}\to K^{3,2}_{\delta/2}$$
$$(f,h)\to h_1$$

such that:

(1) $((Sf)\circ h)(Dh)^2 + \mu(f,h) = (Sh_1)\circ R_\alpha - Sh_1$.

We have $\Phi(R_\alpha,h) = \mathrm{Id}|_{B_{\delta/2}}$ and if $\varepsilon>0$ is sufficiently small, by an immediate calculation of derivatives, we obtain

$$\sup_{(f,h)} |||D_2\Phi(f,h)||| \le 1/2$$

where $D_2\Phi$ is the partial derivative with respect to the variable h and $|||\ |||$ is the norm induced by $\|\ \|_{O^{3,2}_{\delta/2}}$ on the continuous linear operators from $O^{3,2}_{\delta/2}$ to itself. By the implicit function theorem we can find a holomorphic map $F_2: f\to h\in K^{3,2}_{\delta/2}$ such that h is the fixed point for the contracting mapping

$$h\in K^{3,2}_{\delta/2}\to\Phi(f,h)\in K^{3,2}_{\delta/2}$$

and furthermore $F_2(R_\alpha) = \mathrm{Id}|_{B_{\delta/2}}$.

Writing that h is a fixed point we obtain by (1):

(2) $((Sf)\circ h)(Dh)^2+\mu(f,h) = (Sh)\circ R_\alpha - Sh$.

Furthermore, given $\eta>0$, if $\varepsilon>0$ is small enough, then

(3) $\|Dh-1\|_{C^0_{\delta/2}} < \eta$.

Using (2), (3) and 6, by the same proof as in I.11, we conclude that $\mu(f,h)=0$ and $f = R_{-\lambda}\circ h\circ R_\alpha\circ h^{-1}$ on $B_{\delta/4}$, which implies $(f+\lambda)\circ h = h\circ R_\alpha$ on $B_{\delta/2}$ (using (3) with η small enough, h^{-1} is defined on $B_{\delta/4}$ and $h^{-1}(B_{\delta/4})\subset B_{\delta/2}$).

Using I.(2) one sees that $F_1: f\to\lambda$ is holomorphic.

We define now

$$F(f) = (F_1(f),F_2(f))\in\mathbb{C}\times K^{3,2}_{\delta/2}.$$

The local uniqueness in the theorem follows the local uniqueness of the fixed point of a contracting mapping. ∎

10. Remarks

1. We can also apply Schauder-Tychonov's fixed point theorem.

2. If $0 < \delta < 1$, then one can choose $\varepsilon = \varepsilon_1 \delta^6$ with $\varepsilon_1 > 0$ independent of a and δ.

3. If $f \in V_{\varepsilon,\delta}$ and if $f|_{\mathbb{R}}$ is real, then $F(f) = (\lambda, h)$ is such that $\lambda \in \mathbb{R}$ and $h|_{\mathbb{R}}$ is real (cf. [4,VIII.7]).

III. COMPLEXIFICATION OF THE ROTATION NUMBER

1. In the following we will fix the number β, $0 < \beta < 1$. For $0 < \gamma < 1/2$, we define:

$$C_\gamma = \left\{ a \in \mathbb{R} \;\middle|\; \forall\, p/q \in \mathbb{Q},\ |a-(p/q)| \geq \gamma q^{-2-\beta} \right\}.$$

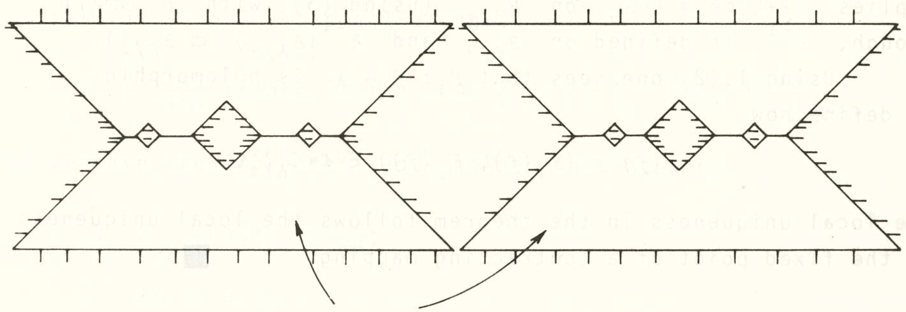
We have for $p \in \mathbb{Z}$, $R_p(C_\gamma) = C_\gamma$.

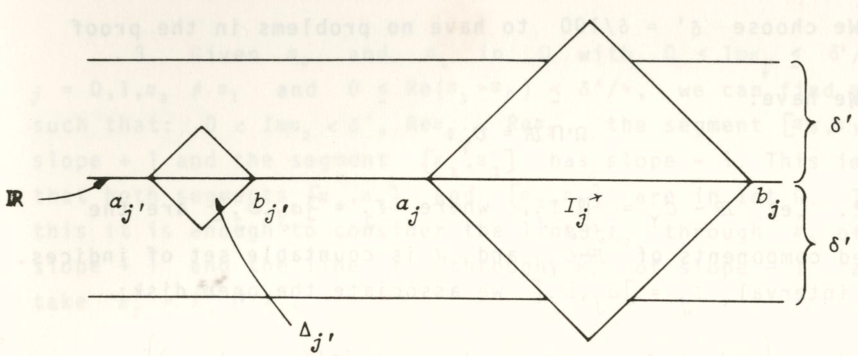
Given ε, $0 < \varepsilon < 1$, if $\gamma > 0$ is small enough the closed set C_γ is non-empty and the Haar measure of C_γ mod 1 is superior to $1 - \varepsilon$. In the following, we will suppose that $\gamma > 0$ is small enough so that C_γ mod 1 has positive Haar measure.

