SOME REMARKS ON MINIMAL IMMERSIONS IN HYPERBOLIC SPACES

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1. Introduction

Throughout the paper differentiable means C^∞, M is an m-dimensional connected and oriented manifold, and Q is the set $\{(x,t) \in \mathbb{R}^n \mid x \in \mathbb{R}^{n-1},\ t > 0\}$ endowed with its usual differentiable structure. Then $\mathbb{R}^n_+$ is the pair $(Q, <\ ,\ >)$, where $<\ ,\ >$ is the usual flat riemannian metric of $\mathbb{R}^n$ and H^n is the pair $(Q, (\,,))$, where $(\,,)$ is the riemannian metric on Q given by $(\,,)_{(x,t)} = \frac{1}{t^2} <,>$. It is well known that H^n is a model for the n-dimensional hyperbolic space having constant sectional curvature and equal to -1. Given an immersion of M into Q, the metrics $<\ ,\ >$ and $(\,,)$ induce on M metrics that are conformal with each other. Comparing the geometric entities induced on M by these two metrics we establish formulas that allow us to get properties that an isometric minimal immersion of M into H^n has to have.

Let ϕ and $\tilde{\phi}$ be the above immersion accordingly we consider it as an isometric immersion into $\mathbb{R}^n_+$ or H^n. Then assuming that $\tilde{\phi} = (x,t)$ is minimal we show that t has to be a superharmonic function with respect to $\tilde{\phi}^*(\,,)$ and $\phi^* <,>$. As a consequence there is no isometric minimal immersion of M into H^n if either M is compact without boundary or if $m=2$ and it is complete and parabolic. Under the hypothesis that the asymptotic boundary of $\tilde{\phi}(M)$ in H^{m+1} omits a point of the ideal boundary of H^{m+1}, we show that there is no complete isometric minimal immersion of M^m having either at all points at least one sectional

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curvature less than $-\frac{m}{4}(m+4)$, or having at all points scalar curvature less than-2.

2. **Basic facts** - If V is an open set of Q we indicate by $\chi(V)$ the set of differentiable tangent vector fields defined on V. Let U_n be the tangent vector field of Q given by $U_n(x,t) = (0,0,\dots,0,1)$ $(\forall\ (x,t) \in Q)$.

Proposition 1. - If ∇ and $\tilde{\nabla}$ are the Levi-Civita connections with respect to $\langle\ ,\ \rangle$ and $(\ ,\ )$ and if V is an open subset of Q we have

$$(2.1)\qquad \tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{t}\langle X,U_n\rangle Y - \frac{1}{t}\langle Y,U_n\rangle X + \frac{1}{t}\langle X,Y\rangle U_n, \quad (\forall\ X,Y \in \chi(V))$$

Proof - Let (e_A), $A = 1,2,\dots,n$, be a local orthonormal referential of $\mathbb{R}^n_+$. Then $(\tilde{e}_A)$, given by $\tilde{e}_A(x,t) = t\,e_A(x,t)$ is a local orthonormal referential of H^n. Let (θ^A) and $(\tilde{\theta}^A)$, $A = 1,2,\dots,n$, be respectively the dual referentials of (e_A) and $(\tilde{e}_A)$. If (θ^A_B) and $(\tilde{\theta}^A_B)$, $A,B = 1,\dots,n$, are the connection 1-forms of (e_A) and $(\tilde{e}_A)$ with respect to ∇ and $\tilde{\nabla}$, from the first structural equations we obtain

$$\sum_B \left[\tilde{\theta}^A_B - \theta^A_B + \frac{1}{t}\langle U_n, e_B\rangle \theta^A\right] \wedge \theta^B = 0.$$

Thus the Cartan's lemma [W] implies

$$\tilde{\theta}^A_B - \theta^A_B + \frac{1}{t}\langle U_n, e_B\rangle\theta^A = \sum_C \alpha^A_{BC}\theta^C, \quad \text{with} \quad \alpha^A_{BC} = \alpha^A_{CB} \quad (\forall\ A,B,C).$$

