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REVIEW AND SOME CRITICAL COMMENTS ON A PAPER OF GRÜN CONCERNING THE DIMENSION SUBGROUP CONJECTURE

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1. The beginning of the dimension subgroup problem

In 1935, Magnus' paper "Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring" [1] appeared, in which he stimulated the famous dimension subgroup conjecture: If G is any group and ΔG the augmentation ideal of the integral group ring, then the n-th integral dimension subgroup $G^{(n)} := G \cap (1+\Delta^n G)$ of G coincides with the n-th term of its lower central series (see [1] p. 260 and p. 265). Although Magnus did not work with the integral group ring but with its augmentation-adic completion — for free groups F, this is the ring of formal power series with a set of free generators of F as variables — this did not make any difference for free groups, which were considered by Magnus and Grün to be the starting point for an attack on this conjecture. In [1], Magnus was able to prove that dimension subgroups of free groups are fully invariant (thereby allowing to call the images of the dimension subgroups of a free group F under $F \twoheadrightarrow G$ to be the "dimension subgroups" of G, see [1] p. 260 and p. 269) and among others, the following theorem, the first of which is translated literally:

P. 265, III. If F_n denotes the n-th subgroup of the descending central series of $F = F_1$, then the dimension of every element $\neq 1$

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of F_n is at least n. If $g \neq 1$ is an element of dimension n, then there exists an element of F_n, which has the same summands of dimension n as g^{δ_n}, where δ_n is a fixed number for every n (We interprete this theorem and conclude from its proof that $F_n \subset F^{(n)}$ and that $F^{(n)}/F_n \cdot F^{(n+1)}$ is a torsion group of bounded exponent. It seemed to be obvious to Magnus that $F^{(n+1)} \subset F_n$, see [1] p. 270, so that he always claimed $F^{(n)}/F_n$ to be a torsion group of bounded exponent, see [6] p. 148. But there is no reasoning for this last statement to be found in [1]. In 1979, Sjögren has shown in [5] that already for arbitrary groups G, $G^{(n)}/G_n$ is a torsion group of bounded exponent.);

P. 266, IVa $F^{(n)}/F^{(n+1)}$ is torsion free abelian.

These were the tools, which Grün found when he started his paper "Über eine Faktorgruppe freier Gruppen I" [2]. Grün's paper appeared one year after Magnus had formulated the dimension subgroup conjecture for finitely generated free groups, and one of its main goals was to prove it. It should be mentioned that the dimension subgroup conjecture was not only of interest in its own right, but also because an affirmative answer would imply (by IVa above) that the quotients F_n/F_{n+1} are torsion-free, which was not clear at that time (see Grün's introduction in [2] and for another approach and a solution [4]).

Still one year later, in 1937, Magnus himself published a combinatorial proof of his conjecture for free groups in his paper "Über Beziehungen zwischen höheren Kommutatoren" [3] (although he points out in [6] that it contains a slight gap, which was filled in by an identity supplied by Witt in [4]). And today, possibly the most convenient way of proving it would be by Lie-theoretic methods which go back to Magnus and Witt (see [3] and [4]).

Magnus' paper was submitted for publication on the 26th of October 1936, and in it he admitted (p. 105) that in the meantime Grün had already shown $F^{(n)} = F_n$ in "a simple way and by other means" in [2]. In the Magnus-Chandler 1982 book [6], Grün is still given the credit: "The first proof that $F^{(n)} = F_n$ was given by Grün (1936). It uses matrix representations for F/F_n and is not easy to follow. Also, it seems to have left no trace in the literature", (see p. 149 in [6]). Nowadays there seems to be some confusion as to whom the credit for the solution in case of finitely generated free groups should be given: either to Magnus (and possibly Witt) or to Grün. The aim of this paper is to make the results of Grün's paper [2] accessible to people, who have had much difficulty with its outmoded German forms, and at times unprecise mathematical expression. It turns out that there are two gaps in Grün's paper, one of which leads him to a serious mistake. Although it is unseemly to critisize the work of a deceased author, I feel that the remaining results are very well worth the effort of being understood and brought back to public notice.