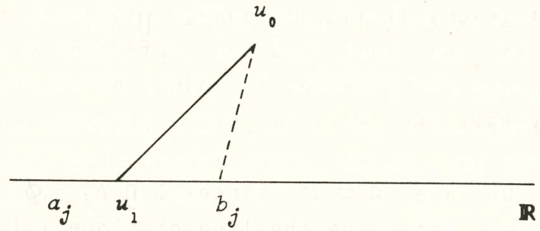
To study the non-tangential limits to C_γ we introduce, for $\delta' = \delta/100 > 0$, the set:

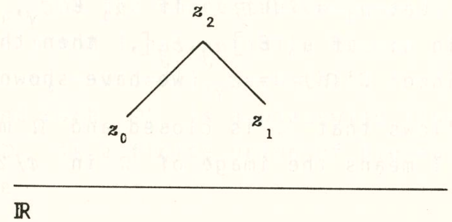
$$\Omega \equiv \Omega_{\gamma,\delta} = \Big\{ a + it \in \mathbb{C} \;\Big|\; a \in \mathbb{R},\ t \in \mathbb{R},$$

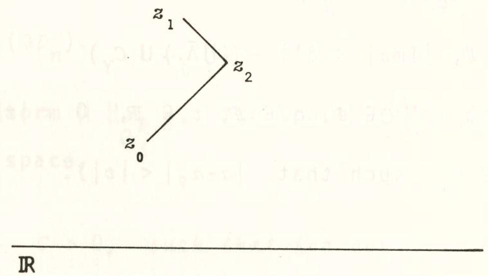
$$0 \leq |t| \leq \delta',\ \exists a_0 \in C_\gamma \text{ such that } |a - a_0| \leq |t| \Big\}.$$











We choose $\delta' = \delta/100$ to have no problems in the proof of 6.

We have:

$$\Omega \cap \mathbb{R} = C_\gamma .$$

2. Let $\mathbb{R} - C_\gamma = \bigcup_{j \in J} I_j$, where $I_j =]a_j, b_j[$ are the connected components of $\mathbb{R} - C_\gamma$ and J is countable set of indices. To each interval $I_j =]a_j, b_j[$ we associate the open disk:

$$\Delta_j = \left\{ x+iy \in \mathbb{C},\ a_j < x < b_j,\ |y| \leq \inf(x-a_j, b_j-x) \right\}.$$

Let $\Omega' = \{z \in \mathbb{C},\ |\mathrm{Im} z| \leq \delta'\} - \bigcup_{j \in J} \Delta_j$.

Lemma.(1) *We have* $\Omega = \Omega'$.

Proof. We have $\Omega \subset \Omega'$ since $\Omega \cap \Delta_j = \emptyset$ for all $j \in J$. If $u_0 \in \Omega' - \mathbb{R}$, let ℓ be the line of slope $+1$ passing through u_0. Let $u_1 = \ell \cap \mathbb{R}$. If $u_1 \in C_\gamma$, then the segment(2) $[u_0, u_1]$ is in Ω; if $u_0 \in]a_j, b_j[$, then the segment $[b_j, u_0]$ is in Ω. Since $\Omega' \cap \mathbb{R} = C_\gamma$ we have shown that $\Omega' \subset \Omega$. ∎

It follows that Ω is closed and Ω mod 1 is compact in $\mathbb{C}/\mathbb{Z}$ (Ω mod 1 means the image of Ω in $\mathbb{C}/\mathbb{Z}$ by the canonical projection).

Since $\overline{\bigcup_{j \in J} \Delta_j} = \bigcup_j \bar{\Delta}_j \cup \mathbb{R}$ we obtain:

$$\mathrm{Int}\, \Omega = \{z \in \mathbb{C},\ |\mathrm{Im} z| < \delta'\} - ((\bigcup_j \bar{\Delta}_j) \cup C_\gamma)$$

$$= \{z = a + it \in \mathbb{C},\ a \in \mathbb{R},\ t \in \mathbb{R},\ 0 < |t| < \delta',$$

$$\exists\, a_0 \in C_\gamma \text{ such that } |a - a_0| < |t|\}.$$

(1) In the definition of Ω we take out the union of the squares Δ_j and not the union of disks as in [1], only to simplify the exposition.

(2) The segment $[u_0, u_1]$ means the closed segment of line in $\mathbb{C}$ joining u_0 to u_1.

3. Given z_0 and z_1 in Ω with $0 < \mathrm{Im} z_j \leq \delta'/4$, for $j = 0, 1, z_0 \neq z_1$ and $0 \leq \mathrm{Re}(z_1 - z_0) \leq \delta'/4$, we can find $z_2 \in \Omega$ such that: $0 < \mathrm{Im} z_2 < \delta'$, $\mathrm{Re} z_0 < \mathrm{Re} z_2$, the segment $[z_0, z_1]$ has slope $+1$ and the segment $[z_2, z_1]$ has slope -1. This implies that both segments $[z_0, z_2]$ and $[z_2, z_1]$ are in $\mathrm{Int}\, \Omega$. To see this it is enough to consider the line ℓ_0 through z_0 of slope $+1$ and the line ℓ_1 through z_1 of slope -1 and to take $z_2 = \ell_0 \cap \ell_1$.

Remark. This implies that the compact set Ω mod 1 has a finite number of connected components. Furthermore, the set Ω mod 1 is locally connected. One also sees that the open set $\mathrm{Int}\, \Omega$ mod 1 has a finite number of connected components U_i. If $\delta > 0$ is sufficiently small, each U_i is simply connected and ∂U_i is a rectifiable Jordan curve. We use the symbol ∂ for the frontier or boundary of a set.