The antisymmetry of $\tilde{\theta}^A_B$ and θ^A_B implies

$$\alpha^A_{BC} = 0, \quad \text{if } C \neq B, \quad \text{and} \quad \alpha^A_{BB} = \frac{1}{t}\langle U_n, e_A\rangle \quad (\forall\ A,B).$$

Then

$$\tilde{\theta}^A_B = \theta^A_B + \frac{1}{t}\{\langle U_n, e_A\rangle\theta^B - \langle U_n, e_B\rangle\theta^A\}.$$

Therefore from

$$\nabla e_A = \sum_B \theta^B_A e_B \quad \text{and} \quad \tilde{\nabla}\tilde{e}_A = \sum_B \tilde{\theta}^B_A \tilde{e}_A$$

we have (2.1).

If $\tilde{A}$ and A, $\tilde{B}$ and B are respectively the Weingarten operators and the second fundamental forms of $\tilde{\phi}$ and ϕ we have

$$\tilde{A}^{\tilde{\nu}}(X) = -(\tilde{\nabla}_{\tilde{\phi}_*(X)}\tilde{\nu})^T, \qquad A^{\nu}(X) = -(\nabla_{\phi_*(X)}\nu)^T$$

$$\tilde{B}(X,Y) = (\tilde{\nabla}_{\tilde{\phi}_*(X)}\phi_*(Y))^N, \quad B(X,Y) = (\nabla_{\phi_*(X)}\phi_*(Y))^N$$

where $\tilde{\nu}$ and $\nu = \frac{1}{t}\tilde{\nu}$ are respectively sections of the normal bundle of $\tilde{\phi}$ and ϕ, and where $X,Y \in \chi(M)$, and where $(\ )^N$ and $(\ )^T$ indicate respectively the orthogonal projections into the normal and tangent bundles of vector fields along $\tilde{\phi}$ and ϕ.

Then if $(\tilde{e}_i)$ and (e_i) are respectively local orthonormal sections of the tangent bundles of $\tilde{\phi}$ and ϕ, and $\tilde{H}$ and H are their respective mean curvature vectors, we have $\tilde{H} = \frac{1}{m}\sum_i \tilde{B}(\tilde{e}_i,\tilde{e}_i)$ and $H = \frac{1}{m}\sum_i B(e_i,e_i)$.

Proposition 2 - If $\tilde{\nu}$ is a section of the normal bundle of $\tilde{\phi}$, $\nu = \frac{1}{t}\tilde{\nu}$, $X,Y \in \chi(M)$ we have

$$(2.2)\qquad \tilde{A}^{\tilde{\nu}}(X) = t\,A^{\nu}(X) + \langle U_n,\nu\rangle\,\phi_*(X);$$

$$(2.3)\qquad \tilde{B}(X,Y) = B(X,Y) + \frac{1}{t}\langle\phi_*(X),\phi_*(Y)\rangle(U_n)^N;$$

$$(2.4)\qquad \tilde{H} = t^2 H + t(U_n)^N;$$

$$(2.5)\qquad \tilde{\phi} \text{ is minimal if, and only if, } H = -\frac{1}{t}(U_n)^N$$

Proof: - It is an easy consequence from (2.1) and of the definitions.

Proposition 3 - Let $\text{grad}_{\tilde{M}}$, $\text{div}_{\tilde{M}}$, $\Delta_{\tilde{M}}$ and grad_M, div_M, Δ_M be respectively the gradient, the divergent and the Laplace-Beltrami operators of M with respect to the riemannian metrics $\tilde{\phi}^*(\ ,\ )$ and $\phi^*\langle,\rangle$. If $f \in C^\infty(M)$ and $X \in \chi(M)$ we have

(2.6) $\mathrm{grad}_{\tilde M} f = t^2\, \mathrm{grad}_M f;$

(2.7) $\mathrm{div}_{\tilde M} X = \mathrm{div}_M X - \frac{m}{t}\langle U_n, \phi_*(X)\rangle$

