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BÄCKLUND'S THEOREM FOR SUBMANIFOLDS OF SPACE FORMS AND A GENERALIZED WAVE EQUATION

Keti Tenenblat

Introduction

In this paper, we extend the results obtained in [5] and [6], by proving a generalization of Bäcklund's theorem and Bianchi's permutability theorem for n-dimensional submanifolds of a $(2n-1)$-dimensional, simply connected, complete, space form. Moreover, we show that the analytic interpretation of these theorems provides a Bäcklund transformation and a superposition formula for systems of partial differential equations which generalize the homogeneous wave equation and the Sine Gordon equation. We observe that the generalized wave equation is nonlinear for $n \geq 3$ and the initial value problem for these generalized equations can be solved by applying the inverse scattering method [1].

In section 1, we review the local theory for n-dimensional Riemannian manifolds, with constant sectional curvature k, isometrically immersed in a $(2n-1)$-dimensional simply connected, complete, space form $\bar{M}_k^{2n-1}$ of curvature k, such that $k < K$. We show that such immersions are in correspondence with the class of orthogonal, $(n\times n)$-matrix functions which satisfy a system of partial differential equations (1.12). For $n = 2$, whenever $k \neq 0$, the system of equations reduces to the Sine-Gordon equation, and when $k = 0$ (the case of flat surfaces contained in a 3-dimensional sphere) it reduces to the homogeneous wave equation.

In section 2, we define a pseudo-spherical geodesic congruence between two n-dimensional submanifolds of a space form $\bar{M}_k^{2n-1}$. We prove a generalization of Bäcklund's theorem (theorem 1), which shows that the existence of such a congruence implies that both submanifolds have constant sectional curvature k, $k < K$. In theorem 2, by proving the complete integrability of

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the differential ideal associated to the existence of a pseudo--spherical congruence, we show that given an n-dimensional submanifold M of $\bar{M}_K^{2n-1}$, with constant curvature k, $k < K$, there exists an n-parameter family of submanifolds, which are related to M by pseudo-spherical geodesic congruences.

In section 3, we prove the permutability property (theorem 4). The geometric theory obtained in sections 2 and 3 is interpreted analytically in the last section. Namely, theorem 5 provides a Backlund transformation for the system of equations (1.12), which generates new solutions from a given one. Theorem 6 provides a superposition formula which generates other solutions algebraically. Moreover, it follows from the characterization of section 1, that given a Riemannian manifold M^n, with constant sectional curvature k, immersed in a space form $\bar{M}_K^{2n-1}$, these theorems provide the first and second fundamental forms of new such submanifolds.

We observe that the geometric theory for $n = 2$, in the non-euclidean cases was obtained by Bianchi [2].

In this paper, M, M', denote n-dimensional submanifolds of a $(2n-1)$-dimensional simply connected, complete, space form $\bar{M}_K^{2n-1}$ of curvature K. Without loss of generality we consider $K = 0, 1, -1$, i.e. $\bar{M}$ is respectively the euclidean space, unit sphere S^{2n-1} and the hyperbolic space H^{2n-1}. For the sake of completeness, we include the euclidean case i.e. $K = 0$, which was treated in [5] and [6].

We will use the following conventions on ranges of indices,

$$1 \leq I,J,L \leq n, \qquad 1 \leq A,B,C \leq 2n-1,$$

$$2 \leq i,j,\ell \leq n, \qquad n+1 \leq \alpha,\beta,\gamma \leq 2n-1$$

Moreover, we shall agree that repeated indices are summed over the respective ranges.

1. Local theory for submanifolds of space form

Let M be an n-dimensional riemannian manifold of constant curvature k, isometrically immersed in a space form $\bar{M}_K^{2n-1}$, such that $k < K$.

Let $e_1, e_2, \ldots, e_{2n-1}$ be a moving orthonormal frame on an open set of $\bar{M}$, so that at points of M, $e_1, \ldots, e_n$ are tangent to M. Let ω_A be the dual orthonormal coframe, and consider ω_{AB} defined by

$$de_A = \sum_B \omega_{AB} e_B.$$

The structure equations of $\bar{M}$ are

$$(1.1) \qquad d\omega_A = \sum_B \omega_B \wedge \omega_{BA}, \quad \omega_{AB} + \omega_{BA} = 0;$$

$$(1.2) \qquad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - K\omega_A \wedge \omega_B.$$

Restricting these forms to M we have $\omega_\alpha = 0$, hence from (1.1) we obtain

$$(1.3) \qquad d\omega_\alpha = \sum_I \omega_I \wedge \omega_{I\alpha} = 0$$

$$(1.4) \qquad d\omega_I = \sum_J \omega_J \wedge \omega_{JI}.$$

The first of these implies via Cartan's lemma, that

$$\omega_{I\alpha} = \sum_J b^\alpha_{IJ} \omega_J, \quad b^\alpha_{IJ} = b^\alpha_{JI}.$$

From (1.2) we obtain, Gauss equation

$$(1.5) \qquad d\omega_{IJ} = \sum_L \omega_{IL} \wedge \omega_{LJ} + \sum_\alpha \omega_{I\alpha} \wedge \omega_J - K\omega_I \wedge \omega_J$$

and Codazzi equation

$$(1.6) \qquad d\omega_{I\alpha} = \sum_A \omega_{IA} \wedge \omega_{A\alpha}.$$

M has constant sectional curvature k, if and only if

$$(1.7) \qquad \Omega_{IJ} = d\omega_{IJ} - \sum_L \omega_{IL} \wedge \omega_{LJ} = -k\omega_I \wedge \omega_J,$$

i.e.

$$(1.8) \qquad \sum_\alpha \omega_{I\alpha} \wedge \omega_{\alpha J} = (K-k)\omega_I \wedge \omega_J.$$

From (1.2) we have

$$d\omega_{\alpha\beta} = \sum_{\gamma} \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} + \Omega_{\alpha\beta},$$

where

$$\Omega_{\alpha\beta} = \sum_{I} \omega_{\alpha I}\wedge\omega_{I\beta}$$

is the normal curvature of M.

We denote by $I = \sum_{I} (\omega_I)^2$ the first fundamental form on M and $II = \sum_{\alpha} H^{\alpha} e_{\alpha}$, where H^{α} is the second fundamental form with respect to e_{α}. Since $k < K$, there exists ([3] [4]) a local orthonormal frame tangent to M, $v_1,\dots,v_n$, called principal directions which diagonalizes the quadratic forms H^{α} simultaneously. Lines of curvature are curves of M which are tangent at each point to principal directions. A tangent vector V is called asymptotic if $H^{\alpha}(V) = 0$, $\forall\,\alpha$. E. Cartan and J.D. Moore proved the following.