4. Let $(D_n)_{n \in \mathbb{N}}$ be the connected components of $\mathbb{C}/\mathbb{Z} - (\Omega \text{ mod } 1)$. Since Ω mod 1 has a finite number of components, there exists n_0 (i.e. $n_0 = 0$ or 1), such that, for $n \geq n_0$, D_n is simply connected and each ∂D_n is a piecewise linear Jordan curve. For $n < n_0$, ∂D_n is a finite union of piecewise linear Jordan curves. We have

$$(*) \quad \sum_{n \geq 0} \ell_n < +\infty$$

where $\ell_n = \text{length}\,(\partial D_n)$.

5. For the norm $\|\ \|_{O^{k,2}_\delta}$, the space $O^{k,2}_{\delta,0}$ (defined in II.7) is a Hilbert space.

Lemma. *There exists* $C > 0$, *such that for every* $\alpha \in \Omega_{\gamma,\delta}$ *and every* $\phi \in O^{k,2}_{\delta,0}$, $k \geq 2$, *there exists a unique* $\psi_\alpha \in O^{k-2,2}_{\delta,0}$ *satisfying*

$$\psi_\alpha \circ R_\alpha - \psi_\alpha = \phi$$

and

$$\|D^{k-2}\psi_a\|_{O_\delta^{0,2}} \leq \frac{c}{\gamma}\|D^k\phi\|_{O_\delta^{0,2}}$$

(the constant $C > 0$ *is independent of* $a \in \Omega_{\gamma,\delta}$*). Furthermore, the mapping:*

$$(a,\phi) \in \Omega_{\gamma,\delta}\times O_{\delta,0}^{k,2} \to \psi_a \in O_{\delta,0}^{k-2,2}$$

is continuous and the mapping:

$$(a,\phi) \in (\mathrm{Int}\,\Omega_{\gamma,\delta})\times O_{\delta,0}^{k,2} \to \psi_a \in O_{\delta,0}^{k-2,2}$$

is holomorphic.

Proof. The proof of the first part is the same as in I.3 using that

$$\sup_{n\neq 0,\, a\in\Omega} \left| n^{1+\beta}\,(1-e^{2\pi i n a})\right|^{-1} \leq C_1'$$

where C_1' is a constant.

The properties of the mapping $(a,\phi) \to \psi_a$, come from the fact that we suppose that $0 < \beta < 1$ and, if

$$S_N(\psi_a) = \sum_{0<|n|\leq N} \frac{\hat{\phi}(n)}{e^{2\pi i n a}-1} e^{2\pi i n\theta}, \quad N \in \mathbb{N}^*,$$

then

$$\|S_N(\psi_a) - \psi_a\|_{O_\delta^{k-2,2}} \leq \frac{c_2}{N^{1-\beta}}\|\phi\|_{O_\delta^{k,2}}$$

where C_2 is a constant (independent of N, ϕ and $a \in \Omega$). Letting $N \to +\infty$, the results follow since $(a,\phi) \to \psi_a$ is the local uniform limit of the mappings $(a,\phi) \to S_N(\psi_a)$. ■

Remark. As $0 < \beta < 1$, the operator

$$\phi \in O_{\delta,0}^{k,2} \to \psi_a \in O_{\delta,0}^{k-2,2}$$

is compact and depends continuously on $a \in \Omega_{\gamma,\delta}$ for the norm topology on linear continuous operators from $O_{\delta,0}^{k,2}$ to $O_{\delta,0}^{k-2,2}$.

6. We will use the same notations as in II.8.

Theorem. *Given* $\delta > 0$ *and* $\gamma > 0$ *there exists* $\varepsilon > 0$, $\varepsilon > 1/2$ *and a continuous map*

$$G:\Omega_{\gamma,\delta}\times V_{\varepsilon,\delta} \to \mathbb{C}\times K_{\delta/2}^{3,2}.$$

such that, for $(a,f) \in \Omega_{\gamma,\delta}\times V_{\varepsilon,\delta}$, $G(a,f) = (\lambda,h)$ *satisfies*

(+) $\quad (f+\lambda)\circ h = h\circ R_a$ *on* $B_{\delta/2}$

and

$$G(a,\mathrm{Id}|_{B_\delta}) = (a,\mathrm{Id}|_{B_{\delta/2}}).$$

Furthermore, if $a \in \Omega \cap \mathbb{R} = C_\gamma$, *then* $G(a,f) = F(f)$, *where* F *is the locally unique holomorphic map given by* II.8; *the mapping*

$$(a,f) \in \mathrm{Int}\,\Omega_{\gamma,\delta}\times V_{\varepsilon,\delta} \to G(a,f)$$

is holomorphic.

Proof. Considering $a \in \Omega$ as a parameter and using that the constant of 5 does not depend on $a \in \Omega$ (by the same proof as for II.8) one obtains, for $\varepsilon > 0$ small enough a map G satisfying (+). The map G is continuous $\Omega\times V_{\varepsilon,\delta}$ because of the following well known fact:

If (Y,d) is a complete metric space, X is a topological space and L is a continuous map from $X\times Y$ into Y satisfying $d(L(x,y_1), L(x,y_2)) \leq kd(y_1,y_2)$, for all $x \in X$ and all y_1 and y_2 in Y for a fixed $0 \leq k < 1$ independent of x, y_1 and y_2, then the unique fixed point of $y \to L(x,y)$ depends continuously on x. (We have taken $|t| \leq \delta'$ in the definition of $\Omega_{\gamma,\delta}$, to have no problems at the end of the proof of II.8.)