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2. From Grün's introduction

Grün's aim is to prove Magnus' dimension subgroup conjecture for groups F, which are free on n generators a_i $(i=1,\ldots,n)$, by the following means:

Let $\underline{\underline{r}}$ be the ring of $(n+1)\times(n+1)$-strictly lower triangular matrices (i.e. zeros on and above the main-diagonal) with integer coefficients, and let G be the (multiplicative) group $E+\underline{\underline{r}}$, where E is the unit matrix. Starting with a representation $F \twoheadrightarrow G$, which is defined by sending the generators of F to appropriate generators of G, he finally constructs a representation

$$\gamma: F \longrightarrow \Pi G \quad \text{(non-restricted direct product)},$$

whose kernel turns out to be $F^{(n+1)}$.

According to this program, the paper consists of two parts: analysing the structure of G and construction of γ.

3. The structure of G

There are many important results in Grün's 1, which are partly well-known nowadays. It seems appropriate to state and outline the proof of some of them.

1. and 4. (p. 773) Defining $\underline{\underline{s}}_k$ $(k=1,\dots,n)$ to be the set of those matrices in $\underline{\underline{r}}$ having non-vanishing entries only in the k-th column, $\underline{\underline{s}}_k$ turns out to be a left-ideal, whereas $\underline{\underline{q}}_k := \underline{\underline{s}}_1 + \dots + \underline{\underline{s}}_k$ is a two-sided ideal.

2. Taking now the k-th row (instead of column) gives right-ideals $\underline{\underline{z}}_k$ $(k=2,\dots,n+1)$.

Note that there is some inconsistency in Grün's indexing in 5., which can be avoided by setting $\underline{\underline{r}}_{n-k} := \underline{\underline{z}}_{n+1} + \dots + \underline{\underline{z}}_{n+1-k}$ (instead of Grün's $\underline{\underline{r}}_k$), which is then a two-sided ideal.

With this correction, 6. (and everything except 7.) remains unchanged: $\underline{\underline{a}}_k := \underline{\underline{r}}_k \cap \underline{\underline{q}}_k$ forms a two-sided ideal with $\underline{\underline{a}}_k^2 = 0$.

Instead of Grün's $\underline{\underline{r}}_2$, one now has to take in 7.: $\underline{\underline{r}}_1 = \underline{\underline{q}}_n$ is a prime-ideal in $\mathbb{Z}E + \underline{\underline{r}}$.

He proceeds by mentioning that $\underline{\underline{r}}^i$ consists of those matrices, which have zero entries in the first $i-1$ diagonals below the maindiagonal.

Corresponding to 6., he obtains

Proposition 3. $A_k := (E+\underline{\underline{r}}_k) \cap (E+\underline{\underline{q}}_k)$ forms an abelian normal subgroup of G, and moreover (Proposition 4): $G = A_1 \cdot \ldots \cdot A_n$.

All of these results are more or less direct conclusions from the relations $A_{ij}A_{kl} = \delta_{jk}A_{il}$ (A_{ij} the matrix with 1 in position (i,j) and zeros elsewhere, δ_{ij} the Kronecker-delta). Another conclusion from these relations together with the observation that "every element of G can be expressed as a product of elements $E+S_i$, where in S_i at most the i-th column is not vanishing" gives the first really important result with respect to the second part of his work; namely,

Proposition 5 (p. 774). G is generated by the n elements $t_i := E + A_{i+1,i}$ $(i=1,\dots,n)$. Abbreviating $t_k^{-1}t_it_k$ by $t_i^{t_k}$ and $s^{-1}t_ist_i^{-1}s^{-1}t_i^{-1}st_i$ by $t_i^{s-1-s+1}$ with s arbitrary in G, the following relations hold in G:

$$t_i^{t_k-1} = 1 \quad (k\neq i\pm 1), \qquad t_i^{t_k-1-t_k+1} = 1 \quad (k=i\pm 1),$$

and generally:

$$t_i^{s-1-s+1} = 1, \quad s \text{ an element of } G.$$

The following two propositions describe the lower central series of G.

