ON A VARIATIONAL INEQUALITY FOR A NONLINEAR OPERATOR OF HYPERBOLIC TYPE

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Introduction

The study of variational inequalities was initiated by Stampacchia [8], Lions-Stampacchia [6], Brezis [2], Browder [1], cfr. also Kinderlehrer-Stampacchia [4]. In Lions ([5], chapter 3), we can find the same type of problem for a non linear operator of hyperbolic type. In this note we study a similar problem for the operator:

$$(*)\qquad \frac{\partial^2 u}{\partial t^2} - (1+M(|u|_0,|u|_1,|u|_2))\Delta u - f(x,t).$$

The plan of this paper is the following

1. Notation and the main result.
2. Proof of the theorem.

1. Notation and the Main Result

We denote by (w_ν) the sequence of eigenfunctions of $-\Delta$ considered on the space $H_0^1(\Omega) \cap H^2(\Omega)$, Ω is a bounded open set of $\mathbb{R}^n$ with regular boundary Γ. We represent by V_k, $k=0,1,2,\ldots$ the domain of the operator $(-\Delta)^{k/2}$ with the inner product and norm:

$$(u,v)_k = ((-\Delta)^{k/2}u,\ (-\Delta)^{k/2}v)_{L^2(\Omega)},\qquad |u|_k^2 = (u,u)_k$$

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The constraint $D \subset V_2$ will be a bounded set, not necessarily a closed convex set containing zero. Local solutions when the coeficient of $-\Delta$ in (*) is $M(|u|_1)$ is studied in Medeiros--Miranda [7] when D is a closed convex set containing zero, and $M(\eta) \geq m_0 > 0$; that is, the non degenerated case. In the present paper we suppose $M(\xi,\eta,\zeta) \geq 0$ and we use the penalty method suggested by Ebihara [3]. For the particular case $k = 0$, we write:

$$(u,v)_0 = (u,v), \quad |u|_0^2 = |u|;$$

the inner product and norm in $L^2(\Omega)$.

We assume that the following spaces are known: $L^p(t_1,t_2;H)$, $C^\ell(t_1,t_2;H)$, $L^p_{\text{loc}}(0,\infty;H)$, $C^\ell(0,\infty;H)$, where t_1, t_2 are real numbers and H is a Hilbert space.

In order to establish the main result in this paper, we assume:

(1.1) $M(\xi,\eta,\zeta) \in C^1(\mathbb{R}^3_+)$ where $\mathbb{R}_+ = [0,\infty[$, and

$M(\xi,\eta,\zeta) \geq 0$ on $\mathbb{R}^3_+$.

Let $F(\xi)$, $0 < \xi < +\infty$, be a real function satisfying the following conditions:

(1.2) $F(\xi) \in C^1(0,\infty)$,

(1.3) there exist real numbers $\alpha > 0$, $\beta \geq 1$ and $\delta > 0$ such that

$$F(\xi) \geq (\alpha/\xi^\beta) \quad \text{for all} \quad \xi \in (0,\delta],$$

(1.4) $F'(\xi) \leq 0$ for all $\xi > 0$,

(1.5) $F(\xi) = 1$ for all $\xi \geq 1$.

Theorem 1. Let $T > 0$ be arbitrarily given number. Suppose

(1.6) $(u_0,u_1) \in V_5 \times V_4$,

(1.7) $f \in L^2_{\text{loc}}(0,\infty;V_4)$, $df/dt \in L^2_{\text{loc}}(0,\infty;L^2(\Omega))$ and $M(\xi,\eta,\zeta)$ satisfies the condition (1.1).

Then, there exists only one function $u\colon [0,\infty[\to L^2(\Omega)$, satisfying the conditions:

(1.8) $u \in L^\infty_{\text{loc}}(0,\infty;V_5)$,

(1.9) $u' \in L^\infty_{\text{loc}}(0,\infty;V_4) \cap C^0(0,\infty;V_2)$,

(1.10) $u'' \in L^\infty_{\text{loc}}(0,\infty;L^2(\Omega))$,

such that

(1.11) $$\int_0^T (u''-\{1+M(|u|,|u|_1,|u|_2)\}\Delta u - f, v-u')dt \geq 0$$

for all v in D, where D is a bounded set in V_2,

(1.12) $u(0) = u_0$, $u'(0) = u_1$.