The fact that the mapping

$$(a,f) \in \mathrm{Int}\,\Omega\times V_{\varepsilon,\delta} \to G(a,f) \in K_{\delta/2}^{3,2}$$

is holomorphic follows from 5, using the usual implicit function theorem in the proof of II.8. ■

7. Remarks

1. If $\varepsilon > 0$ is small enough, (which we will suppose from now on) then for all $(a,f) \in \Omega_{\gamma,\delta} \times V_{\varepsilon,\delta}$, $G(a,f) = (\lambda,h)$ satisfies:

$$\|Dh-1\|_{C^0_{\delta/2}} \le 1/10.$$

(This follows from the continuity of G on Ω mod 1, or also, from the proof of II.8.)

2. If $\varepsilon > 0$ is small enough, then $(a,f)\in(\Omega-C_\gamma)\times V_{\varepsilon,\delta} \to G(a,f)$ extends analytically to a neighborhood of $(\partial\Omega-C_\gamma)\times V_{\varepsilon,\delta}$.

(To see this it is enough to replace, in the proof of 6, the set $\Omega_{\gamma,\delta}$ by the set:

$$\Omega^*_{\gamma,\delta} = \left\{a+it\in\mathbb{C} \mid a\in\mathbb{R},\ t\in\mathbb{R},\ 0\le|t|\le 2\delta',\ \exists a_0\in C_\gamma \text{ such that } |a-a_0| \le \tfrac{1}{2}|t|\right\}$$

and to observe that 5 holds for $\Omega^*_{\gamma,\delta}$ after changing the constant C of 5.)

3. If $p \in \mathbb{Z}$, we have $G(u,p+f) = G(u-p,f)$.

8. Let $\ell \in \mathbb{R}$, $|\ell| < 1$, $a = \ell + i$, $a_0 \in C_\gamma$ and $t \in I_{\delta'} = \{t \in \mathbb{R},\ |t| < \delta'\} \to a_t = a_0 + ta$. For $f \in V_{\varepsilon,\delta'}$ with $\varepsilon > 0$ satisfying 6 and 7, we consider the mapping

$$t \in I_{\delta'} \to G_1(t) \in \mathbb{C}\times D^0_{\delta/4}$$

where $G_1(t) = J\circ G(a_t)$ and J is the inclusion of $\mathbb{C}\times K^{3,2}_{\delta/2}$ into $\mathbb{C}\times D^0_{\delta/4}$.

As $a_t \in \Omega$, the mapping $t \to G_1(t)$ is continuous and $\mathbb{R}$-analytic on $\{t\in I_{\delta'},\ t \ne 0\}$, since, if $t\ne 0$, $a_t \in \operatorname{Int}\Omega$ (cf. 2).

9. **Proposition**. *The mapping*

$$t \in I_{\delta'} \to G_1(t) \in \mathbb{C}\times D^0_{\delta/4}$$

is of class C^∞ *and, for every* $k \ge 1$, *one has:*

$$\sup_{|t|<\delta',\ \alpha\in C_\gamma,\ |\lambda|<1} \left\|\left(\frac{d}{dt}\right)^k G_1(t)\right\| < +\infty$$

where $\|\ \|$ *is the norm* $|\ | + \|\ \|_{C^0_{\delta/4}}$ *on* $C \times O^0_{\delta/4}$.

Proof. We already know that $t \in \{t\in I_{\delta'},\ t\ne 0\} \to G_1(t)$ is $\mathbb{R}$-analytic. We write $G_1(t) = (\lambda(t), h(t))$ with $h(t)(0) = 0$. We differentiate with respect to t, $t \ne 0$,

(+) $\quad f\circ h(t) + \lambda(t) = h(t)\circ R_{a_t}$

hence

(i) $\quad \dfrac{d\lambda(t)}{dt} + Df\circ h(t)\dfrac{dh(t)}{dt} - \dfrac{dh(t)}{dt}\circ R_{a_t} = a\, Dh(t)\circ R_{a_t}$

and

$$\frac{dh(t)}{dt}(0) = 0.$$

If we differentiate (+) with respect to the variable z we obtain (cf. 7):

(ii) $\quad Df\circ h(t) = Dh(t)\circ R_{a_t} / Dh(t),\ t \ne 0.$

We multiply (i) by $1/Dh(t)\circ R_{a_t}$ and we obtain:

(iii) $\quad \dfrac{1}{Dh(t)}\dfrac{dt(t)}{dt} - \left[\dfrac{1}{Dh(t)}\dfrac{dh(t)}{dt}\right]\circ R_{a_t} = a - \dfrac{1}{Dh(t)\circ R_{a_t}}\dfrac{d\lambda(t)}{dt}$

therefore

(iv) $\quad \dfrac{d\lambda(t)}{dt}\displaystyle\int_0^1 \frac{1}{Dh(t)}(\theta+a_t)\,d\theta = a.$

As $t \in I_{\delta'} \to \dfrac{1}{Dh(t)} \in O^\infty_{\delta/3}$ is continuous it follows that the map $t \in \{t\in I_{\delta'},\ t\ne 0\} \to \dfrac{d\lambda(t)}{dt} \in \mathbb{C}$ extends continuously at

$t = 0$ and we conclude that the function $t \to \lambda(t)$ is of class C^1 (we use the well known fact: If B is a Banach space, $\phi: [0,1] \to B$ is a continuous function of class C^1 on $]0,1[$ and $\lim_{t\to 0} \frac{d\phi(t)}{dt}$ exists then ϕ is of class C^1 on $[0,1[$; this comes from Taylor's formula:

$$\phi(t) - \phi(0) = t\int_0^1 \frac{d\phi}{dz}(tu)du,$$

valid for $0 \leq t < 1$.)

