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DIAGONAL EQUATIONS OVER FUNCTION FIELDS

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Abstract: Let K be a function field in one variable over $\mathbb{C}$ and $a_1,\dots,a_m,b$ non-zero elements of K, such that b is linearly independent from $a_1,\dots,a_m$ over $\mathbb{C}$. We show that for n sufficiently large, the equation $\sum_{i=1}^m a_i x_i^n = b$ has no non-constant solutions in K.

§1. Introduction

Let K be a function field in one variable over $\mathbb{C}$. In [S], Silverman proved that, if a,b,c, are non-zero elements of K then for $\max\{m,n\}$ sufficiently large the Cassels-Catalan equation $ax^n+by^m = c$ has no non-constant solutions in K. This result was generalized by Newman and Slater to equations $\sum_{i=1}^m a_i x_i^n = b$, for m arbitrary, when $K = \mathbb{C}(t)$. The main result of this paper is Theorem 1 below which generalizes the results mentioned above to m arbitrary and K arbitrary. We also prove two other results by the same method which deal, respectively, with diagonal equations for subrings of integral functions of K and unit equations.

For $x \in K$, $x \notin \mathbb{C}$ we define $\deg x = [K:\mathbb{C}(x)]$, and if $x \in \mathbb{C}$ we put $\deg x = 0$. Thus $\deg x$ is the number of zeros (or poles) of x counted with multiplicities.

The results are the following

Theorem 1. Let K be a function field in one variable over $\mathbb{C}$ and $a_1,\dots,a_m,b$ non-zero elements of K, such that b is linearly independent from $a_1,\dots,a_m$ over $\mathbb{C}$. If n is

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sufficiently large depending only on $\deg a_1,\ldots,\deg a_m,\deg b$, then the equation

$$\sum_{i=1}^{m} a_i x_i^n = b \qquad (1)$$

has no non-constant solutions $x_1,\ldots,x_m \in K$.

Theorem 2: Let K, $a_1,\ldots,a_m,b$, be as in Theorem 1. Let S be a finite set of places of K such that $a_1,\ldots,a_m,b$ are S-integral. Then given $n > m(m-1)$, any solutions $x_1,\ldots,x_m$ of (1), which are S-integral and such that $a_1x_1^n,\ldots,a_mx_m^n$ are linearly independent over $\mathbb{C}$, satisfy

$$[n-m(m-1)]\max_i \deg x_i \leq \frac{m(m-1)}{2}(2g-2+|S|)+2H,$$

where

$$H = \deg a_1 + \ldots + \deg a_m + \deg b.$$

Corollary 3. With the same notation as in Theorem 2, if $m = 3$, $n \geq 7$ and if a_i/a_j $(i\neq j)$, a_i/b are not n-th powers in K, then all solutions of (1) that are S-integral have bounded degree. If $n \geq 16$ and a_i/a_j $(i\neq j)$, and a_i/b are not n-th powers in K then all solutions of (1) in K have bounded degree.

The following result is due to Mason (see [M] for the case $m = 2$, the general case seems to be unpublished). We give a new proof of this result.

Theorem 4: If K is as above, S is a finite set of places of K, and $u_1,\ldots,u_m$ are S-units, linearly independent over $\mathbb{C}$, satisfying

$$\sum_{i=1}^{m} u_i = 1 \qquad (2)$$

then $\max_i \deg u_i \leq \frac{m(m-1)}{2}(2g-2+|S|)$.

The proof of the above results will be given in §3. It is a generalization of the methods of [NS], where they employ Wronskians of $a_1x_1^n,\ldots,a_mx_m^n$, for solutions $x_1,\ldots,x_m$ of (1). In our case we use the theory of Weierstrass points of projective embeddings as is given for example in [L] or [SV]. The results of this theory are proved by using Wronskians; however, by using only the results we avoid explicit mention of Wronskians in this paper. The results we need on Weierstrass points will be stated in §2.

§2. Weierstrass points

In this section we state the results from the theory of Weierstrass points we need. Proofs for these results can be found in [L] or [SV]. We follow the notation of [SV].

Let K be as in §1 and let X be the algebraic curve (or compact Riemann Surface) with K as function field. If $p \in X$ we denote by v_p the valuation of K associated to p.