(2.8) $\Delta_{\tilde M} f = t^2 \Delta_M f - (m-2)t\ \langle U_n, \phi_*\ (\mathrm{grad}_M f)\rangle$

Proof: - The formulas are consequence of

$$(\tilde\phi_*(\mathrm{grad}_M f),\ \tilde\phi_*(X)) = df(X) = \langle \phi_*(\mathrm{grad}_M f), \phi_*(X)\rangle.$$

Corollary - If $U_1 = (1,0,\dots,0),\dots,U_n = (0,0,\dots,0,1)$ and $\tilde\phi = (x^1,\dots,x^{n-1},t)$,

(2.9) $\Delta_{\tilde M} x^A = t^2 m\langle H, U_A\rangle - (m-2)t\langle U_n, (U_A)^T\rangle, \quad A = 1,\dots,n-1;$

(2.10) $\Delta_{\tilde M} t = t^2 m\langle H, U_n\rangle - (m-2)t\langle (U_n)^T, (U_n)^T\rangle.$

Proof: - It is a consequence of the well known formula

(2.11) $\Delta_M \phi = mH$

Proposition 4: - Let $(\tilde e_i)$, $1 \leq i \leq m$ be a local orthonormal referential of M with respect to $\tilde\phi^*(\ ,\ )$, and assume that $\tilde\phi = (x,t)$ and $e_i = \frac{1}{t}\tilde e_i$. If $i \neq j$, let $\tilde K(\tilde e_i, \tilde e_j)$ be the sectional curvature of $(M, \tilde\phi^*(\ ,\ ))$ determined by $(\tilde e_i, \tilde e_j)$ and let $K(e_i, e_j)$ be the sectional curvature of $(M, \phi^*\langle,\rangle)$ determined by (e_i, e_j). Then

(2.12) $\tilde K(\tilde e_i, \tilde e_j) = t^2 K(e_i, e_j) + \langle \tilde H, B(e_i,e_i) + B(e_j,e_j)\rangle - t^2\langle H, B(e_i,e_i) + B(e_j,e_j)\rangle + \frac{1}{t^2}\langle\tilde H, \tilde H\rangle + t^2\langle H,H\rangle - 2\langle\tilde H, H\rangle - 1.$

Proof: - Let (u_A), $1 \leq A \leq n$, be a local orthonormal referential of H^n such that $u_i(\tilde\phi(p)) = \tilde\phi_{*p}(\tilde e_i(p))$ if $p \in M$ and $1 \leq i \leq m$. Let $\tilde\Omega^i_j$ be the differential 2-forms of curvature of $(\tilde e_i)$ and

suppose that $B(e_i, e_j) = \sum_{\alpha=m+1}^{n} \tilde h^\alpha_{ij} \tilde u_\alpha$. The Gauss equation implies,

$$\tilde K(\tilde e_i, \tilde e_j) = \tilde\Omega^i_j(\tilde e_i, \tilde e_j) = -1 + \sum_{\alpha=m+1}^{n} \det\begin{bmatrix} \tilde h^\alpha_{ii} & \tilde h^\alpha_{ij} \\ \tilde h^\alpha_{ji} & \tilde h^\alpha_{jj}\end{bmatrix}.$$

Then if $u_\alpha = \frac{1}{t}\tilde u_\alpha$ and $B(e_i, e_j) = \sum_{\alpha=m+1}^{n} h^\alpha_{ij} u_\alpha$

(2.1) implies $\tilde h^\alpha_{ij} = t h^\alpha_{ij} + \langle u_i, u_j\rangle\langle U_n, u_\alpha\rangle.$

Therefore

$$\tilde K(\tilde e_i, \tilde e_j) = t^2 K(e_i, e_j) + t\langle B(e_i,e_i) + B(e_j,e_j), U_n\rangle - \langle (U_n)^T, (U_n)^T\rangle.$$

Since $(U_n)^N = \frac{1}{t}\tilde H - tH$ and

$$\langle (U_n)^T, (U_n)^T\rangle = 1 - \frac{1}{t^2}\langle\tilde H, \tilde H\rangle - t^2\langle H, H\rangle + 2\langle H, \tilde H\rangle,$$

we have (2.12).