2. Proof of the Theorem 1

We shall use in the proof the penalization method of Ebihara [3], and Galerkin approximations. In fact, let us consider a real number $K > 0$, such that

(2.1) $K > |u_1|_2^2$.

With this hypothesis, for each $\varepsilon > 0$ we define the penalized problem associated to (*), by:

(2.2) $$\begin{cases} u''-\{1+M(|u_\varepsilon|,|u_\varepsilon|_1,|u_\varepsilon|_2)\}\Delta u_\varepsilon + \varepsilon F\left(\dfrac{K-|u'_\varepsilon|_2^2}{\varepsilon}\right)\Delta^2 u'_\varepsilon = f, \\ u_\varepsilon = 0 \text{ on } \Gamma, \\ u_\varepsilon(x,0) = u_0(x), \quad u'_\varepsilon(x,0) = u_1(x). \end{cases}$$

In the next step we shall define the approximated problem associated to (2.2). It consists in finding a function

$$u_{\varepsilon m}(t) = \sum_{\nu=1}^{m} g_{\varepsilon m\nu}(t) w_\nu$$

defined by:

$$(2.3) \qquad (u''_{\varepsilon m} - \{1+M(|u_{\varepsilon m}|,|u_{\varepsilon m}|_1,|u_{\varepsilon m}|_2)\}\Delta u_{\varepsilon m} + \varepsilon F\left(\frac{K-|u'_{\varepsilon m}|_2^2}{\varepsilon}\right)\Delta^2 u'_{\varepsilon m} - f, v) = 0,$$

for each $v \in [w_1, w_2, \dots, w_m]$, the subspace of dimension m generated by the m first eigenvectors (w_ν),

$$(2.4) \qquad u_{\varepsilon m}(0) = u_{om}, \quad u_{om} \to u_0 \text{ in } V_5,$$

$$(2.5) \qquad u'_{\varepsilon m}(0) = u_{1m}, \quad u_{1m} \to u_1 \text{ in } V_4.$$

Note that u_{om}, u_{1m} belong to $[w_1, w_2, \dots, w_m]$. We know that $u_{\varepsilon m}(t)$ is defined in some interval $[0, \delta_{\varepsilon m}]$, $\delta_{\varepsilon m} > 0$.

We shall obtain a priori estimates for the approximated solutions $u_{\varepsilon m}$ that permit to pass to the limits in (2.3) in order to get a solution for the penalized problem (2.2), that will be the solution which we are looking for.

The proof of the Theorem 1 shall be divided in the following eight lemmas.

Lemma 1. We have

$$|u'_{\varepsilon m}(t)|_2^2 < K$$

for all $t > 0$, $\varepsilon > 0$ and $m \geq 1$.

Proof. Let us fix ε and m. In order to work with an easy notation, we omit ε, m in $u_{\varepsilon m}$. The proof shall be done by contradiction. In fact, suppose there exists $0 < t^* \leq \delta_{\varepsilon m}$ such that:

$$(2.6) \qquad |u'(t)|_2^2 < K, \quad \text{for } 0 \leq t < t^* \quad \text{and} \quad |u'(t^*)|_2^2 = K.$$

Taking $v = u''$ in the equation (2.3) and integrating from t_1 to t, $0 \leq t_1 < t \leq t^*$, we obtain:

$$(2.7) \qquad \int_{t_1}^{t} \varepsilon F\left(\frac{K-|u'(\tau)|_2^2}{\varepsilon}\right)(u'(\tau), u''(\tau))_2 d\tau = \int_{t_1}^{t} (f(\tau), u''(\tau)) d\tau + \int_{t_1}^{t} \{1+M(|u(\tau)|,|u(\tau)|_1,|u(\tau)|_2)\}(\Delta u(\tau), u''(\tau)) d\tau - \int_{t_1}^{t} |u''(\tau)|^2 d\tau.$$

We observe that the right side of (2.7) is bounded by a constant $c(t^*)$, which depends on ε and m, but is independent of t and t. We then obtain:

$$(2.8) \qquad \varepsilon \int_{t_1}^{t} F\left(\frac{K-|u'(\tau)|_2^2}{\varepsilon}\right)(|u'(\tau)|_2^2)' d\tau < 2c(t^*),$$

for all $0 \leq t_1 < t \leq t^*$.