Let $\phi: X \to \mathbb{P}^{m-1}$ be a morphism, which we assume to be non-degenerate; i.e., $\phi(X)$ is not contained in a hyperplane. By choosing coordenates in $\mathbb{P}^n$, ϕ is given by $(f_1:\ldots:f_m)$, with $f_i \in K$ for all i. So if $p \in X$ and t is local parameter at p, $\phi(p) = (t^{e_p}f_1(p):\ldots:t^{e_p}f_m(p))$ where $e_p = -\min\{v_p(f_1),\ldots,v_p(f_m)\}$.

We define the divisor E on X by $E = \sum_{p\in X} e_p p$. This depends only on ϕ and we define $\deg\phi = \deg E = \sum_{p\in X} e_p$. If ϕ is an embedding, $\deg\phi = \deg\phi(X)$ (the degree of $\phi(X)$ as a curve on $\mathbb{P}^{m-1}$).

For $p \in X$, the set $\left\{v_p\left(\sum_{i=1}^{m}\alpha_i t^{e_p} f_i\right) \mid \alpha_i \in \mathbb{C}\right\}$ consists of m integers $0 = j_0 < j_1 < \ldots < j_{m-1} \leq \deg\phi$. (The j_i depend on p, but the notation should cause no confusion). The integers $j_0,\dots,j_{m-1}$ are called the (ϕ,p)-orders, and $\{j_0,\dots,j_{m-1}\} = \{0,\dots,m-1\}$ for all but finitely many $p \in X$. These finitely many exceptions are called Weierstrass points of ϕ. The number $w_\phi(p) := \sum_{i=0}^{m-1}(j_i - i)$ is called the weight of p and we have

$$\sum_{p\in X} w_\phi(p) = m(m-1)(g-1) + m \deg \phi. \qquad (3)$$

We also have that

$$\dim_{\mathbb{C}}\left\{f = \sum_{i=1}^{m} \alpha_i f_i,\ \alpha_i \in \mathbb{C},\ v_p(f) \geq j_r - e_p\right\} = m - r.$$

We need the following.

Lemma 5: If $v_p(f_1) \leq \dots \leq v_p(f_m)$ then $j_i \geq v_p(f_{i-1}) + e_p$, $i = 0,\dots,m-1$.

Proof: The lemma is clear for $i = m-1$ since j_{m-1} is the largest order that $\sum \alpha_i t^{e_p} f_i$ can assume for $\alpha_i \in \mathbb{C}$. Assume that for some $0 \leq k < m-1$ the result is true for $i > k$ and that $j_k < v_p(f_{i-1}) + e_p$. We have that

$$\dim_{\mathbb{C}}\left\{f = \sum \alpha_i f_i \mid \alpha_i \in \mathbb{C},\ v_p(f) > j_k - e_p\right\} =$$

$$= \dim_{\mathbb{C}}\left\{f = \sum \alpha_i f_i \mid \alpha_i \in \mathbb{C},\ v_p(f) \geq j_{k+1} - e_p\right\} =$$

$$= m - (k+1).$$

But, by assumption, this first space contains the $m-k$ linearly independent functions $f_m,\dots,f_{k-1}$. We have reached a contradiction and so the lemma is established.

We shall use constantly the following two trivial consequences of the lemma which are valid for any $p \in X$,

$$w_\phi(p) \geq \sum_{i=1}^{m}(v_p(f_i) + e_p) - \frac{m(m-1)}{2} \qquad (4)$$

$$w_\phi(p) \geq \sum_{i\in I}\left[(v_p(f_i) + e_p) - (m-1)\right] \text{ for any } I \subseteq \{1,\dots,m\} \qquad (5)$$

§3. Proof of the results

We start by proving Therem 2. Let X be as in §2, $x_1,\dots,x_m$ a solution of (1) satisfying the hypotheses of Theorem 2, and $\phi: X \to \mathbb{P}^{m-1}$ the morphism given by $(a_1x_1^n : \dots : a_mx_m^n)$ which is non-degenerate by hypothesis. The plan of the proof is first to find lower bounds for $w_\phi(p)$ for $p \in X$ and then deduce Theorem 2 from (3).