Corollary - Let $\tilde K$ and K be respectively the scalar curvatures of $(M, \tilde\phi^*(,))$ and $(M, \phi^*\langle,\rangle)$. Then

(2.13) $\tilde K = t^2 K + \frac{1}{t^2}\langle\tilde H, H\rangle - t^2\langle H, H\rangle - 1.$

In particular, $\tilde H \equiv 0$ implies

(2.14) $K \leq 0$ if $\tilde K \leq -2$,

(2.15) $K < 0$ if $\tilde K < -2$.

Proof. Given $p \in M$,

$$\tilde K(p) = \frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \tilde K(\tilde e_i, \tilde e_j) \quad \text{and} \quad K(p) = \frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} K(e_i, e_j),$$

where $(\tilde{e}_i)$ and (e_i) are respectively orthonormal basis of $T_p(M)$ with respect to $\tilde{\phi}^*(,)$ and $\phi^*<,>$. Then (2.13) follows from (2.12). The inequalities follow from (2.13) and from $<H,H> \leq \frac{1}{t^2}$, which is a consequence of (2.5).

3. Minimal immersions in H^n

The formulas (2.5), (2.9), (2.10) and (2.11) imply the following result.

Proposition 6. - If $\tilde{\phi} = (x^1,\dots,x^{n-1},t)$ is minimal we have

(3.1) $\quad \Delta_{\tilde{M}} x^A = -2t <(U_n)^N, U_A>,$

(3.2) $\quad \Delta_M x^A = -\frac{m}{t} <(U_n)^N, U_A>,$ when $A = 1,2,\dots,n-1;$

(3.3) $\quad \Delta_{\tilde{M}} t = -t\left[m-2 <(U_n)^T,(U_n)^T>\right]$

(3.4) $\quad \Delta_M t = -\frac{m}{t} <(U_n)^N, (U_n)^N>.$

Therefore, if $m \geq 2$, t is superharmonic with respect to the riemannian metrics $\tilde{\phi}^*(,)$ and $\phi^*<,>$.

Now let us collect some geometric consequences of the superharmonicity of t in the following result.

Theorem 7 - Let M be an m-dimensional connected and oriented riemannian manifold. Then

1) if M is compact without boundary, there is no minimal isometric immersion of M into H^n;

2) if $m = 2$ and M is parabolic there is no minimal isometric immersion of M into H^n;

3) if $m = 2$ and M is complete and has finite total curvature, there is no minimal isometric immersion of M into H^n.

Proof.

1) Assume that $\tilde{\phi}$ is minimal. Then the divergence theorem and (3.4) imply

$$-\int_M \frac{m}{t} <(U_n)^N, (U_n)^N> dM = 0. \quad \text{Thus } (U_n)^N \equiv 0.$$

Therefore (2.5) implies that ϕ is a minimal immersion in $\mathbb{R}^n_+$. This is a contradiction, because there is no isometric minimal immersion of a compact, without boundary, riemannian manifold in $\mathbb{R}^n$.

2) Consider the conformal structure induced by $\tilde{\phi}^*(,)$ on M.

Going to the universal covering space, we may suppose that M is globally parametrized by isothermal parameters $z = u+iv$. Then

$$\Delta_M = \frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \quad \text{where } \lambda = <\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial u}> = <\frac{\partial\phi}{\partial v}, \frac{\partial\phi}{\partial v}>.$$

Therefore (3.4) implies that $\frac{1}{\lambda^2}\left(\frac{\partial^2 t}{\partial u^2} + \frac{\partial^2 t}{\partial v^2}\right) \leq 0$. So t is a positive superharmonic function globally defined on M. Since on a connected Riemann surface of parabolic type, there is no non-constant superharmonic function bounded below, we have that t is constant [A-S]. Thus $\phi(M)$ is contained in a hyperplane orthogonal to U_n and so $U_n = (U_n)^N$ and $<\Delta_M\phi, U_n> = 0$. But (2.1) and (2.5) imply that $\Delta_n\phi = -\frac{m}{t}(U_n)^N$. Then we have a contradiction.