By the change of variable

$$\xi(\tau) = \frac{K-|u'(\tau)|_2^2}{\varepsilon}$$

in the integral (2.8), choosing t_1 such that $\xi(t_1) \leq \delta$ (δ as in the (1.3) of the definition of $F(\xi)$) and using (1.3) we obtain:

$$\alpha\varepsilon^2 \int_{\xi(t)}^{\xi(t_1)} \xi^{-\beta} d\xi \leq \varepsilon^2 \int_{\xi(t)}^{\xi(t_1)} F(\xi) d\xi \leq 2c(t^*)$$

for all $0 \leq t_1 < t < t^*$. It follows that we get a contradiction, because by (2.6) $\xi(t) \to 0$ when $t \to t^*$, which implies the divergence of the integral of $\xi^{-\beta}$ on $(\xi(t),\xi(t_1))$, and it cannot be bounded by $c(t^*)$. This contradiction proves the Lemma 1.

It follows from the Lemma 1 that $u_{\varepsilon m}(t)$ is defined on all the semi line $[0,\infty)$. By the Lemma 1 and the fundamental theorem of the calculus it follows that:

$$|u_{\varepsilon m}(t)|_2 \le c(T) \tag{2.9}$$

for all $t \in [0,T]$, $\varepsilon > 0$ and $m \ge 1$.

Lemma 2. We have

$$|u'_{\varepsilon m}(t)|_4^2 + |u_{\varepsilon m}(t)|_5^2 \le C(T)$$

for all t in $[0,T]$, $\varepsilon > 0$ and $m \ge m_0$.

Proof. Omitting ε, m in $u_{\varepsilon m}$ and doing $v = 2(-\Delta)^4 u'$ in the approximated equation (2.3), we obtain:

$$(|u'(t)|_4^2 + |u(t)|_5^2 + M(|u(t)|,|u(t)|_1,|u(t)|_2)|u(t)|_5^2)' \le$$

$$\le |f(t)|_4^2 + |u'(t)|_4^2 + [\tfrac{d}{dt} M(|u(t)|,|u(t)|_1,|u(t)|_2)]|u(t)|_5^2.$$

It follows from Lemma 1 and (2.9), that

$$|\tfrac{d}{dt} M(|u(t)|,|u(t)|_1,|u(t)|_2)| \le C(T).$$

By the Gronwall inequality the proof of the Lemma 2 follows.

Lemma 3. We have

$$|u''_{\varepsilon m}(t)| \le C(T)$$

for all t in $[0,T]$, $\varepsilon > 0$ and $m \ge m_0$.

Proof. In order to obtain an estimate for $u''_{\varepsilon m}$, we first find an estimate for $u''_{\varepsilon m}(0)$. In fact, taking $t = 0$ in the approximated equation (2.3) and doing $v = u''(0)$, applying Cauchy-Schwarz inequality, we obtain:

$$|u''(0)| \le \{1+M(|u(0)|,|u(0)|_1,|u(0)|_2)\}|\Delta u(0)| + \tag{2.10}$$

$$+ \varepsilon F(\frac{K-|u'(0)|_2^2}{\varepsilon})|\Delta^2 u'(0)| + |f(0)|.$$

By the conditions (2.4), (2.5) on the initial data, it follows that the right side of (2.10) is bounded by a constant C independent of ε and $m \ge m_0$. We then have:

$$|u''_{\varepsilon m}(0)| \le C \text{ for all } \varepsilon > 0 \text{ and } m \ge m_0. \tag{2.11}$$