If one applies 5 to (iii), using 7 and Cauchy's inequalities, we conclude that the mapping $t \in \{t \in I_{\delta'},\ t \neq 0\} \to \frac{dh(t)}{dt}(t) \in O^{\infty}_{\delta/4}$ extends continuously at $t = 0$ and hence the mapping $t \to h(t) \in D^{\infty}_{\delta/4}$ is of class C^1.

The inequality with $k = 1$ follows from 5 and 7 (using Cauchy's inequalities) applied to (3).

The case $k \geq 2$ follows by induction on k and the same argument as above (after differentiating (iii), using 5, 7 and Cauchy's inequalities). ■

10. Let $f \in V_{\varepsilon,\delta}$ with $\varepsilon > 0$ satisfying 6 and 7. Let $W = (\mathrm{Int}\,\Omega) \cup \mathbb{C}_\gamma$. For $u_0 \in \mathrm{Int}\,\Omega$ and $k \geq 1$, we define $\left(\frac{d}{du}\right)^k G(u_0,f)$ as the $\mathbb{C}$-derivative of the $\mathbb{C}$-analytic mapping $u \to J\circ G(u,f)$.

For $u_0 = a_0 \in C_\gamma$ we define:

$$\left(\frac{d}{du}\right)^k G(a_0,f) = a^{-k}\left(\frac{d}{dt}\right)^k J\circ G(a_t,f)\Big|_{t=0}$$

where a_t is defined in 8.

The function $u \in W \to \left(\frac{d}{du}\right)^k G(u,f) \in \mathbb{C}\times O^0_{\delta/4}$ is well defined and by 9 and 7.2 it is bounded.

11. **Proposition**. *Given $f \in V_{\varepsilon,\delta'}$, where $\varepsilon > 0$ satisfies 6 and 7, and $k \in \mathbb{N}$, there exists $K_k > 0$ such that, for all u_0 and u_1 in W, we have:*

$$(1)\qquad \left\|\left(\frac{d}{du}\right)^k G(u_0,f) - \left(\frac{d}{du}\right)^k G(u_1,f)\right\| \leq K_k|u_0-u_1|$$

where $\|\ \|$ is the norm defined in 9 and with the convention $\left(\frac{d}{du}\right)^0 G(u,f) = J\circ G(u,f)$.

Proof. (a) Case where u_0, u_1 satisfy $\mathrm{Im}\,u_j > 0$, for $j = 0,1$.

By interchanging u_0 and u_1 we can suppose that

$$(2)\qquad \mathrm{Re}(u_1-u_0) \geq 0.$$

By the $\mathbb{Z}$-periodicity and the boundedness (in norm) of the function:

$$u \in W \to \left(\frac{d}{du}\right)^k (J\circ G(u,f) - \mathrm{Id}|_{\mathbb{C}\times B_{\delta/4}}) \in \mathbb{C}\times O^0_{\delta/4}$$

we can suppose that

$$(3)\qquad \mathrm{Re}(u_1-u_0) < \delta'/4$$

and

$$(4)\qquad |\mathrm{Im}\,u_j| < \delta'/4, \quad \text{for } j = 0,1.$$

Using 3, we can find $u_2 \in \mathrm{Int}\,\Omega$ such that the segment $[u_0,u_2]$ has slope $+1$ and $[u_2,u_1]$ has slope -1. By Taylor's formula applied on $[u_0,u_2] \subset \mathrm{Int}\,\Omega$ and on $[u_2,u_1] \subset \mathrm{Int}\,\Omega$, using that $u \to \left(\frac{d}{du}\right)^k G(u,f)$, $k \geq 1$, is bounded, we obtain:

$$\left\|\left(\frac{d}{du}\right)^k G(u_0,f) - \left(\frac{d}{du}\right)^k G(u_2,f)\right\| \leq K'_k\,|u_0-u_2|$$

and

$$\left\|\left(\frac{d}{du}\right)^k G(u_2,f) - \left(\frac{d}{du}\right)^k G(u_1,f)\right\| \leq K_k' \, |u_2-u_1|$$

where $K_k' > 0$ is a constant. Now (1) follows from

$$|u_1-u_0|^2 = |u_0-u_2|^2 + |u_2-u_1|^2$$

which implies

(5) $$2\,|u_1-u_0| \geq |u_0-u_2| + |u_2-u_1| \geq \sup(|u_0-u_2|,|u_1-u_2|).$$

(b) Case where u_0, u_1 satisfy $\mathrm{Im}\, u_j \geq 0$ (resp. $\mathrm{Im}\, u_j \leq 0$), for $j = 0,1$.

This follows from (a) (resp. the same proof as (a)).

(c) Case where $\mathrm{Im}\, u_0 > 0$ and $\mathrm{Im}\, u_1 < 0$.

Let $u_2' \in C_\gamma$ such that $[u_2',u_0]$ has slope ≥ 1. By (a), we have:

$$\left\|\left(\frac{d}{du}\right)^k G(u_2',f) - \left(\frac{d}{du}\right)^k G(u_0,f)\right\| \leq K_k' \, |u_2'-u_1|$$

and by (b):

$$\left\|\left(\frac{d}{du}\right)^k G(u_2',f) - \left(\frac{d}{du}\right)^k G(u_1,f)\right\| \leq K_k' \, |u_2'-u_1|$$

where K_k' is a constant.