Theorem A ([3] [4]) - *Let M be a Riemannian n-dimensional manifold with constant sectional curvature k, isometrically immersed in a $(2n-1)$-dimensional space form $\bar{M}_K^{2n-1}$, such that $k < K$. Then locally there exists coordinates $(u_1,\dots,u_n)$ parametrized by lines of curvature such that*

$$I = \sum_{I} a_I^2 du_I^2,$$

$$II = \sum_{I,\alpha} b_I^{\alpha} a_I du_I^2 e_{\alpha},$$

where $a_I > 0$ for all I and $\sum_{I} a_I^2 = 1$. Moreover, the vector $V = \sum_{I} a_I v_I$ is the unique unit asymptotic vector such that $a_I > 0$, $\forall\, I$.

It follows from this theorem that the normal bundle of M in $\bar{M}$ has zero curvature. Hence, without loss of generality, we will assume that e_{α} have been chosen so that $\omega_{\alpha\beta} = 0$.

Consider the moving frame $v_1,\dots,v_n, e_{\alpha}$, let ϕ_I be the dual forms to v_I and ϕ_{AB} the connection forms. Then, $\phi_I = a_I du_I$, $\phi_I = b_I^{\alpha} a_I du_I$ and (1.8) takes the simpler form

$$(1.9) \qquad \sum_{\alpha} b_J^{\alpha} b_I^{\alpha} = k-K, \qquad I \neq J.$$

Since $\sum_{I} \frac{\partial}{\partial u_I} = \sum_{I} a_I v_I$ is a unit asymptotic vector, we get

$$(1.10) \qquad \sum_{I} b_I^{\alpha} a_I^2 = 0, \quad \sum_{I} a_I^2 = 1.$$

From (1.9) and (1.10), it follows that for each I

$$(1.11) \qquad \sum_{\alpha} b_I^{\alpha} b_I^{\alpha} = (K-k)(1-a_I^2)/a_I^2.$$

Therefore there is a matrix function $\bar{A}$ associated to M, with respect to e_{α},

$$\bar{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 b_1^{n+1} & a_2 b_2^{n+1} & \cdots & a_n b_n^{b+1} \\ \vdots & & & \\ a_1 b_1^{2n-1} & a_2 b_2^{2n-1} & \cdots & a_n b_n^{2n-1} \end{bmatrix}$$

satisfying (1.9)-(1.11). It follows from these properties, that if we consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{K-k}} I_{n-1} \end{bmatrix} \bar{A},$$

where I_{n-1} is the $(n-1)\times(n-1)$ unit matrix, we obtain an *orthogonal matrix function A, defined on an open set U, associated to M, with respect to e_{α}.*

Moreover if we denote $A = (a_{IJ})$, it follows from (1.4) that

$$\phi_{IJ} = \frac{1}{a_{1I}} \frac{\partial a_{1J}}{\partial u_I} du_J - \frac{1}{a_{1J}} \frac{\partial a_{1I}}{\partial u_J} du_I.$$

Therefore, from (1.6) and (1.7) we conclude that A satisfies the following system of partial differential equations

$$
(1.12)\quad \begin{cases} \dfrac{\partial}{\partial u_I}\left(\dfrac{1}{a_{1I}}\dfrac{\partial a_{1J}}{\partial u_I}\right) + \dfrac{\partial}{\partial u_J}\left(\dfrac{1}{a_{1J}}\dfrac{\partial a_{1I}}{\partial u_J}\right) + \displaystyle\sum_{L\neq I, L\neq J} \dfrac{1}{a_{1L}^2}\dfrac{\partial a_{1I}}{\partial u_L}\dfrac{\partial a_{1J}}{\partial u_L} = -k a_{1I} a_{1J}, & I\neq J \\[2ex] \dfrac{\partial}{\partial u_L}\left(\dfrac{1}{a_{1I}}\dfrac{\partial a_{1J}}{\partial u_I}\right) = \dfrac{1}{a_{1I}a_{1L}}\dfrac{\partial a_{1L}}{\partial u_I}\dfrac{\partial a_{1J}}{\partial u_L} & L\neq I\neq J, \\[2ex] \dfrac{a_{iI}}{u_J} = \dfrac{a_{iJ}}{a_{1J}}\dfrac{\partial a_{1I}}{\partial u_J} & \forall\, i,I,J;\ I\neq J \end{cases}
$$

where $1 \leq I,J,L \leq n$, $2 \leq i \leq n$.

Conversely, given an orthogonal matrix function A satisfying the above system, it follows from the fundamental theorem for submanifolds of a space form, that for a fixed K, such that $k < K$, the matrix

$$
\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{K-k}\; I_{n-1} \end{bmatrix} A
$$

determines the existence of an n-dimensional manifold M, with constant sectional curvature k, isometrically immersed in a space form $\bar{M}_K^{2n-1}$.

We observe, that for $n = 2$, whenever $k \neq 0$ the above system of equations reduces to the Sine-Gordon equation, and when $k = 0$ it reduces to the homogeneous wave equation. In fact, consider

$$
A = \begin{bmatrix} \cos f & \sin f \\ -\sin f & \cos f \end{bmatrix},
$$

where f is a differentiable function of u_1, u_2. Then (1.12) reduces to

$$
(1.13)\qquad f_{u_1u_1} - f_{u_2u_2} = -k \sin f \cos f,
$$

which is the homogeneous wave quation when $k = 0$. For $k \neq 0$, we define

$$
\psi(\bar{u}_1,\bar{u}_2) = \begin{cases} 2f\left(\dfrac{\bar{u}_1}{\sqrt{-k}}, \dfrac{\bar{u}_2}{\sqrt{-k}}\right), & \text{if } k < 0; \\[2ex] 2f\left(\dfrac{\bar{u}_1}{\sqrt{k}}, \dfrac{\bar{u}_2}{\sqrt{k}}\right) - \pi, & \text{if } k > 0. \end{cases}
$$

Then (1.13) is equivalent to

$$
\psi_{\bar{u}_1\bar{u}_1} - \psi_{\bar{u}_2\bar{u}_2} = \sin\psi,
$$

which is the Sine-Gordon equation.