Proposition 6+7. G_i is the set of all matrices in G having zeros on the first $i-1$ diagonals below the main-diagonal. Thus

$$G_i = E + \underline{\underline{r}}^i .$$

To show that this set A_i contains G_i, Grün simply makes use of the fact that $A_i = E + \underline{\underline{r}}^i$. The proof of the reverse inclusion is nice: Letting H (resp. K) be generated by $t_1,\dots,t_{n-1}$ (resp. $t_2,\dots,t_n$), one finds $H_iK_i = A_i$, and on the other hand obviously $H_iK_i \subset G_i$.

He then mentions that it can be shown along the same lines that the i-th term of the derived series of G is just G_{2^i} (*Proposition* 8).

Possibly, the most interesting result of the first section (although it has nothing to do with dimension subgroups) is

Proposition 10 (p. 775). Let G^m be the subgroup of G generated by all m-th powers. Then $G^m \supset G_m$.

The proof is constructive and proceeds in three steps

Lemma 1. $E + V \in G \Longrightarrow E + mV \in G^m$.

This is simply a consequence of the fact that one can express V in the form $S_1+\dots+S_n$, where in S_i all columns except the i-th one are null, and

$$E + mV = (E + mS_1)\cdot\ldots\cdot(E + mS_n) = (E + S_1)^m \cdot\ldots\cdot (E + S_n)^m.$$

Lemma 2. $E + V \in G_2 \Longrightarrow E + \binom{m}{2} V \in G^m$.

Since one has

$$[(E+A_{i+1,i})(E+A_{i,i-1})]^m = (E+A_{i+1,i}+A_{i,i-1}+A_{i+1,i-1})^m$$

$$= [E + \binom{m}{1}(A_{i+1,i}+A_{i,i-1}+A_{i+1,i-1})][E+\binom{m}{2}A_{i+1,i-1}],$$

where the left hand side as well as the first factor of the right hand side (by Lemma 1) are contained in G^m, this gives $E + \binom{m}{2}A_{i+1,i-1} \in G^m$. In the same way, one can show successively that $E+\binom{m}{2}A_{i+1,i-2} \in G^m$, $E + \binom{m}{2}A_{i+1,i-3}\in G^m$ etc., which generate the subgroup of all matrices of the form $E+\binom{m}{2}V$ with $V \in \underline{\underline{r}}^2$.

Lemma 3. $E+V\in G_s \Longrightarrow E+\binom{m}{s}V \in G^m$ (and taking $s=m$ then shows Proposition 10).

Grün proceeds by induction on s:

Let $W := \sum_{k=1}^{s} A_{i+k,i+k-1}$. Then one has

$$W^{s+1} = 0,\quad W^s = \prod_{k=s}^{1} A_{i+k,i+k-1} = A_{i+s,i}. \qquad (*)$$

Now let $\underline{\underline{i}}$ be the ideal generated by all B, where $E+B\in G^m$. With the above W, one then obtains $(E+W)^m = E + \sum_{k=1}^{s}\binom{m}{k}W^k \in G^m$, and hence $\sum_{k=1}^{s}\binom{m}{k}W^k \in \underline{\underline{i}}$. By the inductive hypothesis, one already has $\binom{m}{k}W^k \in \underline{\underline{i}}$ for $k = 1,\ldots,s-1$, and whence: $\binom{m}{s}W^s \in \underline{\underline{i}}$.

It is easy to see that $\binom{m}{s}\underline{\underline{r}}^s$ is generated by the $\binom{m}{s}A_{i+s,i}$ $(i=1,\ldots,n)$; hence $\binom{m}{s}\underline{\underline{r}}^s \subset \underline{\underline{i}}$, and the assertion follows from Proposition 6+7.

Grün finishes this section by remarking that the Propositions 1-10 carry over to matrices with coeficients from some finite field, but there will be some further relations in Proposition 5 arising from the modular situation.