Now (1) follows from

(7) $$|u_2'-u_0| \leq (2)^{1/2}\, \mathrm{Im}\, u_0 \leq (2)^{1/2}\, |u_1-u_0|$$

and

(8) $$|u_1-u_2'| \leq |u_1-u_0| + |u_2'-u_0|.$$ ■

12. It follows that, for $k \geq 0$, the mappings $u \in W \to \left(\frac{d}{du}\right)^k G(u,f)$ extend to Lipschitz functions on Ω that

satisfy (1) and one easily sees that $u \in \Omega \to J \circ G(u,f) \in \mathbb{C} \times D^0_{\delta/4}$ is of class C^∞ on $\Omega - C_\gamma$ (see also 7.2).

This implies that the C^∞ function:

$$(\Omega - C_\gamma) \times B_{\delta/4} \to \mathbb{C} \times \mathbb{C}$$

$$(u,z) \to G(u,f)(z) = (\lambda, h(z)),$$

is $\mathbb{C}$ analytic on $\mathrm{Int}\, \Omega \times \mathrm{Int}\, B_{\delta/4}$ and all its partial derivatives $\dfrac{\partial^{k_1+k_2}}{(\partial u)^{k_1}(\partial z)^{k_2}} G(u,f)(z)$, $k_1 + k_2 \geq 1$, are bounded.

By a proof similar to 11, these partial derivatives extend to Lipschitz functions on $\Omega \times B_{\delta/4}$.

13. Let B be a Banach space. We denote by $\|\ \|$ its norm consider on $\mathbb{R}^n$ a norm $|\ |$. We denote by $|||\ |||$ the induced norm the induced norm on the space $L(\mathbb{R}^n, B)$ of linear mappings from $\mathbb{R}^n$ into B: if $u \in L(\mathbb{R}^n, B)$, then

$$|||u||| = \sup_{|x|<1} \|u(x)\|.$$

Definition: *Let $C \subset \mathbb{R}^n$ a closed subset and $0 < \ell < 1$ (respectivel $\ell = 1$). We say that the mapping $H : C \to B$ is of class $C^{1+\ell}$ in the sense of Whitney (respectevly $C^{1+\mathrm{Lip}_1}$) if there exists a mapping $DH : C \to L(\mathbb{R}^n, B)$, such that for any compact set $K \subset C$, there exists $\ell_K > 0$, satisfying for every $x, y \in K$:*

i) $\|H(x) - H(y)\| \leq \ell_K\, |x-y|$;

ii) $|||DH(x) - DH(y)||| \leq \ell_K\, |x-y|^\ell$;

iii) $\|H(x) - H(y) - DH(y)(x-y)\| \leq \ell_K\, |x-y|^{1+\ell}$.

We refer the reader to [7] for some properties of functions of class $C^{1+\ell}$ in the sense of Whitney.

The following theorem generalizes a theorem of V. I. Arnold [1, Theorem 3, p. 252]. Furthermore it solves a question left open by V. I. Arnold 1, p. 251 .

Theorem. *Let $f \in V_{\varepsilon,\delta}$, with $\varepsilon > 0$ given by 6 and satisfying 7. Then the mapping*

$$u \in \Omega \to J\circ G(u,f) \in C \times D^0_{\delta/4}$$

is of class $C^{1+\ell}$, $0 < \ell < 1$ *in the sense of Whitney.*

Remark. Since $u \in \operatorname{Int}\Omega \to J\circ G(u,f)$ is $\mathbb{C}$-analytic, by the definition we will give in 16 and by the above theorem the mapping $u \in \Omega \to J\circ G(u,f)$ is C^1-*holomorphic*.

Proof. We want to prove that for $u_0, u_1 \in \Omega$, one has:

$$(9) \qquad \|A(u_1,u_0)\| \le K\ |u_1-u_0|^2$$

for some constant $K > 0$ independent of u_1 and $u_0 \in \Omega$ where

$$A(u_1,u_0) = J\circ G(u_1,f) - J\circ G(u_0,f) - \frac{d}{du}\, G(u_0,f)(u_1-u_0).$$

Once we will have proved (9) then with (1) we can conclude, by definition, that the mapping $u \in \Omega \to G(u,f)$ is of class $C^{1+\ell}$, $0 < \ell < 1$, in the sense of Whitney (and even $C^{1+\mathrm{Lip}_1}$).

To prove (9), we will follow what we have done in the proof of 11.

(a) Case where $\operatorname{Im} u_j > 0$, for $j = 0,1$.

We use the same notations as in 11.(a) and can suppose that (2), (3), (4) are satisfied.

We use Taylor's formula on the segment $[u_0,u_2]$ and on $[u_2,u_1]$ and hence obtain, using that $u \in \Omega \to \left(\frac{d}{du}\right)G(u,f)$ is bounded:

$$\|A(u_2,u_0)\| \le K''\ |u_2-u_0|^2$$

$$\|A(u_2,u_1)\| \le K''\ |u_2-u_1|^2$$

for $K'' > 0$ some constant.

We have:

$$(10) \qquad A(u_1,u_0) = A(u_2,u_0) - A(u_2,u_1) + B(u_0,u_1,u_2)$$

where

$$B(u_0,u_1,u_2) = \left(\frac{d}{du}\, G(u_1,f) - \frac{d}{du}\, G(u_0,f)\right)(u_2-u_1).$$

Now (9) follows from (1) and (5).

(b) Same case as case (b) of 11.

(c) Case where $\operatorname{Im} u_0 > 0$ and $\operatorname{Im} u_1 < 0$.