To find lower bounds for $w_\phi(p)$ assume first that $p \notin S$, and let $I_p \subseteq \{1,\dots,m\}$ be the set for which $v_p(x_i) > 0$ if and only if $i \in I_p$. It follows from (4) that

$$w_\phi(p) \geq \sum_{i\in I_p}(nv_p(x_i) + e_p - (m-1)) \geq$$

$$\geq \sum_{i\in I_p}(n-m+1)v_p(x_i) + |I_p|e_p = \sum_{i=1}^{m}(n-m+1)v_p(x_i) + |I_p|e_p.$$

Since $|I_p| \leq m$ we get

$$w_\phi(p) \geq (n-m+1)\sum_{i=1}^{m} v_p(x_i) + me_p. \qquad (6)$$

If $p \in S$, define $i(p)$ such that

$$v_p(a_{i(p)}x_{i(p)}^n) \leq v_p(a_ix_i^n), \qquad i = 1,\dots,m.$$

To bound $w_\phi(p)$ for $p \in S$ we make a change of coordinates in $\mathbb{P}^{m-1}$ such that ϕ is given by $(a_1x_1^n:\dots:b:\dots:a_mx_m^n)$

where b occurs in the $i(p)$-th place. From (4) it follows that (note that $e_p = -v_p(a_{i(p)}x^n_{i(p)})$)

$$w_\phi(p) \geq \sum_{i \neq i(p)} (v_p(a_i x_i^n) - v_p(a_{i(p)}x^n_{i(p)}) +$$

$$+ v_p(b) - v_p(a_{i(p)}x^n_{i(p)}) - \frac{m(m-1)}{2},$$

which we rewrite as

$$w_\phi(p) \geq \sum_{i=1}^m v_p(a_i x_i^n) - (m+1)v_p(a_{i(p)}x^n_{i(p)}) + v_p(b) - \frac{m(m-1)}{2}. \quad (7)$$

We now are going to substitute inequalities (6) and (7) into (3), but before let's notice that, by definition

$$\deg \phi = -\sum_{p \in S} v_p(a_{i(p)}x_{i(p)}) + \sum_{p \notin S} e_p.$$

We then get

$$(n-m+1) \sum_{p \notin S} v_p(x_i) + m \sum_{p \notin S} e_p +$$

$$+ \sum_{p \in S} \left\{ \sum_{i=1}^m v_p(a_i x_i^n) - (m+1)v_p(a_{i(p)}x^n_{i(p)}) + v_p(b) \right\}$$

$$- \frac{m(m-1)}{2}|S| \leq m(m-1)(g-1) + m \sum_{p \notin S} e_p - m \sum_{p \in S} v_p(a_{i(p)}x^n_{i(p)}).$$

This reduces to,

$$(n-m+1) \sum_{p \notin S} v_p(x_i) + \sum_{p \in S} (nv_p(x_i) + v_p(a_i)) - \sum_{p \in S} v_p(a_{i(p)}x^n_{i(p)}) + \sum_{p \in S} v_p(b) \leq$$

$$\leq \frac{m(m-1)}{2}((2g-2)+|S|)$$

Using now that $\sum_{p \in X} v_p(x_i) = 0$ and

$$-\sum_{p \in S} (v_p(a_1) + \ldots + v_p(a_m) + v_p(b)) \leq \deg a_1 + \ldots + \deg a_n + \deg b = H,$$

we obtain

$$-(m-1) \sum_{p \notin S} \sum_{i=1}^m v_p(x_i) - \sum_{p \in S} v_p(a_{i(p)}x^n_{i(p)}) \leq \frac{m(m-1)}{2}(2g-2+|S|) + H.$$

To complete the proof of Theorem 2 its suffices now to prove that

$$[n-m(m-1)] \max_i \deg x_i \leq$$

$$\leq -(m-1) \sum_{p \notin S} \sum_{i=1}^m v_p(x_i) - \sum_{p \in S} v_p(a_{i(p)}x^n_{i(p)}) + H. \quad (8)$$