2. Generalization of Bäcklund's theorem

In this section, we define a pseudo-spherical geodesic congruence between two n-dimensional submanifolds M and M' of a space form $\bar{M}_K^{2n-1}$ with constant sectional curvature K. We prove a generalization of Backlund's theorem, for such submanifolds and the complete integrability of the differential ideal associated to the existence of a pseudo-spherical congruence.

In what follows we need the notion of angles between two k-planes in a $2k$-dimensional inner product space. Let E and E be two k-planes in a $2k$-dimensional inner product space $(V, <,>)$ and $\pi: V \to E_1$ the orthogonal projection. Define a symmetric bilinear form on E_2 by $(v_1,v_2) = \langle \pi(v_1), \pi(v_2)\rangle$. The *$k$ angles between E_1 and E_2* are defined to be $\theta_1,\ldots,\theta_k$ where $\cos^2\theta_1,\ldots,\cos^2\theta_k$ are the k-eigenvalues for the self-adjoint operator $A: E_2 \to E_2$ such that $(v_1,v_2) = \langle Av_1, v_2\rangle$.

Definition 1. A *geodesic congruence* between two n-dimensional submanifolds M and M' of a $(2n-1)$-dimensional space form $\bar{M}$ is a diffeomorphism $\ell: M \to M'$, such that for $P \in M$ and $P' = \ell(P)$, there exists a unique geodesic γ in $\bar{M}$ joining P and P', whose tangent vectors at P and P' are in T_PM and $T_{P'}M'$ respectively.

Given a geodesic congruence $\ell: M \to M'$, we remark that the normal spaces ν_P and $\nu_{P'}$ at corresponding points P and P'

are $(n-1)$ dimensional and orthogonal to the plane determined by the position vector X of M and the tangent vector of γ at P. Therefore, ν_P and $\nu_{P'}$ lie in a $2n-2$ dimensional vector space, i.e. there are $(n-1)$ angles between ν_P and $\nu_{P'}$.

Definition 2. A *geodesic congruence* $\ell: M \to M'$ between two n-dimensional submanifolds of $\bar{M}$ is called *pseudo-spherical* if:

(1) the distance between P and $P' = \ell(P)$ on $\bar{M}$, is a constant r, independent of P;

(2) the $(n-1)$ angles between ν_P and $\nu_{P'}$ are all equal to a constant θ, independent of P;

(3) the normal bundles ν and ν' are flat;

(4) the bundle map $\Gamma: \nu \to \nu'$ given by the orthogonal projection commutes with the normal connections.

The above definition generalizes the notion of pseudo-spherical line congruence introduced in [5]. The following result is a generalization of Bäcklund's theorem for submanifolds for a space form. Without loss of generality, we consider $K \equiv 0$, 1 or -1 i.e. $\bar{M}$ is respectively the $(2n-1)$-dimensional euclidean space, unit sphere H^{2n-1} or the hyperbolic space H^{2n-1}. Moreover, we consider H^{2n-1} as being in a Minkowski space $\tilde{\mathbb{R}}^{2n}$.

Theorem 1. *Suppose there is a pseudo-spherical geodesic congruence $\ell: M \to M'$ between two n-dimensional submanifolds of $\bar{M}_K^{2n-1}$, with constants r and $\theta \neq 0$. Then, both M and M' have constant sectional curvature k, where*

$$(2.1) \qquad k = \begin{cases} -\dfrac{\sin^2\theta}{r^2}, & \text{if } K \equiv 0; \\ 1 - \dfrac{\sin^2\theta}{\sin^2 r}, & \text{if } K \equiv 1; \\ -1 - \dfrac{\sin^2\theta}{\sinh^2 r}, & \text{if } K \equiv -1. \end{cases}$$

Proof: i) The case $K \equiv 0$ was proved in [5] by considering local orthonormal frames $e_1, \ldots, e_{2n-1}$ for M and $e'_1, \ldots, e'_{2n-1}$ for M' such that,

$$(2.2) \qquad \omega_{n+i-1,n+j-1} = 0,$$

$$(2.3) \qquad \begin{aligned} e'_{n+i-1} &= -\sin\theta\, e_i + \cos\theta\, e_{n+i-1}, \\ e'_i &= \cos\theta\, e_i + \sin\theta\, e_{n+i-1}, \end{aligned}$$

$$(2.4) \qquad e'_1 = -e_1,$$

where e_1 is the unit direction of the line PP'. For such, frames we show that

$$\omega'_1 = -\omega_1$$

$$(2.5) \qquad \begin{aligned} \cos\theta\,\omega'_i &= \omega_i + r\omega_{1i} \\ \sin\theta\,\omega'_i &= r\omega_{1,n+i-1}, \end{aligned}$$

and therefore

$$(2.6) \qquad \omega_i + r\omega_{1i} = r \operatorname{cotg}\theta\,\omega_{1,n+i-1}.$$

Moreover, we prove that

$$(2.7) \qquad \omega_{ij} = \operatorname{cotg}\theta(\omega_{i,n+j-1} - \omega_{j,n+i-1}),$$

and

$$(2.8) \qquad \begin{aligned} \omega'_{1,n+k-1} &= -\frac{\sin\theta}{r}\,\omega_k \\ \omega'_{i,n+k-1} &= \omega_{k,n+i-1}. \end{aligned}$$

Finally, it follows that the sectional curvature of M' is constant equal to $-\sin^2\theta/r^2$. By symmetry, M has the same constant sectional curvature.

ii) Let M and M' be submanifolds of the unit sphere S^{2n-1} contained in the euclidean space $\mathbb{R}^{2n}$. Consider M locally given by $X:U \to M \subset S^{2n-1} \subset \mathbb{R}^{2n}$, where U is an open subset of $\mathbb{R}^n$. Since there is a pseudo-spherical congruence between M and M', there exist local orthonormal frames $e_1, e_2, \dots, e_{2n-1}$ for M and $e_1', \dots, e_{2n-1}'$ for M' such that (2.2), (2.3) are verified and

$$e_1' = \sin r X - \cos r\, e_1, \tag{2.9}$$

where e_1 at $P \in M$, is the unit vector tangent to the geodesic from P to $P' = \ell(P)$.