4. A representation of F

Grün starts this part of his work by introducing some terminology from Magnus' paper [1]: Let O be the ring of formal power-series in the non-commuting variables s_i $(i=1,\ldots,n)$ with integer coefficients. Then one can extend $a_i \longmapsto 1+s_i$ $(i=1,\ldots,n)$ to a faithful representation of F (which was freely generated by the a_i) into the group of units of O. Hence one may identify F with the subgroup of the unit group of O generated by all $1+s_i$ $(i=1,\ldots,n)$. Denoting by $\mathbb{Z}F$ the integral group ring of F and by ΔF its augmentation ideal, it turns out that

$$O = \lim_i \operatorname{inv} \mathbb{Z}F/\Delta^i F.$$

Grün then defines L to be the augmentation ideal of O (i.e. the ideal generated by all s_i $(i=1,\ldots,n)$) and I_1 to be the ideal generated by all $s_i^2, s_i s_k$ $(i,k=1,\ldots,n,\ k\neq i-1)$ and denotes by $F(I_1)$ the image of F under

$$F \longrightarrow O \longrightarrow O/I_1$$

$$a_i \longmapsto 1+s_i \longmapsto 1+s_i + I_1.$$

Before we come to the heart of the paper — Proposition 11+12 - I would like to point out the following: Throughout the whole paper, Grün carefully avoids the use of a ringhomomorphism $O \longrightarrow \mathbb{Z}E + \underline{\underline{r}}$ (remember that O is not the group ring but its augmentation-adic completion, so that the existence of this homomorphisms is not immediate) and calculates instead — for example in the proof of

Proposition 11 - with the relations holding in $F(I_1)$. But if he wants to establish some kind of connection between the dimension subgroups of F and those of G, he has to use this homomorphism somewhere. So we will use it right from the beginning to streamline his proofs to some extent. Further on, since Grün's language is not only outmoded but sometimes very unprecise, the proof of Proposition 12 given here (though still sketchy) may look very different from the original one. But, it is essentially Grün's proof!

Proposition 11 (p. 777). $a_i \longmapsto t_i$ induces an isomorphism $F(I_1) \xrightarrow{\sim} G$.

Proof: $a_i \longmapsto t_i$ can be extended to a ringhomomorphisms $\mathbb{Z}F \longrightarrow \mathbb{Z}E+\underline{\underline{r}}$, and since $\underline{\underline{r}}$ is nilpotent so that $\mathbb{Z}E + \underline{\underline{r}}$ is already $\underline{\underline{r}}$-adic complete, this can be extended to the completion of $\mathbb{Z}F$ to give a ring-homomorphism

$$O \longrightarrow \mathbb{Z}E + \underline{\underline{r}},$$

whose kernel obviously contains the ideal I_1 generated by all s_i^2, $s_i s_k$ $(k \neq i-1)$. Thus there is an induced homomorphism

$$O/I_1 \longrightarrow \mathbb{Z}E + \underline{\underline{r}},$$

and passing to unit groups, gives

$$F(I_1) \longrightarrow G \text{ with } a_i \longmapsto t_i,$$

which is surjective by Proposition 5.

It remains to show that it is injective. For this purpose, it is enough to prove this for its restriction to the center of $F(I_1)$. This is so, because in L/I_1 every product of more than n factors vanishes. Hence $F(I_1)$ is nilpotent and its center is met by every normal subgroup, in particular by the kernel.

The next step consists of showing that the center of $F(I_1)$ is generated by the residue class $1+s_n s_{n-1}\cdot\ldots\cdot s_1 + I_1$. Obviously, this element generates a central subgroup. Now if $1 + r + I_1$ is

any central element, then $rs_k - s_k r \in I_1$ for all k. By the definition of I_1, it is easy to see that this already implies $s_k r, rs_k \in I_1$, and the only elements r with this property, which are not already contained in I_1, are of the form $m.s_n\cdot\ldots\cdot s_1+a$, $a \in I_1$, $m \in \mathbb{Z}$. Since one has

$$1 + ms_n\cdot\ldots\cdot s_1 \equiv (1 + s_n\cdot\ldots\cdot s_1)^m \bmod I_1,$$

this proves the assertion concerning the center.

On the other hand, the center of G is G_n, which is generated by

$$E + A_{n+1,1} = E + A_{n+1,n}\cdot A_{n,n-1}\cdot\ldots\cdot A_{2,1}.$$

And since

$$1 + s_n\cdot\ldots\cdot s_1 \longmapsto E + A_{n+1,n}\cdot\ldots\cdot A_{2,1},$$

the above map is injective, when restricted to the center of $F(I_1)$.