Differentiating the approximated equation (2.3) and taking $v = 2u''$, we have:

$$(|u''|^2)' + 2[\tfrac{d}{dt} M(|u|,|u|_1,|u|_2)](-\Delta u,u'') + \tag{2.12}$$

$$+ 2(1+M(|u|,|u|_1,|u|_2))(-\Delta u',u'') - 2(f',u'') +$$

$$- 4F'(\frac{K-|u'|_2^2}{\varepsilon})[(u',u'')_2]^2 + 2\varepsilon F(\frac{K-|u'|_2^2}{\varepsilon})|u''|_2^2 = 0.$$

Noting that the last two terms of (2.12) are non negative, by Lemma 1 and the estimate (2.9) we obtain:

$$(|u''(t)|^2)' \le C_1(T) + C_2|f'(t)|^2 + |u''(t)|^2,$$

whence by the estimate (2.11) and Gronwall inequality the proof of the Lemma 3 follows.

Lemma 4. We have:

$$|u'_{\varepsilon m}(t_1)-u'_{\varepsilon m}(t_2)|_2 \leq C(T)|t_1-t_2|$$

for all $0 \leq t_1 < t_2 \leq T$, $\varepsilon > 0$ and $m \geq m_0$.

Proof: We have

$$|u'(t_1)-u'(t_2)|_2^2 = \int_{t_1}^{t_2} (u''(\tau),u'(t_1)-u'(t_2))_2 d\tau \leq$$

$$\leq \int_{t_1}^{t_2} |u''(\tau)||u'(t_1)-u'_1(t_2)|_4 d\tau.$$

By the Lemmas 2 and 3, the proof is done.

Lemma 5. The following estimate holds:

$$\varepsilon F(\frac{K-|u'_{\varepsilon m}(t)|_2^2}{\varepsilon}) \leq C(T)$$

for all t in $[0,T]$, $\varepsilon > 0$ and $m \geq m_0$.

Proof. Taking $v = u'$ in (2.3) we have:

$$(2.13)\qquad F(\frac{K-|u'|_2^2}{\varepsilon})|u'|_2^2 = (f+\{1+M(|u|,|u|_1,|u|_2)\}\Delta u-u'',u').$$

If $|u'(t_0)|_2^2 \leq K/2$, then by the properties (1.4), (1.5) of $F(\xi)$, we have for sufficiently small $\varepsilon > 0$:

$$\varepsilon F(\frac{K-|u'(t_0)|_2^2}{\varepsilon}) = \varepsilon.$$

If $|u'(t_0)|_2^2 > K/2$, then by (2.13), the Lemma 1, (2.9) and Lemma 3, we have:

$$\varepsilon F(\frac{K-|u'(t_0)|_2^2}{\varepsilon}) \leq$$

$$\leq \frac{2}{K} \Big[|f||u'| + \{1+M(|u|,|u|_1,|u|_2)\}|u|_2|u'|+|u''||u'|\Big]_{t=t_0} \leq C(T).$$

Thus, Lemma 5 is proved.

From Lemmas 2, 3 and 5 we can extract a subsequence of $(u_{\varepsilon m})$, still denoted by $(u_{\varepsilon m})$, and a function $u(x,t)$ satisfying the conditions:

$$(2.14)\qquad u_{\varepsilon m} \to u \text{ weak star in } L^\infty(0,T;V_5),$$

$$(2.15)\qquad u'_{\varepsilon m} \to u' \text{ weak star in } L^\infty(0,T;V_4),$$

$$(2.16)\qquad u''_{\varepsilon m} \to u'' \text{ weak star in } L^\infty(0,T;L^2(\Omega)),$$

$$(2.17)\qquad \varepsilon F(\frac{K-|u'_{\varepsilon m}|^2}{\varepsilon}) \to X \text{ weak star in } L^\infty(0,T).$$

As the embedding of V_5 in V_2 is compact, it follows from (2.14), (2.15) and Aubin-Lions Theorem that

$$u_{\varepsilon m} \to u \text{ strongly in } L^2(0,T;V_2),$$

whence

$$(2.18)\qquad M(|u_{\varepsilon m}|,|u_{\varepsilon m}|_1,|u_{\varepsilon m}|_2) \to M(|u|,|u|_1,|u|_2) \text{ strongly in } L^2(0,T).$$

As the embedding of V_4 in V_2 is compact, by (2.15), (2.16) we obtain:

$$(2.19)\qquad u'_{\varepsilon m} \to u' \text{ strongly in } L^2(0,T;V_2).$$

By the compactness of the embedding of V_4 in V_2 and Lemma 2, we have that

$u'_{\varepsilon m}(t)$ is relatively compact in V_2 for each t in $[0,T]$, and by the Lemma 4,

$u'_{\varepsilon m}$ is equicontinuous on $[0,T]$ with values in V_2.