To prove (8) let j be such that $\deg x_j \geq \deg x_i$ $i = 1,\ldots,m$. As $\sum_{p \notin S} v_p(x_i) \leq \deg x_i$ we have

$$\sum_{i=1}^m \sum_{p \notin S} v_p(x_i) \leq m \deg x_j. \quad (9)$$

By definition of $i(p)$ we have

$$-v_p(a_j x_j^n) \leq -v_p(a_{i(p)}x^n_{i(p)}).$$

Let S_1 be the subset of S where x_j has poles. Then

$$n \deg x_j = -\sum_{p \in S_1} v_p(x_j^n) = -\sum_{p \in S_1} v_p(a_j x_j^n) + \sum_{p \in S_1} v_p(a_j) \leq$$

$$\leq -\sum_{p \in S_1} v_p(a_{i(p)}x^n_{i(p)}) + \sum_{p \in S_1} v_p(a_j). \quad (10)$$

Let $S_2 = \{p \in S \mid v_p(a_{i(p)}x^n_{i(p)}) < 0\}$ and $S_3 = S - S_2$ then

$$\sum_{p \in S_1} -v_p(a_{i(p)}x^n_{i(p)}) \leq \sum_{p \in S_1 \cap S_2} -v_p(a_{i(p)}x^n_{i(p)}) \leq$$

$$\leq \sum_{p \in S_2} -v_p(a_{i(p)}x^n_{i(p)}). \quad (11)$$

If $p \in S_3$ it is clear that $v_p(a_{i(p)}x^n_{i(p)}) \le v_p(b)$, hence

$$\sum_{p\in S} - v_p(a_{i(p)}x^n_{i(p)}) \ge \sum_{p\in S_2} -v_p(a_{i(p)}x^n_{i(p)}) - \sum_{p\in S_3} v_p(b) \qquad (12)$$

So, by (10), (11) and (12)

$$n \deg x_j \le -\sum_{p\in S} v_p(a_{i(p)}x^n_{i(p)}) + \sum_{p\in S_3} v_p(b) + \sum_{p\in S_1} v_p(a_j) \le$$

$$\le -\sum_{p\in S} v_p(a_{i(p)}x^n_{i(p)}) + \deg b + \deg a_j \le$$

$$\le -\sum_{p\in S} v_p(a_{i(p)}x^n_{i(p)}) + H. \qquad (13)$$

Now, (8) follows from (9) and (13) so Theorem 2 is proved.

We now prove Theorem 1, by induction on m, the case $m=1$ being trivial.

Suppose $a,\dots,a_m,b$, are given and $n > m(m-1)$, suppose that $x_1,\dots,x_m$ is a solution of (1).

If $a_1x_1^n,\dots,a_mx_m^n$ are linearly dependent over C, we have (say) that $a_mx_m^n = \sum \alpha_i a_i x_i^n$, $\alpha_i \in C$, and so

$$\sum_{i=1}^{m-1} (1+\alpha_i)a_i x_i^n = b,$$

which is impossible by the induction hypothesis if n is sufficiently large.

If $a_1x_1^n,\dots,a_mx_m^n$ are linearly independent over $\mathbb{C}$, let be the minimal set os places of K for which $a_1,\dots,a_m$, $x_1,\dots,x_m,b$ are all S-integral. Then

$$|S| \le H + \sum_{i=1}^{m} \deg x_i \le H + m \deg x_j$$

if $\deg x_j \ge \deg x_i$, $i = ,\dots,m$.

To Theorem 2 gives

$$[n-m(m-1)]\deg x_j \le \frac{m^2(m-1)}{2}\deg x_j + m(m-1)(g-1) + \left[\frac{m(m-1)}{2} + 2\right]H. \qquad (14)$$

Hence if n is so large that

$$\frac{m(m-1)(g-1) + \left[\frac{m(m-1)}{2} + 2\right]H}{n - \frac{m(m-1)(m+2)}{2}} < 1,$$

$\deg x_j = 0$. So $\deg x_i = 0$ for $i = 1,\dots,n$, and $x_i \in \mathbb{C}$ for $i = 1,\dots,m$, which is impossible by hypothesis.