Qed.

Remark. Actually, the proof shows somewhat more, which may be of interest to those involved in circle and unit groups. Replacing $F(I_1)$ by O/I_1 and accordingly the center by the annihilator of L/I_1, the above reasoning applies to give:

$$a_i \longmapsto t_i \text{ induces a ring-isomorphism } O/I_1 \xrightarrow{\sim} \mathbb{Z}E + \underline{\underline{r}}.$$

Having identified F with its image in O, we may form $N := (1 + I_1) \cap F$, which turns out to be a normal subgroup of F, namely the kernel of $F \longrightarrow G$ with $a_i \longmapsto t_i$. Grün now proceeds by constructing representations of F from the above one, by means of endomorphisms of F, which do not leave N invariant. To supply these endomorphisms, he shows

Lemma 4. Let $(c_{i,k}) \in Gl(n,\mathbb{Z})$ and define $a_i' := \prod_{k=1}^{n} a_k^{c_{i,k}}$ $(i=1,\ldots,n)$. Then F is generated by the $a_1',\ldots,a_n'$ together with F_q, q arbitrary.

The proof is an application of the theory of linear equations.

Since $\det(c_{ik}) = \pm 1$, the above expression - looked at modulo F_2 - allows a unique solution for the a_i:

$a_i = f_i \cdot X_i$, f_i a word in the a_i', and $X_i \in F_2$ expressed in the a_i.

Feeding these expressions into the X_i and collecting the f_i's then gives

$a_i = g_i \cdot Y_i$, g_i a word in the a_i' and $Y_i \in F_3$ expressed in the a_i; and iteration leads to the assertion.

Grün now simply states that this implies that

$$\tau_C : F \longrightarrow F, \quad C = (c_{i,k}) \in Gl(n,\mathbb{Z})$$
$$a_i \longmapsto a_i'$$

induces an automorphism of F/F_k for all k. Surjectivity is indeed obvious, and injectivity follows then, since F/F_k is hopfian (a weak version of this last argument may have been known to Grün). Anyway, with each matrix $C \in Gl(n,\mathbb{Z})$ there is associated an endomorphism τ_C of F, which induces an automorphism of F/F_{n+1}. Hence, $a_i'F_{n+1} \longmapsto t_i$ can be extended to a homomorphism $F/F_{n+1} \longrightarrow G$, which - when composed with the natural map $F \to F/F_{n+1}$ - gives a representation

$$\gamma_C : F \longrightarrow G$$
$$a_i' \longmapsto t_i.$$

With γ_1 as our basic representation $a_i \longmapsto t_i$, one has $\gamma_C \tau_C = \gamma_1$; and hence $\ker \gamma_C = F_{n+1} \cdot \tau_C(\ker \gamma_1) = F_{n+1} \cdot \tau_C(N)$.

As Grün put it, "adding up" these representations will then yield a representation of F with kernel $\bigcap_{C \in Gl(n,\mathbb{Z})} \ker \gamma_C$, and the main work consists in showing that this intersection is equal to $F^{(n+1)}$. So the outcome of our considerations will be

$\gamma : F \longrightarrow \prod_{C \in Gl(n,\mathbb{Z})} G$, where $\gamma(x) = (\gamma_C(x))_{C \in Gl(n,\mathbb{Z})}$, and

Proposition 12 (p.781). $\ker \gamma = F^{(n+1)}$.

Proof: Since one can extend $\gamma_C : F \longrightarrow G$ to a ring homomorphism $\mathcal{O} \longrightarrow \mathbb{Z}E + \underline{\underline{r}}$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\gamma_C} & \mathbb{Z}E + \underline{\underline{r}} \\ \downarrow & \nearrow & \\ \mathcal{O} & & \end{array}$$

commutes, one obviously has $F^{(n+1)} \subset \ker \gamma_C$ for all $C \in Gl(n,\mathbb{Z})$, and thus "$\supset$" in our assertion is already clear.