Therefore, by Arzela-Ascoli theorem,

$$u'_{\varepsilon m} \to u' \quad \text{in} \quad C^0([0,T];V_2). \tag{2.20}$$

By (2.14) - (2.19) it is permissible to pass to the limits in the approximated equation (2.3), obtaining that

$$u'' - \{1+M(|u|,|u|_1,|u|_2)\}\Delta u + \chi\Delta^2 u' - f = 0 \tag{2.21}$$

in the distributional sense on $Q \equiv (0,T)\times\Omega$, and by (2.14), (2.16) we have that

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.22}$$

Lemma 6. If $|u(t_0)|_2^2 < K$, $t_0 \in]0,T[$, there exists some interval $]t_0-\rho, t_0+\rho[$ where $\chi(t) = 0$ almost everywhere.

Proof: We use the following notation,

$$a = \frac{K+|u'(t_0)|_2^2}{2}, \qquad b = \frac{K-|u'(t_0)|_2^2}{4} > 0.$$

Then there exists some interval $I(t_0) =]t_0-\rho, t_0+\rho[$ such that

$$|u'(t)|_2^2 < a \quad \text{for all} \quad t \in I(t_0).$$

By (2.20) it follows that $|u'_{\varepsilon m}|_2^2$ converges to $|u'|_2^2$ uniformly in $[0,T]$, therefore,

$$\left| |u'_{\varepsilon m}(t)|_2^2 - |u'(t)|_2^2 \right| \le b$$

for all $t \in I(t_0)$ and $\varepsilon < \varepsilon_0$, $m \ge m_0$. Thus, for $t \in I(t_0)$, we have that

$$|u'_{\varepsilon m}(t)|_2^2 \le |u'(t)|_2^2 + b < a+b < K-b$$

or

$$K - |u'_{\varepsilon m}(t)|_2^2 > b > 0 \quad \text{for all} \quad t \in V(t_0),$$

and $\varepsilon < \varepsilon_0$, $m \ge m_0$. Therefore, for ε small enough, we have that

$$\varepsilon F\left(\frac{K-|u'_{\varepsilon m}|_2^2}{\varepsilon}\right) = \varepsilon \quad \text{in} \quad I(t_0),$$

which implies the proof of the Lemma 6.

As a consequence of the Lemma 6, we obtain that if U is an open set of points $t \in]0,T[$ where $|u'(t)|_2^2 < K$, then

$$\chi(t) = 0 \quad \text{a.e. for} \quad t \in U. \tag{2.23}$$

Lemma 7. If we choose $K > 0$ large enough such that

$$K > \max\{\sup_{v\in D} |v|_2^2, \ |u_1|_2^2\},$$

then the function $u(t)$ constructed above satisfies the inequality (1.11).

Proof: We have from (2.21):

$$\int_0^T (u'' - \{1+M(|u|,|u_1|,|u_2|)\}\Delta u - f, \ v-u')dt$$

$$= \int_0^T \chi(u', \ u'-v)_2\, dt$$

$$\geq \int_0^T \chi\{|u'|_2^2 - |u'|_2|v|_2\}dt$$

for every $v \in \bar{D}_K \equiv \{v \in V_2;\ |v|_2^2 \leqq K\}$. We observe that

(2.24) if $|u'(t_0)|_2^2 = K$ then $|u'(t_0)|_2^2 - |u'(t_0)|_2 |v|_2 \geqq 0$.

Thus, from (2.23) and (2.24), it follows that the last integral is nonnegative, and then (1.11) holds for $v \in D \subset \bar{D}_K$. Therefore the part of existence of Theorem 1 is proved.

We finally have the uniqueness:

Lemma 8. The function $u(t)$ is a unique solution of (1.11) in this class in which $u(t)$ belongs.