3) As a consequence of theorem 15 of [H], if M is a complete 2-dimensional riemannian manifold having finite total curvature it has to be parabolic.

Remark. The assertion 1 of the above theorem was first proved by [O'N]. See also [M]. The proof above was included due to its simplicity.

Proposition 8 - Suppose that $\tilde{\phi}$ is minimal and let $s(p) = \langle \phi(p), U_n \rangle$, $p \in M$. If q is a local maximum of s, (e_i), $i=1,\dots,m$, is an orthonormal basis of $T_q(M)$ with respect to $\phi^*\langle , \rangle$ and $\tilde{e}_i = t\, e_i$ for all i, we have

$$(3.5) \qquad K(e_i,e_j) \leqq \frac{1}{t^2}\left[\tilde{K}(\tilde{e}_i,\tilde{e}_j) + m\right] \quad \text{for all } i \neq j.$$

In particular, if $n = m+1$ we have

$$(3.6) \qquad K(e_i,e_j) \geqq -\frac{m^2}{4t^2(q)},$$

$$(3.7) \qquad \tilde{K}(\tilde{e}_i,\tilde{e}_j) \geqq -\frac{m}{4}(m+4).$$

Moreover, if $n = m+1$ and (e_i) diagonalizes the second fundamental form of ϕ, we have

$$(3.8) \qquad K(e_i,e_j) \geqq 0,$$

$$(3.9) \qquad \tilde{K}(\tilde{e}_i,\tilde{e}_j) \geqq -m.$$

Proof - Since the result is local, we may identify $\phi(p)$ with p in a neighborhood V of q. Thus if $X \in \mathfrak{X}(V)$ we have

$$(X[s])(q) = \langle X(q), U_n \rangle = 0. \text{ Then } U_n(q) = (U_n)^N(q)$$

and

$$(X[X[s]])(q) = \langle \nabla_{X(q)} X, U_n \rangle = \langle B(X(q),X(q)), U_n \rangle \leqq 0.$$

Now let (e_i) be an orthonormal basis of $T_q(M)$ with respect to $\langle , \rangle$. Then $\langle B(e_i,e_i), U_n \rangle \leqq 0$ $(\forall\, i)$, and from

$$H(q) = -\frac{(U_n)^N(q)}{t(q)} = -\frac{U_n(q)}{t(q)}, \text{ we have}$$

$$-\sum_{i=1}^{m} \langle B(e_i,e_i), U_n \rangle = -m\langle H(q), U_n(q) \rangle = \frac{m}{t(q)}.$$

From (2.12)

$$\tilde{K}(\tilde{e}_i,\tilde{e}_j) = t^2(q)K(e_i,e_j) + t(q)\langle U_n, B(e_i,e_i) + B(e_j,e_j) \rangle.$$

Then

$$K(e_i,e_j) = \frac{1}{t^2(q)}\{\tilde{K}(\tilde{e}_i,\tilde{e}_j) - t(q)\langle U_n, B(e_i,e_i) + B(e_j,e_j) \rangle\} \leqq$$

$$\leqq \frac{1}{t^2(q)}\{\tilde{K}(\tilde{e}_i,\tilde{e}_j) + t(q)\frac{m}{t(q)}\} = \frac{1}{t^2(q)}\{\tilde{K}(\tilde{e}_i,\tilde{e}_j) + m\}.$$

If $n = m+1$, we have

$$K(e_i,e_j) = \langle B(e_i,e_i), U_{m+1} \rangle \langle B(e_j,e_j), U_{m+1} \rangle - \langle B(e_i,e_j), U_{m+1} \rangle^2.$$