To prove the reverse inclusion, let I_C be the ideal generated by all $x-1$, $x \in \ker \gamma_C$ and

$$I := \bigcap_{C \in Gl(n,\mathbb{Z})} I_C .$$

Because of $\ker \gamma = \bigcap \ker \gamma_C \subset 1 + I$, it suffices to show $I \subset L^{n+1}$. Denoting the image of I_C under the natural map $\mathcal{O} \longrightarrow \mathcal{O}/L^{n+1}$ by $\underline{\underline{i}}_C$, this will follow once we have shown

$$\underline{\underline{j}} := \bigcap \underline{\underline{i}}_C = 0.$$

Let $\mathcal{O}_k$ be the set of homogeneous elements of $\mathcal{O}$ of degree k, then

$$\mathcal{O} = \bigoplus_{k \in \mathbb{N}_0} \mathcal{O}_k, \qquad L^{n+1} = \bigoplus_{k>n} \mathcal{O}_k,$$

and thus $\mathcal{O}/L^{n+1}$ can be identified with the Z-module

$$A := \bigoplus_{k=0}^{n} \mathcal{O}_k .$$

Structure transport via this identification then makes A a ring.

Now let $x \in \underline{\underline{j}}$. Since $\underline{\underline{j}}$ is an ideal in A, there exists already a $y \in \underline{\underline{1}}^n \cap \underline{\underline{j}}$ ($\underline{\underline{1}}$ the image of L in A), which does not vanish, provided $x \neq 0$. Hence, y is an integral linear combination of homogeneous elements of degree n

$$y = \sum a_{j_1,\dots,j_n} s_{j_1} \cdot \ldots \cdot s_{j_n}$$

where the summation is taken over all n-tuples $(j_1,\dots,j_n)$, $1 \le j_i \le n$. We want to show $y = 0$.

Now, I_C is obtained from I_1 by applying an endomorphism τ_C of O (namely the extension of $a_i \longmapsto a_i'$). This τ_C induces an endomorphism $\hat{\tau}_C$ of O/L^{n+1} so that $\underline{\underline{i}}_C = \hat{\tau}_C(\underline{\underline{i}}_1)$. We claim that $\hat{\tau}_C$ is bijective: τ_C induces an automorphism of F/F_{n+1} and hence an automorphism of $\mathbb{Z}(F/F_{n+1})$, which gives by $\mathbb{Z}(F/F_{n+1}) \xrightarrow{\sim} \mathbb{Z}F/\Delta(F,F_{n+1})$ ($\Delta(F,F_{n+1})$ the kernel of $\mathbb{Z}F \longrightarrow \mathbb{Z}(F/F_{n+1})$) an automorphism of the latter quotient. Factoring out $\Delta^{n+1}F/\Delta(F,F_{n+1})$ still yields an automorphism, which carries over to one of O/L^{n+1} via

$$(\mathbb{Z}F/\Delta(F,F_{n+1}))/(\Delta^{n+1}F/\Delta(F,F_{n+1})) \xrightarrow{\sim} \mathbb{Z}F/\Delta^{n+1}F \xrightarrow{\sim} O/L^{n+1}$$

So if $y \in \hat{\tau}_C(\underline{\underline{i}}_1)$ for all $C \in Gl(n,\mathbb{Z})$, then

$$\hat{\tau}_C^{-1}(y) \in \underline{\underline{i}}_1 \quad \forall\, C \in Gl(n,\mathbb{Z}).$$

For $C = (c_{i,k})$, $\hat{\tau}_C$ can be described in A by

$$s_i \longmapsto \left(\prod_{k=1}^{n} (1+s_k)^{c_{i,k}}\right) - 1;$$

on expanding this product we get

$$\hat{\tau}_C : s_i \longmapsto \sum_{k=1}^{n} c_{i,k}s_k + \text{terms of higher degree.}$$

For $(c'_{i,k}) := C^{-1}$, it is easy to see that $\hat{\tau}_C^{-1}$ is given by

$$\hat{\tau}_C^{-1} : s_i \longmapsto \sum_{k=1}^{n} c'_{i,k}s_k + \text{terms of higher degree.}$$

Since if C runs through $Gl(n,\mathbb{Z})$, the same happens to C^{-1}, we have to show that: $\hat{\tau}_C(y) \in \underline{\underline{i}}_1$ $\forall\, C \in Gl(n,\mathbb{Z})$ implies $y = 0$.