We choose u_2' as in 11.(c) and we write

$$A(u_1,u_0) = A(u_2,u_0) - A(u_2,u_1) + B(u_0,u_1,u_2').$$

Using cases (a) and (b) as well as (1), (7) and (8) the inequality (9) follows. ■

14. Remarks. 1. The same proof, using 9 and 12, shows that, for every $k \in \mathbb{N}$, the mapping $u \to \left(\frac{d}{du}\right)^k G(u,f)$ is of class $C^{1+\ell}$, $0 < \ell < 1$, in the sense of Whitney.

2. The same proof shows also that the function:

$$(u,z) \in \Omega \times B_{\delta/4} \to G(u,f)(z) \in \mathbb{C}^2$$

is of class $C^{1+\ell}$, $0 < \ell < 1$, in the sense of Whitney, as well as all its partial derivatives.

To the *function* $(u,z) \to G(u,f)(z) \in \mathbb{C}^2 \cong \mathbb{R}^4$, we can apply Whitney's extension theorem and conclude that it is the restriction on $\Omega \times B_{\delta/4}$ of a function $\tilde{G}\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}^2$ of class $C^{1+\ell}$ (cf. [7]). We can even suppose that: $\tilde{G}(u+p_1, z+p_2) = (p_1-p_2, p_2) + G(u,z)$, for every p_1 and $p_2 \in \mathbb{Z}$

It follows that the function $u\in\Omega \to G(u,f)-(u,0) \in \mathbb{C}\times D^\infty_{\delta/4}$ is the restriction to Ω of a $\mathbb{Z}$-periodic mapping

$$H\colon \mathbb{C} \to \mathbb{C}\times D^0_{\delta/4}$$

of class $C^{1+\ell}$.

15. Examples. 1. Let $f_b(z) = z + a\sin(2\pi z) + b$ where

$a,b \in \mathbb{R}$ and $0 < a < (2\pi)^{-1}$. By [2,III], there exists an open dense set, $U \subset \mathbb{R}$, such that, if $b \in U$, the rotation number $\rho(f_{b|\mathbb{R}}) \in \mathbb{Q}$ and $f_{b|\mathbb{R}}$ is not topologically conjugate (in the group $D^0(\mathbb{T}^1)$) to a translation (more precisely for $b \in U$, $\rho(f_{b|\mathbb{R}}) = p/q$, the function $f^q - R_p$ changes sign on $\mathbb{R}$). If $b \in U$, $\rho(f_{b|\mathbb{R}}) = p/q$, then on $\mathbb{C}/\mathbb{Z}$ f_b has a unique attracting periodic cycle $(f_b^i(z_0))_{i\geq 0}$ necessarily contained in $\mathbb{T}^1$ and of period q (attracting means $|Df_b^q(z_0)| < 1$). On $\mathbb{C}/\mathbb{Z}$, f_b has a unique repulsive periodic cycle contained in $\mathbb{T}^1$. [This follows from the theory of iteration of entire functions of Julia and Fatou:(3) In each immediate invariant basin of attraction of an attracting or parabolic fixed point of f_b^n, $n \geq 1$, (i.e. an invariant Fatou domain of f_b^n), there exists a critical point of f_b^n (f_b^n has no finite asymptotic value). One easily deduces the result from this, using that f_b commutes with $z \to \bar{z}$, f_b has two critical points in $\mathbb{C}/\mathbb{Z}$ and that $b \in U$.] As the property of having an attracting periodic orbit is stable under perturbations, even when b becomes complex, this explains why we considered Ω such that $\Omega \cap \mathbb{R} = C_\gamma$ is nowhere dense in $\mathbb{R}$. The reader should also consult [2,XII].

2. Let $f_b(z) = z + b + \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ where $\ell(z) = \sum_{n=1}^{\infty} a_n z^n$ converges on $\{z, |z| < R\}$, for some $R > 1$.

Let $g_b(z) = e^{2\pi i b}(z + \eta(z))$, with $\eta(z) = z(e^{2\pi i \ell(z)} - 1)$. We have:

$$g_b(e^{2\pi i z}) = \exp(2\pi i f_b(z)).$$

Any $\mathbb{C}$-analytic function g_b can be obtained in this way, if $\|\eta\|_{C^0(\{|z|<R\})}$ is small enough and $\eta(z) = O(z^2)$ (this will always be the case for $\frac{1}{t} g_b(tz)$, $t > 0$, $t \to 0$).

The solutions of (+), in 6, are closely related to the linearization of the holomorphic map g_b at $z = 0$. If $b = a+it$,

(3) For more details see [4_1].

$a \in \mathbb{R}$, $t \neq 0$, the linearization is possible by Poincare's theorem and if $b = a \in DC_\beta$, it follows from Siegel's theorem. To have a solution to (+), we need that the linearizing map h_1 of g_b at $z = 0$ (i.e. $h_1(e^{2\pi i b} z) = g_b \circ h_1(z)$ and $Dh_1(0) = 1$) has a radius of convergence $R_0 > 1$ and satisfies $h_1(\{|z|<R_0\}) \supset \{|z| < 1\}$. This explains why we had to consider Ω with $\Omega \cap \mathbb{R} = C_\gamma$ nowhere dense in $\mathbb{R}$.

What we did in 9 and 13 gives information on the dependence on b of the linearizing map of g_b at $z = 0$, when $e^{2\pi i b}$ crosses the unit circle ($b \in \Omega$).

16. **C^1-Holomorphic Maps and Monogenic Functions.** In this last section following V. I. Arnold [1] and A. Denjoy [D] (see also [B]), we propose to discuss E. Borel monogenic functions ([B_1] and [B_2]).