Let X' denote the position vector for M'. Then locally

$$X' = \cos r X + \sin r\, e_1.$$

We remark that since $dX = \omega_1 e_1 + \omega_i e_i$ and $\langle X, e_1 \rangle = 0$, it follows that $\langle de_1, X \rangle = -\omega_1$. Hence,

$$\begin{aligned} dX' &= \cos r\, dX + \sin r\, de_1 \\ &= \cos r\, \omega_1 e_1 + \cos r\, \omega_i e_i + \sin r\, \omega_{1i} e_i + \\ &\quad \sin r\, \omega_{1,n+i-1} e_{n+i} - \sin r\, \omega_1 X. \end{aligned}$$

On the other hand

$$\begin{aligned} dX' &= \omega_1' e_1' + \omega_i' e_i' = \\ &= \omega_1' (\sin r X - \cos r\, e_1) + \omega_i' (\cos\theta\, e_i + \sin\theta\, e_{n+i-1}). \end{aligned}$$

Comparing coefficients of X, $e_1, \dots, e_{2n-1}$ we get

$$\omega_1' = -\omega_1$$

$$\cos\theta\, \omega_i' = \cos r\, \omega_i + \sin r\, \omega_{1i} \tag{2.10}$$

$$\sin\theta\, \omega_i' = \sin r\, \omega_{1,n+i-1}.$$

Therefore we obtain

$$\cos r\, \omega_i + \sin r\, \omega_{1i} = \sin r \operatorname{cotg}\theta\, \omega_{1,n+i-1}. \tag{2.11}$$

Since $\omega'_{n+i-1,n+j-1} = 0$, it follows that

$$0 = \sin^2\theta\, \omega_{ij} - \sin\theta \cos\theta (\omega_{i,n+j-1} - \omega_{j,n+i-1})$$

i.e.

$$\omega_{ij} = \operatorname{cotg}\theta (\omega_{i,n+j-1} - \omega_{j,n+i-1}). \tag{2.12}$$

Now we want to compute Ω'_{1i} and Ω'_{ij}. From (2.3) and (2.9)-(2.12) we get

$$\begin{aligned} \omega'_{1,n+k-1} &= -\sin r \sin\theta\, \omega_k + \cos r (\sin\theta\, \omega_{1k} - \cos\theta\, \omega_{1,n+k-1}) \\ &= -\frac{\sin\theta}{\sin r}\, \omega_k \end{aligned} \tag{2.13}$$

and

$$\omega'_{i,n+k-1} = \omega_{k,n+i-1}. \tag{2.14}$$

Therefore by (1.7) and (2.13) we get

$$\begin{aligned} \Omega'_{1i} &= -\omega'_{1,n+k-1} \wedge \omega'_{i,n+k-1} - \omega_1' \wedge \omega_i' = \\ &= \frac{\sin\theta}{\sin r}\, \omega_k \wedge \omega_{k,n+i-1} - \omega_1' \wedge \omega_i' \\ &= -\left(1 - \frac{\sin^2\theta}{\sin^2 r}\right) \omega_1' \wedge \omega_i', \end{aligned}$$

where last equality follows from (1.3) and (2.10). Similarly,

$$\begin{aligned} \Omega'_{ij} &= -\omega'_{i,n+k-1} \wedge \omega'_{j,n+k-1} - \omega_i' \wedge \omega_j' = \\ &= -\omega_{k,n+i-1} \wedge \omega_{k,n+j-1} - \omega_i' \wedge \omega_j' = \\ &= \omega_{1,n+i-1} \wedge \omega_{1,n+j-1} - \omega_i' \wedge \omega_j' \\ &= -\left(1 - \frac{\sin^2\theta}{\sin^2 r}\right) \omega_i' \wedge \omega_j' \end{aligned}$$

where the last equality follows from (2.10).

Therefore M' has constant sectional curvature $1 - \frac{\sin^2\theta}{\sin^2 r}$. By symmetry M also has the same constant sectional curvature.

iii) Let M and M' be submanifolds of $H^{2n-1} \subset \tilde{\mathbb{R}}^{2n}$ where $\tilde{\mathbb{R}}^{2n}$ is a Minkowski space (cf [7], p. 66). Consider M locally given by $X: U \to M \subset H^{2n-1} \subset \tilde{\mathbb{R}}^{2n}$ where U is an open subset of $\mathbb{R}^n$. Then X is normal to H^{2n-1} and $\|X\| = -1$.

Since there is a pseudo-spherical congruence between M and M', there exist local orthonormal frames $e_1, e_2, \ldots, e_{2n-1}$ for M and $e'_1, \ldots, e'_{2n-1}$ for M' such that (2.2), (2.3) are verified and

$$e'_1 = \sinh r\, X - \cosh r\, e_1, \tag{2.15}$$

where e_1 at $P \in M$ is the unit vector tangent to the geodesic from P to $P' = \ell(P)$. Let X' be the position vector for M', then locally

$$X' = \cosh r\, X + \sinh r\, e_1.$$

With arguments analogue to the previous case we get

$$\begin{aligned} \omega'_1 &= -\omega_1 \\ \cos\theta\, \omega'_i &= \cosh r\; \omega_i + \sinh r\; \omega_{1i} \\ \sin\theta\, \omega'_i &= \sinh r\; \omega_{1,n+i-1}. \end{aligned} \tag{2.16}$$

Therefore, we obtain

$$\cosh r\; \omega_i + \sinh r\; \omega_{1i} = \sinh r\, \operatorname{cotg}\theta\; \omega_{1n+i} \tag{2.17}$$

$$\omega_{ij} = \operatorname{cotg}\theta(\omega_{i,n+j-1} - \omega_{j,n+i-1}). \tag{2.18}$$

Moreover

$$\begin{aligned} \omega'_{1,n+k-1} &= -\frac{\sin\theta}{\sinh r}\,\omega_k \\ \omega'_{i,n+k-1} &= \omega_{k,n+i-1}. \end{aligned} \tag{2.19}$$

Finally, we obtain that the sectional curvature of M', and therefore of M, is constant equal to $-1 - \sin^2\theta/\sinh^2 r$.

q.e.d.

The following theorem shows that, given an n-dimensional submanifold M of a space form $\bar{M}_K^{2n-1}$, with constant curvature $k < K$, there exists an n-parameter family of submanifolds M', which are related to M by pseudo-spherical geodesic congruences.