Every product in $A = O/L^{n+1}$ of more than n factors in s_i vanishes, and $\hat{\tau}_C$ is multiplicative. This implies

$$\hat{\tau}_C\left(\prod_{r=1}^{n} s_{i_r}\right) = \prod_{r=1}^{n}\left(\sum_{k=1}^{n} c_{i_r,k}s_k\right).$$

Hence, the contribution of any constituent $\prod_{r=1}^{n} s_{j_r}$ forming y to the coefficient of $s_n \cdot s_{n-1} \cdot \ldots \cdot s_1$ after having applied $\hat{\tau}_C$ is precisely

$$\prod_{r=1}^{n} c_{j_r,n+1-r}.$$

Now $\underline{\underline{i}}_1$ (as the image of I_1, which is generated by s_i^2, s_is_k; $k \neq i-1$) contains every product of n factors except $s_n \cdot s_{n-1} \cdot \ldots \cdot s_1$; and since $A/\underline{\underline{i}}_1 \xrightarrow{\sim} O/I_1 \xrightarrow{\sim} \mathbb{Z}E + \underline{\underline{r}}$ (by Proposition 11) is torsion-free, $b \cdot s_n \cdot \ldots \cdot s_1 \in \underline{\underline{i}}_1$ with $b \in \mathbb{Z}$ implies $b = 0$. Hence, the coefficient of $s_n \cdot \ldots \cdot s_1$ in $y = \hat{\tau}_1(y) \in \underline{\underline{i}}_1$ must vanish; and the same happens to the coefficient of every $s_{j_1} \cdot \ldots \cdot s_{j_n}$, where $(j_1,\dots,j_n)$ is a permutation of $(1,\dots,n)$, since exactly $s_{j_1} \cdot \ldots \cdot s_{j_n}$ contributes (and is sent) to $s_n \cdot \ldots \cdot s_1$ by an appropriate permutation matrix.

Let $a_{i_1,\dots,i_n} \cdot s_{i_1} \cdot \ldots \cdot s_{i_n}$ be one of the remaining constituents of y. We construct a matrix $C \in Gl(n,\mathbb{Z})$ such that the contribution say a of this term under $\hat{\tau}_C$ to the coefficient of $s_n \cdot \ldots \cdot s_1$ will be sufficiently large to force $a_{i_1,\dots,i_n} = 0$.

One has $a = a_{i_1,\dots,i_n} \cdot \prod_{r=1}^{n} c_{i_r,n+1-r}$. Let C' be the matrix with 1's precisely in all positions $(i_r, n+1-r)$, $r=1,\dots,n$, and zeros elsewhere. By (only!) interchanging the rows, C' can be brought into reduced row echelon form, which amounts to form $D \cdot C'$ with an appropriate $D \in Gl(n,\mathbb{Z})$. DC' is an upper triangular matrix, not all of its main-diagonal entries equal to 1, since $(i_1,\dots,i_n)$ is not a permutation of $(1,\dots,n)$.

Define $c := \sum |a_{j_1,\dots,j_n}|$, replace the non-zero entries in DC' outside the main-diagonal by c and fill up the main-diagonal with 1's. With the resulting matrix C'' define $C := D^{-1}C''$. Then one has det C = det D^{-1}.det C'' = ±det C'' = ±1, hence $C \in Gl(n,\mathbb{Z})$. Further on, $\prod_{r=1}^{n} c_{i_r,n+1-r} = c^m$, where m is the number of elements not on the main-diagonal of C'', and forming now any other product $\prod_{r=1}^{n} c_{j_r,n+1-r}$ (with precisely one factor from each column) gives $c^{m_{j_1,\dots,j_n}}$ (or zero), which is at most equal to c^{m-1}.

The coefficient of $s_n\cdot\ldots\cdot s_1$ in $\hat{\tau}_C(y)$ is given by

$$\sum a_{j_1,\dots,j_n}\cdot\prod_{r=1}^{n} c_{j_r,n+1-r} =: b,$$

and if $\hat{\tau}_C(y) \in \underline{i}_1$, then $b = 0$. Hence,

$$0 = |b| \geq \left| |a_{i_1,\dots,i_n}|\cdot c^m - \sum |a_{j_1,\dots,j_n}| c^{m_{j_1,\dots,j_m}} \right|$$

(the summation taken over all $(j_1,\dots,j_n) \neq (i_1,\dots,i_n)$), which by the choice of c is only possible for $a_{i_1,\dots,i_n} = 0$. Qed.