Proof: If we have another function $\bar{u}(t)$ which satisfies (1.11) and belongs to the same class of $u(t)$, setting for each $0 < t \leqq T$

$$v_1(s) = \begin{cases} \bar{u}'(s) & 0 < s < t \leqq T \\ u'(s) & t < s < T, \end{cases}$$

$$v_2(s) = \begin{cases} u'(s) & 0 < s < t \leqq T \\ \bar{u}'(s) & t \leqq s \leqq T, \end{cases}$$

we know that $v_1(s) \in \bar{D}_K$, $v_2(s) \in \bar{D}_K$, where $\bar{D}_K = \{v \in V_2;\ |v|_2 \leqq K\}$. Therefore we have

$$\int_0^t (u'' - \{1+M(|u|,|u|_1,|u|_2)\}\Delta u - f,\ \bar{u}' - u')\,ds \geqq 0,$$

$$\int_0^t (\bar{u}'' - \{1+M(|\bar{u}|,|\bar{u}|_1,|\bar{u}|_2)\}\Delta \bar{u} - f,\ u' - \bar{u}')\,ds \geqq 0.$$

Then, $w(t) \equiv u(t) - \bar{u}(t)$ satisfies $w(0) = 0$, $w'(0) = 0$ and

$$\int_0^t (w'' - \Delta w - M(|u|,|u|_1,|u|_2)\Delta u + M(|\bar{u}|,|\bar{u}|_1,|\bar{u}|_2)\Delta \bar{u},\ w')\,ds \leqq 0.$$

Therefore,

$$|w'|^2 + |w|_1^2 + \int_0^t M(|u|,|u|_1,|u|_2)(|w|_1^2)'\,ds$$

$$+ 2\int_0^t \{M(|u|,|u|_1,|u|_2) - M(|\bar{u}|,|\bar{u}|_1,|\bar{u}|_2)\}(-\Delta \bar{u}, w')\,ds \leqq 0;$$

then

$$|w'|^2 + |w|_1^2 + M(|u|,|u|_1,|u|_2)|w|_1^2$$

$$\leqq \int_0^t \left|\frac{d}{ds} M(|u|,|u|_1,|u|_2)\right| |w|_1^2\,ds$$

$$+ 2\int_0^t |M(|u|,|u|_1,|u|_2) - M(|\bar{u}|,|\bar{u}|_1,|\bar{u}|_2)|\ |\bar{u}|_2 |w'|\,ds.$$

Here we know from our assumptions that

$$\max_{t \in [0,T]} \left|\frac{d}{dt} M(|u|,|u|_1,|u|_2)\right| \leqq C(T),$$

and

$$|M(|u|,|u|_1,|u|_2) - M(|\bar{u}|,|\bar{u}|_1,|\bar{u}|_2)|$$

$$\leqq C(T)\{|w| + |w|_1 + ||u|_2 - |\bar{u}|_2|\},$$

and

$$||u|_2 - |\bar{u}|_2| = \frac{||u|_2^2 - |\bar{u}|_2^2|}{|u|_2 + |\bar{u}|_2} = \frac{||w|_2^2 + 2(w,\bar{u})_2|}{|u|_2 + |\bar{u}|_2}$$

$$\leqq \frac{|w|_3|w|_1 + 2|\bar{u}|_3|w|_1}{|u|_2 + |\bar{u}|_2} \leqq C(T)\frac{|w|_1}{|u|_2 + |\bar{u}|_2};$$

Thus, we have

$$|w'|^2 + |w|_1^2 + M(|u|,|u|_1,|u|_2)|w|_1^2$$

$$\leq C(T)\int_0^t |w|_1^2 ds + C(T)\int_0^t \{|w| + |w|_1 + \frac{|w|_1}{|u|_2+|\bar{u}|_2}\}|\bar{u}|_2|w'|ds$$

$$\leq C(T)\int_0^t \{|w'|^2+|w|_1^2+M(|u|,|u|_1,|u|_2)|w|_1^2 ds.$$

This implies $w'(t) \equiv 0$, $w(t) \equiv 0$, $t \in [0,T]$. Consequently, we have completed the proof of Theorem 1.

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