Theorem 2. *Let M be an n-dimensional submanifold of a space form $\bar{M}_K^{2n-1}$, with constant sectional curvature k given by (2.1), where $r > 0$ (and $r < \pi$, whenever $K \equiv 1$) and $\theta \neq 0$ are constants. Let $v_1^0, \ldots, v_n^0$ be an orthonormal basis of the tangent space to M at P_0, given by principal directions. Given a unit vector $v_0 = \sum_{I=1}^n C_I v_I^0$, $C_I \neq 0$ for all $1 \leq I \leq n$, there exists an n-dimensional submanifold M' of $\bar{M}$ and a pseudo-spherical geodesic congruence $\ell: M \to M'$ such that the geodesic joining P_0 to $P'_0 = \ell(P_0)$ is tangent to v_0 at P, the distance in $\bar{M}$ between P_0 and P'_0 is r and θ is the angle between ν_{P_0} and $\nu_{P'_0}$.*

Proof: i) The case $K \equiv 0$ was proved in [5] by showing the complete integrability of the differential ideal generated by (2.2), (2.6) and (2.7).

ii) When $K \equiv 1$ we consider (2.2), (2.11) and (2.12). More precisely, let J be the ideal generated by the forms

$$\alpha_i = \cos r\; \omega_i + \sin r\; \omega_{1i} - \sin r\, \operatorname{cotg}\theta\; \omega_{1,n+i-1}$$

$$\beta_{ij} = \omega_{ij} - \operatorname{cotg}\theta\,(\omega_{i,n+j-1} - \omega_{j,n+i-1})$$

$$\gamma_{ij} = \omega_{n+i-1,n+j-1}.$$

We first prove that J is a closed differential ideal

$$d\alpha_i = \cos r\, \omega_1 \wedge \omega_{1i} + \cos r\, \omega_k \wedge \omega_{ki} + \sin r\, \omega_{1k} \wedge \omega_{ki} +$$

$$+ \sin r\, \omega_{1,n+k-1} \wedge \omega_{n+k-1,i} - \sin r\, \omega_1 \wedge \omega_i$$

$$- \sin r \operatorname{cotg}\theta\, \omega_{1k} \wedge \omega_{k,n+i-1} - \sin r \operatorname{cotg}\theta\, \omega_{1,n+k-1} \wedge \omega_{ki}$$

$$\equiv - \frac{\cos^2 r}{\sin r}\, \omega_1 \wedge \omega_i + \cos r \operatorname{cotg}\theta\, \omega_1 \wedge \omega_{1,n+i-1}$$

$$+ \sin r \operatorname{cotg}^2\theta\, \omega_{1,n+k-1} \wedge (\omega_{k,n+i-1} - \omega_{i,n+k-1})$$

$$+ \sin r\, \omega_{1,n+k-1} \wedge \omega_{n+k-1,i} - \sin r\, \omega_1 \wedge \omega_i$$

$$+ \operatorname{cotg}\theta\, (\cos r\, \omega_k - \operatorname{cotg}\theta \sin r\, \omega_{1,n+k-1}) \wedge \omega_{k,n+i-1}$$

$$= - \frac{1}{\sin r}\, \omega_1 \wedge \omega_i - \frac{\sin r}{\sin^2\theta}\, \omega_{1,n+k-1} \wedge \omega_{i,n+k-1}$$

$$= - \frac{1}{\sin r}\left(1 - \frac{\sin^2 r}{\sin^2\theta}\right)\omega_1 \wedge \omega_i + \frac{\sin r}{\sin^2\theta}\, \Omega_{1i}.$$

By hypothesis $\Omega_{1i} = -\left(1 - \frac{\sin^2\theta}{\sin^2 r}\right)\omega_1 \wedge \omega_i$, therefore $d\alpha_i \equiv 0 \pmod J$ i.e. $d\alpha_i \in J$.

A similar computation will show that $d\beta_{ij} \in J$. Finally, we remark that

$$d\gamma_{ij} \equiv \Omega_{n+i-1,\, n+j-1} \pmod J.$$

Since M has flat normal curvature we get $d\gamma_{ij} \equiv 0 \pmod J$. Therefore, it follows from Frobenius theorem, that there exists an orthonormal frame $e_1, \ldots, e_{2n-1}$ on a neighborhood of P_0 on M, such that $e_1(P_0) = v_0$, and (2.2), (2.11) and (2.12) are satisfied.

Let $X : U \to M \subset S^{2n-1} \subset \mathbb{R}^{2n}$ be the position vector of M. We define

$$X' = \cos r\, X + \sin r\, e_1.$$

Then $X'(U) = M'$ is contained in S^{2n-1}. We want to prove that M' is n-dimensional and $\ell : M \to M'$ defined by $\ell(X) = X'$ is a pseudo-spherical geodesic congruence. Consider

$$dX' = \cos r\, dX + \sin r\, de_1$$

$$= \omega_1(\cos r\, e_1 - \sin r\, X) + \frac{\sin r}{\sin\theta}\, \omega_{1,n+i-1}(\cos\theta\, e_i + \sin\theta\, e_{n+i-1})$$

where last equality follows from (2.11). Since $\omega_1, \omega_{1,n+i-1}$ ar linearly independent it follows that X' defines an n-dimension submanifold of S^{2n-1}. Moreover, $\cos r\, e_1 - \sin r\, X$, and $\cos\theta\, e_i + \sin\theta\, e_{n+i-1}$ are tangent vectors to M'. The geodesic joining P to P' is given by

$$\cos t\, X + \sin t\, e_1 \qquad t \in [0, r]$$

which is tangent to e_1 and $\cos r\, e_1 - \sin r\, X$ at P and P' respectively. The distance between P and P' is r and the $(n-1)$ angles between ν_P and $\nu_{P'}$ are all equal to θ. Finally, it follows from (2.2) and (2.12) that $\omega'_{n+i-1,n+j-1} = 0$ hence ν and ν' are flat and Γ commutes with the normal connection. This completes the proof for the case $K \equiv 1$.

iii) Similar arguments prove the theorem for $M \subset H^{2n-1}$, by considering the differential ideal generated by (2.2), (2.17) and (2.18).

Remark 1. We observe that the pairs of equations (2.6), (2.7); (2.11), (2.12) and (2.17), (2.18) can be written in matrix notation as

$$(2.20) \qquad \omega = WD - DW^t,$$

where the matrices ω, W and D are defined by

$$\omega = (\omega_{IJ});$$

$$W_{I1} = \omega_I, \quad W_{IJ} = \omega_{I,n+j-1};$$

$$(2.21) \qquad D = \begin{bmatrix} f_K(r) & 0 \\ 0 & \operatorname{cotg}\theta\, I_{n-1} \end{bmatrix};$$

$$f_K(r) = \begin{cases} 1/r & \text{if } K \equiv 0; \\ \operatorname{cotg} r, & \text{if } K \equiv 1; \\ \operatorname{cotgh} r, & \text{if } K \equiv -1; \end{cases}$$

and W^t denotes the transpose of W.