We now prove Corollary 3. In the case $n \ge 7$, let x_1,x_2,x_3 be an S-integral solution of (1) If $a_1x_1^n$, $a_2x_2^n$, $a_3x_3^n$ are linearly independent over $\mathbb{C}$, the result follows from Theorem 2. So we may assume that $a_1x_1^n$, $a_2x_2^n$, $a_3x_3^n$ are linearly dependent over $\mathbb{C}$. We claim that two among $a_1x_1^n$, $a_2x_2^n$, $a_3x_3^n$ are linearly independent. For, otherwise, we have that $a_2x_2^n = \alpha\, a_1x_1^n$, $a_3x_3^n = \beta\, a_1x_1^n$, say. If $\alpha \ne 0$, a_2/a_1 is an n-th power, which contradicts the hipothesis, so $\alpha = 0$. Similarly, $\beta = 0$. But then, $a_1x_1^n = b$ so a_1/b is an n-th power, which again contradicts the hypothesis and proves the claim.

We may then assume that $a_1x_1^n$, $a_2x_2^n$ are linearly independent over C and

$$a_3x_3^n = \alpha a_1x_1^n + \beta a_2x_2^n, \quad \alpha,\beta \in C \qquad (15)$$

then

$$(1+\alpha)a_1x_1^n + (1+\beta)a_2x_2^n = b \qquad (16)$$

If $(1+\alpha)(1+\beta) \ne 0$, we can bound $\deg x_1$, $\deg x_2$ from Theorem 2 applied to (16) and so bound $\deg x_3$ from (15). The first part of Corollary 3 will be proved if we show that $(1+\alpha)(1+\beta) \ne 0$. But, if $1+\alpha = 0$, say, then $1+\beta \ne 0$, since $b \ne 0$; so it follows from (16) that $b/a_2 = (1+\beta)x_2^n$ is an n-th power, which contradicts the hypothesis and shows that $(1+\alpha)(1+\beta)\ne 0$ as desired.

The proof of the second part is similar. One has to use the proof of Theorem 2, especially inequality (14).

We now prove Theorem 4.

We consider $\phi: X \to \mathbb{P}^{m-1}$ given by $(u_1:\ldots:u_m)$ and estimate $w_\phi(p)$ for $p \in S$. Given p, let $i(p)$ be such that $v_p(u_{i(p)}) \le v_p(u_i)$. Changing coordinates of $\mathbb{P}^{m-1}$ we may assume that ϕ is given by $(u_1:\ldots:1:\ldots:u_n)$ with 1 in the $i(p)$-th place. Then by (4)

$$w_\phi(p) \ge \sum_{i \ne i(p)} [v_p(u_i) - v_p(u_{i(p)})] - v_p(u_{i(p)}) - \frac{m(m-1)}{2} =$$

$$= \sum_{i=1}^{m} v_p(u_i) - (m+1) v_p(u_{i(p)}) - \frac{m(m-1)}{2}.$$

Hence, by (3)

$$\sum_{p \in S} \sum_{i=1}^{m} v_p(u_i) - (m+1) \sum_{p \in S} v_p(u_{i(p)}) - \frac{m(m-1)}{2} |S| \le$$

$$\le m(m-1)(g-1) - m \sum_{p \in S} v_p(u_{i(p)}).$$

As $\sum_{p \in S} v_p(u_i) = \sum_{p \in X} v_p(u_i) = 0$, we get

$$-\sum_{p \in S} v_p(u_{i(p)}) \le \frac{m(m-1)}{2} (2g-2+|S|) \qquad (17)$$

define

$$S_1 = \{p \in S,\ v_p(u_i) < 0\},$$ we then have

$$\deg u_i = \sum_{p \in S_1} -v_p(u_i) \le \sum_{p \in S_1} -v_p(u_{i(p)}). \qquad (18)$$

On the other hand if $v_p(u_{i(p)}) > 0$ then $v_p(u_j) > 0$ for $j = 1, \ldots, m$; so, $v_p(\sum u_i) > 0$. But, as $\sum u_i = 1$, this is absurd. So $v_p(u_{i(p)}) \le 0$ for all p, and we conclude that

$$-\sum_{p \in S_1} v_p(u_{i(p)}) \le -\sum_{p \in S} v_p(u_{i(p)})$$

and this inequality together with (17) and (18) give Theorem 4.

Remark: Theorem 4 has applications to several equations over function fields like norm form equations and those considered by Vojta ([V]), i.e., those equations which define a variety whose divisor at infinity has many irreducible components.

The methods of this paper apply also to equation like $\sum a_i x_i^{n_i} = b$ and some other equations $f(X_1, \ldots, X_m) = b$ where f has "few" monomials.

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