Proposition 12 says in other words that the free nilpontent group on n generators and of class n is residually a G-group.

Having established now Proposition 12, everything happens very fast. Since Grün uses the term "representation" somewhat loosely, let us give a literal translation of his Proposition 12, which may serve as a definition of his use of the term "representation":

"The representation Γ of F formed by adding all or at least sufficiently many representations of F, which are obtained from the representation G of F given in 1 by automorphisms of F/F_{n+1}, is isomorphic to $F/F^{(n+1)}$." Hence, $\Gamma = \gamma(F)$ by what he has proved; and he concludes: "If we denote by $\Gamma^{(k)}$ the k-th dimension subgroup of Γ, then on the one hand by Proposition 7 $\Gamma^{(k)} = \Gamma_k$ $(k=1,\dots,n)$ and on the other hand by Proposition 12 $\Gamma = F/F^{(n+1)}$ and hence $F^{(k)} = F_k\cdot F^{(n+1)}$. By Proposition III of the cited paper of Magnus (see [1] p. 265 or 1 of this work), this implies $F^{(k)} = F_k$ $(k=1,\dots,n)$."

There are two objections to his conclusion:

1) $\gamma\colon F \longrightarrow \Pi G$ is not surjective so that Proposition 7 does not apply. Taking a closer look at the preimage of $(\Pi G)^{(k)} = \Pi G^{(k)} = \Pi G_k = (\Pi G)_k$, one obtains

$$F^{(k)} \subset \bigcap_C (F_k\cdot\ker\ \gamma_C),$$

and it remains open, whether the desired result

$$\bigcap (F_k\cdot\ker\ \gamma_C) \subset F_k(\bigcap \ker\ \gamma_C) = F_k\cdot\ker\gamma = F_k\cdot F^{(n+1)}$$

really holds. The same kind of argument on p. 778 leads to an error: Taking it for granted that $F^{(n+1)} = F_{n+1}$, Grün concludes from Proposition 10 and Proposition 12:

$$\Gamma^m \supset \Gamma_m \Longrightarrow F^m F_{n+1} \supset F_m.$$

This holds for all m and n. But remember that F is free on n generators, and one certainly would like to make this inclusion independent of the number of generators of F. The argument given by Grün to conclude $F^{(k)} = F_k\cdot F^{(n+1)}$ for arbitrary finitely generated free groups from the same equality for free n-generator groups applies also here: If m is greater than the number of free generators of F, choose K free on $a_1,\dots,a_n,\dots,a_r$, where $m \leq r$. Then one has an injection

$$j: F \longrightarrow K, \quad j(a_i) = a_i$$

and a projection

$$\pi: K \longrightarrow F \quad \text{by setting} \quad \pi(a_{n+i}) = 1.$$

Apparently $\pi j = \mathrm{Id}_F$. And if Grün' conclusion

$\Gamma^m \supset \Gamma_m \Longrightarrow K^m K_{r+1} \supset K_m$ would be true, then

$$F_m = \pi j(F_m) \subset \pi(K_m) \subset \pi(K^m K_{r+1}) = F^m \cdot F_{r+1}.$$

Hence, the assumption $F^{(k)} = F_k$ for all k implies along Grün's arguments:

$F_m \subset F^m \cdot F_{n+1}$ for all $m \le n$ and all finitely generated free groups.

In particular, it would follow that every finitely generated nilpotent group of exponent m would be of nilpotency class less than m, which – as is well-known – is wrong already for $m = 4$.

2) But even if one accepts $F^{(k)} = F_k \cdot F^{(n+1)}$, I don't see a possibility to conclude $F^{(k)} = F_k$ from Magnus' Proposition III without serious further effort (at least as long as one wants to avoid the use of $F^{(n+1)} \subset F_n$, which is contained – but not proved – in Magnus' paper [1], see p. 270).

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