We conclude this section obtaining a result that will be used later. Suppose there is a pseudo-spherical geodesic congruence $\ell : M \to M'$, with constants r and $\theta \neq 0$. Let $v_1,\dots,v_n$ be the local frame given by principal directions on M. Choose an orthonormal frame e_α normal to M such that $\omega_{n+i-1,n+j-1} = 0$ and define $e_1, e_i, e'_1, e'_i, e'_\alpha$ as in theorem 1. Let A' be the orthogonal matrix function given by

$$e_I = A'_{IJ} v_J.$$

Then we obtain:

Theorem 3. *A' is the orthogonal matrix function associated to M' with respect to e'_α.*

Proof: Let ϕ_I be the dual forms to v_I and ϕ_{AB} the connection forms with respect to the frame v_I, e_α, i.e.

$$\phi_I = a_I du_I \quad \text{and} \quad \phi_{I\alpha} = b^\alpha_I a_I du_I.$$

It follows from (2.5) for $K \equiv 0$, (2.10) for $K \equiv 1$, (2.16) for $K \equiv -1$ and theorem 1, that the first fundamental form on M' is given by

$$\mathrm{I}' = \sum_I \omega_I'^2 = \omega_1^2 + \frac{1}{K-k}\sum_i \omega^2_{1,n+i-1}.$$

Therefore

$$\mathrm{I}' = A'_{1I}A'_{1J}\left(1 + \frac{1}{K-k} b_I^{n+i-1} b_J^{n+i-1}\right)\phi_I\phi_J.$$

Hence, from (1.9) and (1.11) we obtain

$$(2.22) \qquad \mathrm{I}' = \sum_I (A'_{1I})^2 du_I^2.$$

In order to obtain the second fundamental form on M', we consider its normal componentes. Using (2.5), (2.8) for $K \equiv 0$; (2.10), (2.13), (2.14) for $K \equiv 1$; (2.16), (2.19) for $K \equiv -1$ and theorem 1 we have

$$\begin{aligned} \mathrm{II}' \cdot e'_{n+j-1} &= \omega_{I,n+j-1}\omega'_I = \\ &= \sqrt{K-k}\,\omega_j\omega_1 + \frac{1}{\sqrt{K-k}}\,\omega_{j,n+i-1}\,\omega_{1,n+i-1} = \\ &= A'_{jI}A'_{1J}\left(\sqrt{K-k} + \frac{1}{\sqrt{K-k}}\, b_I^{n+i-1} b_J^{n+i-1}\right)\phi_I\phi_J. \end{aligned}$$

Hence, it follows from (1.9) and (1.11) that

$$(2.23) \qquad \mathrm{II}' \cdot e'_{n+j-1} = \sqrt{K-k}\sum_I A'_{jI}A'_{1I} du_I^2.$$

From (2.22) and (2.23) we conclude that

$$\bar{A}' = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{K-k}\, I_{n-1} \end{bmatrix} A'$$

is the matrix function associated with M', satisfying (1.9)-(1.11). Therefore, A' is the orthogonal matrix function associated to M' with respect to e'_α.

q.e.d.

3. Permutability property

In this section, we prove the permutatility property of pseudo-spherical geodesic congruences. More precisely:

Theorem 4. *Let* M, M', M'' *be* n*-dimensional submanifolds of a space form* $\bar M_K^{2n-1}$. *Suppose there exist pseudo-spherical geodesic congruences* $\ell_1 : M \to M'$ *and* $\ell_2 : M \to M''$ *with constants* r_1, θ_1 *and* r_2, θ_2 *respectively,* $\theta_1 \neq \theta_2$. *Then there exists* $M^* \subset \bar M$ *and pseudo-spherical congruences* $\ell_2^* : M' \to M^*$, $\ell_1^* : M'' \to M^*$ *with constants* r_2, θ_2 *and* r_1, θ_1 *respectively, such that*

$$\ell_2^* \circ \ell_1 = \ell_1^* \circ \ell_2 .$$

Proof: Let $v_1, \dots, v_n$ be a local frame given by principal directions on M. Choose an orthonormal frame e_{n+i-1} normal to M, such that $\omega_{n+i-1,n+j-1} = 0$. Consider frames e_1, e_i and $\bar e_1, \bar e_i$ for M, e_1', e_i', e_{n+i-1}' for M' and $e_1'', e_i'', e_{n+i-1}''$ for M'' as in theorem 1, i.e.

$$e_i' = \cos\theta_1\, e_i + \sin\theta_1\, e_{n+i-1} \qquad e_i'' = \cos\theta_2\, \bar e_i + \sin\theta_2\, e_{n+i-1}$$

$$e_{n+i-1}' = -\sin\theta_1\, e_i + \cos\theta_1\, e_{n+i-1} \qquad e_{n+i-1}'' = -\sin\theta_2\, \bar e_i + \cos\theta_2\, e_{n+i-1},$$

(3.1)

$$e_1' = \begin{cases} -e_1, & \text{if } K \equiv 0; \\ \sin r_1 X - \cos r_1 e_1, & \text{if } K \equiv 1; \\ \sinh r_1 X - \cosh r_1 e_1, & \text{if } K \equiv -1; \end{cases} \qquad e_1'' = \begin{cases} -\bar e_1, & \text{if } K \equiv 0; \\ \sin r_2 X - \cos r_2 \bar e_1, & \text{if } K \equiv 1; \\ \sinh r_2 X - \cosh r_2 \bar e_1, & \text{if } K \equiv -1; \end{cases}$$

where X denotes the position vector for M and e_1, $\bar e_1$ are the unit vectors, tangent to the geodesic from P to $P' = \ell_1(P)$ and $P'' = \ell_2(P)$ respectively.

We denote by

$$X' = \begin{cases} X + r_1 e_1, & \text{if } K \equiv 0; \\ \cos r_1 X + \sin r_1 e_1, & \text{if } K \equiv 1; \\ \cosh r_1 X + \sinh r_1 e_1, & \text{if } K \equiv -1. \end{cases}$$

(3.2)

$$X'' = \begin{cases} X + r_2 \bar e_1 & \text{if } K \equiv 0; \\ \cos r_2 X + \sin r_2 \bar e_1, & \text{if } K \equiv 1; \\ \cosh r_2 X + \sinh r_2 \bar e_1, & \text{if } K \equiv -1, \end{cases}$$

the position vectors of M' and M'' respectively.