But $\langle B(e_i \pm e_j, e_i \pm e_j), U_{m+1} \rangle \leqq 0$ implies

$$\pm 2\langle B(e_i,e_j), U_{m+1} \rangle \leqq -\langle B(e_i,e_i), U_{m+1} \rangle - \langle B(e_j,e_j), U_{m+1} \rangle \leqq \frac{m}{t(q)}.$$

Then

$$|\langle B(e_i,e_j), U_{m+1} \rangle| \leqq \frac{m}{2t(q)}.$$

Therefore

$$K(e_i,e_j) \geqq -\langle B(e_i,e_j), U_{m+1} \rangle^2 \geqq -\frac{m^2}{4t^2(q)}.$$

Then, from (3.5), we have

$$\tilde{K}(\tilde{e}_i,\tilde{e}_j) \geqq t^2(q)\, K(e_i,e_j) - m \geqq -\frac{m}{4}(m+4).$$

If $n = m+1$ and (e_i) diagonalizes the second fundamental form of ϕ, we have

$$K(e_i,e_j) = \langle B(e_i,e_i), U_{m+1} \rangle \langle B(e_j,e_j), U_{m+1} \rangle \geqq 0$$

and $\tilde{K}(\tilde{e}_i,\tilde{e}_j) \geqq -m$.

The ideal boundary of H^n is the compactification by a point of $\mathbb{R}^{n-1} = \{(x,t) \in \mathbb{R}^n / t = 0\}$. It is well known that it has a natural conformal structure and that conformal diffeomorphisms of it extend to isometries of H^n. We indicate the ideal boundary of H^n by $\partial_\infty H^n$ and the added point by ∞.

The asymptotic boundary of a given set $S \subset H^n$ is the set $\bar{S} \cap \partial_\infty H^n$, where the closure of S is considered in $H^n \cup \partial_\infty H^n$.

We now have the following results.

Theorem 9 - Let M be an m-dimensional riemannian manifold, assume that it is complete, non compact and oriented and its scalar curvature is always less than -2. Then there is no isometric minimal immersion of M into H^{m+1}, if the asymptotic boundary of its image omits a point of $\partial_\infty H^{m+1}$.

Proof - Suppose that $\tilde{\phi}: M \to H^{m+1}$ is one such immersion. Let $a \in \partial_\infty H^{m+1} - \partial_\infty \tilde{\phi}(M)$ and let F be a conformal diffeomorfism of $\partial_\infty H^{m+1}$ that sends a into ∞. Let f be the natural extension of F to H^{m+1} such that f restricted to H^{m+1} is an isometry. Since $\infty \notin \partial_\infty f(\tilde{\phi}(M))$ we have that $s(p) = \langle f(\tilde{\phi}(p)), U_{m+1}\rangle$ has a maximum $C > 0$ at some $q \in M$ and the hyperplane $t \equiv C$ is the tangent space to $f(\tilde{\phi}(M))$ at $f(\tilde{\phi}(q))$.

Now let ψ be the immersion of M into $\mathbb{R}^{m+1}_+$ given by $\psi(p) = f(\tilde{\phi}(p))$ $(\forall\ p \in M)$. Let K be the scalar curvature of M with respect to the riemannian metric $\psi^*\langle,\rangle$. Since the tangent space to $\psi(M)$ at $\psi(q)$ is the hyperplane $t \equiv C$, and since $s(p) = \langle\psi(p), U_n\rangle \leqq C$ $(\forall\ p \in M)$, we have by (3.8) that $K(q) \geqq 0$. This gives a contradiction, because (2.15) implies that $K(q) < 0$.

Theorem 10 - Let M be an m-dimensional riemannian manifold. Assume that M is connected, complete, non compact and oriented. Then there is no isometric minimal immersion of M into H^{m+1} having at every point at least one sectional curvature less than $-\frac{m}{4}(m+4)$ if the asymptotic boundary of its image omits a point of $\partial_\infty H^{m+1}$.

Proof - It is similar to the one of theorem 9 noting that the hypothesis $\tilde{K}(q) < -\frac{m}{4}(m+4)$ is a contradiction with (3.7).

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