Let $K \subset \mathbb{C}$ be a compact set, B a complex Banach space and $R(K,B)$ the space of functions that are uniform units on K of B-valued rational functions with poles off K. The reader can consult [G, Ex. 19 and 20, p. 238-239] and use the fact that every B-valued function, holomorphic on a neighborhood V of K, can be approximated uniformly on K by elements of $R(K,B)$ (i.e. the standard proof of Runge's theorem, using Cauchy's formula, works for B-valued functions, after choosing a smaller neighborhood V of K, such that $V_1 \subset V$ and V_1 is a manifold whose boundary ∂V_1 is a disjoint union of a finite number of C^∞-embedded circles).

Definition. *For C a closed subset of $\mathbb{C}$ we say that the function $g: C \to B$ is C^k-holomorphic, for $k \geq 1$, if g is of class C^k in the sense of Whitney and $\bar{\partial} g = 0$ where* $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

By 13, an *example* of $C^{1+\ell}$ ($0 < \ell < 1$) *holomorphic* function on the compact set $K_1 = \Omega$ mod. 1 is:

$$u \in K_1 \xrightarrow{g_1} J\circ G(u,f) - \mathrm{Id}\Big|_{C\times B_{\delta/4}} \in C \times \overset{0}{\delta/4}.$$

If $B = C$ and g is C^1-holomorphic on K, then using Whitney's extension theorem and [G,1., p. 26], we conclude that $g \in R(K,C)$.

Using 14.2 and [G,Ex. 20, p. 238] it is also true that the mapping g_1 defined above is such that $g_1 \in R(K_1, C\times O^0_{\delta/4})$.

We suppose that $K = K_1 = \Omega \bmod 1 \subset \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$, and we consider as in 4, the union of piecewise linear Jordan curves $(\partial D_n)_{n\in\mathbb{N}}$ with the condition (*) of 4.

(a) If $g \in R(K,B)$ then *Cauchy's theorem holds*:

$$\sum_{n=0}^{\infty} \int_{\partial D_n} g(z)dz = 0$$

the boundaries $(\partial D_n)_{n\geq 0}$ being oriented in the standard way. (This follows from the fact that we can approximate g by a sequence $(g_k)_k$ of rational function with poles off K in the uniform topology on K. For each g_k, Cauchy's theorem is true and one can pass to the uniform limit using (*).)

(b) Let $x \in K$ such that:

$$\sum_{n\geq 0} \int_{\partial D_n} \frac{|dz|}{|z-x|} < +\infty.$$

For $g_k \in R(K,B)$ we have Cauchy's formula:

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial D_n} \frac{g(z)}{z-x}\, dz.$$

(If $g_k \in R(K,B)$ is a rational function, then Cauchy's formula holds and the result follows by letting $g_k \to g$ in the uniform topology.)

The reader should also consult [D, p. 139-146].

Remark. For $K = \Omega \bmod 1$, since

$$\partial K - \bigcup_j \partial\Delta_j = C_\gamma \bmod 1 \subset \mathbb{C}/\mathbb{Z},$$

by Milnikov's result (see [Z,p. 112]) we have $R(K,\mathbb{C}) = A(K,\mathbb{C})$, where $A(K,C)$ is the space of continuous complex valued functions on K, $\mathbb{C}$-analytic on Int K.

A *monogenic function* in the sense of E. Borel is a tuple $(g,(C_n)_{n\geq 1})$, where (C_n) is an increasing sequence of closed subsets of $\mathbb{C}$, g is a function defined on $\bigcup_{n\geq 1} C_n$ and for every n, $g|_{C_n}$ is a C^1-holomorphic function.

The function $(g,(C_n)_n)$ and $(g,(C'_n)_n)$ are considered equivalent, if $C_{\alpha n} \subset C'_n \subset C_{\beta n}$, when $n \geq n_0$ $(0 < \alpha < 1 < \beta)$.

Example. Let $g(z) = \sum_{p=1}^{\infty} \frac{A_p}{z-a_p}$ (A_p and $a_p \in \mathbb{C}$) and $C_n = \mathbb{C} - \bigcup_{p\geq 1} D_n^p$ where $(D_n^p)_p$ are open disks centered around a_p. We suppose that $D_n^p \supset D_{n+1}^p$ and the sequence $(A_p)_p$ decreases sufficiently rapidly (depending the choice of the sequence $(C_n)_n$). (If $\sup_p |a_p| < +\infty$, then the closed set C_n will be connected, if for $p_1 > p$, either $D_n^{p_1} \subset D_n^p$ or $D_n^{p_1} \cap D_n^p = \emptyset$.) The fact of considering examples of monogenic functions as limit of rational functions is very natural taking into account what we recalled earlier.

Every C^1-holomorphic function g on a compact set K defines a monogenic function $(g,(K_n)), K_n = K$.

We have adopted the terminology "C^1-holomorphic" (instead of monogenic), for we believe that E. Borel. by choosing the sequence $(C_n)_n$ in an appropriate way depending on the function considered (in the example if the radii of the disks D_n^p and the sequence $|A_p|$ decrease fast enough), wanted his monogenic functions to have quasi-analytic properties (i.e. monogenic continuation) (cf. [D, p. 139-146] and [C, ch. IX]). We believe that this last point is one of the main reasons of E. Borel's work on monogenic functions (which is anterior to the work of Denjoy-Carleman on quasi-analytic functions [D], [C]).

In this respect we can ask the following question: Let $t \in I_\delta, \to G(t)$ be the function defined in 8.

Question. *Is the function $G_1(t)$ always "determined" by its Taylor series at $t = 0$?*

(I think that the answer is negative, for the linearized equation does not seem to belong to any quasi-analytic class.)

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