We consider the following matrix notation

$$(3.3) \qquad \Lambda = \begin{bmatrix} \frac{1}{\sqrt{K-k}} & 0 \\ 0 & I_{n-1} \end{bmatrix} \qquad \bar\Lambda = \begin{bmatrix} -\frac{1}{\sqrt{K-k}} & 0 \\ 0 & I_{n-1} \end{bmatrix}.$$

Moreover, we denote by D_i the matrix introduced in (2.21) for the constants r_i, θ_i, where $i = 1,2$.

Let C be the orthogonal matrix defined by

$$(3.4) \qquad \bar e_I = C_{IJ} e_J ,$$

and consider the matrix B such that

$$(3.5) \qquad BF = E,$$

where

$$F = \bar\Lambda (D_2 - D_2 C)$$

$$E = \Lambda (D_1 C - D_2)$$

We remark that since

$$(3.6) \qquad \Lambda^2 (D_1^2 - D_2^2) = (\cot g^2 \theta_1 - \cot g^2 \theta_2) I_n$$

it follows that $E^t E = F^t F$, hence B is an orthogonal matrix.

We define tangent frames $\bar e_I'$ on M' and $\bar e_I''$ on M'' by

$$(3.7) \qquad \bar e_I' = B_{IJ} e_J' \qquad \bar e_I'' = B_{IJ} e_J'' .$$

We claim that the maps

$$\ell_2^*(X') = \begin{cases} X' + r_2 \bar e_1' & \text{if } K \equiv 0; \\ \cos r_2 X' + \sin r_2 \bar e_1', & \text{if } K \equiv 1; \\ \cosh r_2 X' + \sinh r_2 \bar e_1', & \text{if } K \equiv -1. \end{cases}$$

(3.8)

$$\ell_1^*(X'') = \begin{cases} X'' + r \bar e_1'', & \text{if } K \equiv 0; \\ \cos r_1 X'' + \sin r_1 \bar e_1'', & \text{if } K \equiv 1; \\ \cosh r_1 X'' + \sinh r_1 \bar e_1'', & \text{if } K \equiv -1, \end{cases}$$

define M^* and pseudo-spherical geodesic congruences which satisfy the theorem.

First we prove that $\ell_2^* \circ \ell_1 = \ell_1^* \circ \ell_2$. In fact, using (3.1), (3.2), (3.4) and (3.7), we obtain $\ell_2^* \circ \ell_1(X)$ and $\ell_1^* \circ \ell_2(X)$. The equality follows from (3.5) and Theorem 1. Next, we prove that the frame $\bar{e}_1', \bar{e}_i', e_{n+i-1}'$ on M' satisfy (2.16) with constants r_2, θ_2. We denote by ω and W the matrices of 1-forms introduced in (2.21) associated to the frame e_1, e_i, e_{n+i-1} on M. Similarly, we denote by $\bar{\omega}$, $\bar{W}$ and ω', W' the matrices associated to the frames $\bar{e}_1, \bar{e}_i, e_{n+i-1}$, e_1', e_i', e_{n+i-1}' respectively on M and M'.

Since ℓ_1 is a pseudo-spherical geodesic congruence it follows from Remark 1 that

(3.9) $$\omega = WD_1 - D_1W^t.$$

From (3.4) we get

$$\bar{\omega} = (dC)C^t + C\omega C^t$$

$$\bar{W} = CW.$$

Therefore, it follows from the fact that ℓ_2 is a pseudo-spherical congruence that

(3.10) $$(dC)C^t + C\omega C^t = CWD_2 - D_2W^tC^t.$$

We have to prove that

$$(dB)B^t + B\omega'B^t = BW'D - DW'^tB^t$$

which is equivalent to proving

(3.11) $$(dB)F + B\omega'F = BW'D_2E - D_2W'^tF.$$

From (3.5), we get

(3.12) $$(dB)F = (B\bar{\Lambda}D_2 + \Lambda D_1)dC.$$

It follows from (3.10) and (3.9) that

$$dC = -CW(D_1 - D_2C) + (CD_1 - D_2)W^t.$$

Replacing this expression in (3.12) and considering from (3.5) that

$$B\bar{\Lambda}D_1 + \Lambda D_2 = (B\bar{\Lambda}D_2 + \Lambda D_1)C$$

we obtain

(3.13) $$(dB)F = -(B\bar{\Lambda}D_1 + \Lambda D_2)W(D_1 - D_2C) + B\bar{\Lambda}(D_1^2 - D_2^2)W^t$$

We remark that from (2.5) - (2.8) for $K \equiv 0$; (2.10)-(2.14) for $K \equiv 1$; and (2.16)-(2.19) for $K \equiv -1$ and (3.1) it follows that

(3.14) $$\omega' = D_1\bar{\Lambda}W\bar{\Lambda}^{-1} - \bar{\Lambda}^{-1}W^t\bar{\Lambda}D_1$$

and

(3.15) $$W' = \bar{\Lambda}^{-1}W^t\Lambda.$$

We conclude the proof of (3.11) by considering (3.13)-(3.15) and (3.6).

It follows from the proof of theorem 2, that $\ell_2^*(X')$ as defined in (3.8) is a pseudo-spherical geodesic congruence with constants r_2, θ_2.

Similarly, if we denote by ω'', W'' the matrices of 1-forms (2.21) associated to the frame $e_1'', e_i'', e_{n+i-1}''$ on M'', we prove that

$$(dB)B^t + B\omega''B^t = BW''D_1 - D_1W''^tB^t$$

and therefore that $\ell_1^*(X'')$ is a pseudo-spherical congruence with constants r_1, θ_1. We conclude the proof of the theorem considering

$$X^* = \ell_2^* \circ \ell_1(X) = \ell_1^* \circ \ell_2(X)$$

which defines the position vector of M^*. q.e.d.

Remark 2. The normal frames obtained on M^* considering the orthogonal projections of e_{n+i-1}' and e_{n+i-1}'' on ν^* coincide, i.e.

$$-\sin\theta_2 e'_i + \cos\theta_2 e'_{n+i-1} = -\sin\theta_1 e''_i + \cos\theta_1 e''_{n+i-1}.$$

This follows easily from (3.1)-(3.5) and (3.7).

4. Analytic interpretation

In the first section, we have seen that n-dimensional Riemannian manifolds with constant curvature k, isometrically immersed in a space form $\bar{M}_K^{2n-1}$, $k < K$, are in correspondence with the class of orthogonal, $n\times n$ matrix functions, which satisfy the system of partial differential equations (1.12).

The geometric theory presented in sections 2 and 3 can be interpreted in terms of solution for this system of equations, namely theorems 5 and 6 below provide new solutions from a given one. Moreover, given a Riemannian manifold M^n, with constant curvature k, immersed in a space form $\bar{M}^{2n-1}$, $k < K$, these theorems provide the first and second fundamental forms of new such submanifolds of $\bar{M}$.

As in the previous sections, without loss of generality, we consider $K = 0$, 1 or -1. If $A = (a_{IJ})$ is an orthogonal matrix function of $u_1,\ldots,u_n$, we define $\phi = (\phi_{IJ})$ as

$$\phi_{IJ} = \frac{1}{a_{1I}}\frac{\partial a_{1J}}{\partial u_I}du_J - \frac{1}{a_{1J}}\frac{\partial a_{1I}}{\partial u_J}du_I,$$

and the diagonal matrix

$$\delta = \begin{bmatrix} du_1 & & \\ & \ddots & \\ & & du_n \end{bmatrix}$$

Moreover, for any real number $k < K$, we consider constants θ, r such that, $0 < \theta < \pi$, $r > 0$,

$$\sqrt{K-k} = \begin{cases} \sin\theta/r, & \text{if } K \equiv 0; \\ \sin\theta/\sin r, & \text{if } K \equiv 1; \\ \sin\theta/\sinh r, & \text{if } K \equiv -1, \end{cases}$$

and $r < \pi$ whenever $K \equiv 1$. With the notation (3.3) and (2.21) introduced before, the next theorem gives the analytic version of the geometrical results of section 2.

Theorem 5. *Let* $A:\mathbb{R}^n \to O(n)$ *be a solution of* (1.12), *where the real number* $k < K$. *Then the following first order completely integrable system of equations for* $X:\mathbb{R}^n \to O(n)$,

$$(BT(r,\theta)) \qquad (dX)X^t + X\phi X^t = \sqrt{K-k}(X\delta\Lambda A^t D - D A\Lambda\delta X^t)$$

gives a new solution for (1.12).

Proof: Let A be a solution of (1.12), then $\bar{A} = \sqrt{K-k}\Lambda A$ determines a manifold $M^n \subset \bar{M}_K^{2n-1}$ with constant sectional curvature k.

Let $v_1,\ldots,v_n$ be a locally defined orthonormal frame of principal vectors on M, and $e_{n+1},\ldots,e_{2n-1}$ normal to M such that the normal connection $\omega_{n+i-1,n+j-1} = 0$ $\forall\ 2\le i,j\le n$. We denote by ϕ_I, ϕ_{IJ}, $\phi_{I,n+j-1}$, the 1-forms associated to the above frame.

Let X be a solution of $(BT(r,\theta))$. We consider the tangent frame defined by $e_I = X_{IJ}v_J$, and denote by ω_I, ω_{IJ}, $\omega_{I,n+j-1}$ the 1-forms associated to e_I, e_{n+i-1}. Then,

$$\omega = (dX)X^t + X\phi X^t$$

$$W = X\delta\bar{A}.$$

Since X satisfies $BT(r,\theta)$ it follows that

$$\omega = WD - DW^t$$

i.e. the frame e_I, e_{n+i-1}, satisfies (2.3) (2.4) and $\omega_{n+i-1,n+j-1} = 0$ hence from the proof of theorem 2, there exists a manifold $M' \subset \bar{M}_K^{2n-1}$ and a pseudo-spherical geodesic congruence $\ell:M \to M'$ with constants r, θ. Moreover, if we consider e'_i, e'_{n+i-1} defined on M' as in theorem 1, it follows from Theorem 3 that X is the orthogonal matrix function associated to M' with respect to e'_{n+i-1}. Therefore, X satisfies (1.12).

q.e.d.

The following theorem is the analytic interpretation of section 3, and it says that given a solution A of (1.12) and A_1, A_2 new solutions obtained by solving $(BT(r_i,\theta_i))$ for constants r_i, θ_i, $i=1,2$, $\theta_1 \neq \theta_2$, then a fourth solution can be obtained algebraically. With the same notation as before, we denote by D_i the matrix D defined by (2.21), for the constants r_i, θ_i.

Theorem 6. *Let A be a solution of (1.12) and A_i, $i=1,2$ solutions of the same system obtained from A by solving $(BT(r_i,\theta_i))$. Then a fourth solution A^* can be obtained by solving*

$$A^*A^t = \Lambda(D_1A_2A_1^t - D_2)(D_1 - D_2A_2A_1^t)^{-1}\bar{\Lambda}^{-1}.$$

Proof: Let $M^n \subset \bar{M}_K^{2n-1}$ be the manifold with constant curvature k, associated to A. Let $v_1,\dots,v_n$, $e_{n+1},\dots,e_{2n-1}$ be as in the proof of theorem 5. We consider the frames $e_I = (A_1)_{IJ}v_J$, $\bar{e}_I = (A_2)_{IJ}v_J$ on M. Since A_i satisfies $(BT(r_i,\theta_i))$, there exist M' and M'' submanifolds of $\bar{M}_K^{2n-1}$, and pseudo-spherical geodesic congruences $\ell_1: M \to M'$ and $\ell_2: M \to M''$, with constants r_1, θ_1 and r_2, θ_2 respectively. We consider frames e'_I, e'_{n+i-1} on M' and e''_I, e''_{n+i-1} on M'' as in (3.1). Then A_1 and A_2 are the orthogonal matrix functions, associated to M' and M'', with respect to e'_{n+i-1} and e''_{n+i-1}. We denote by

$$C = A_2A_1^t$$

i.e.

$$\bar{e}_I = C_{IJ}e_J.$$

We consider

$$E = \Lambda(D_1C - D_2),$$

$$F = \bar{\Lambda}(D_1 - D_2C),$$

and B defined by $BF = E$. It follows from the hypothesis, that A^* satisfies

$$A^*A^t = B. \tag{4.1}$$

We have to prove that A^* is a solution of (1.12). It follows from theorem 4, that there exists $M^* \subset \bar{M}_K^{2n-1}$ and pseudo-spherical congruences $\ell_2^*: M' \to M^*$, $\ell_1^*: M'' \to M^*$ with constants r_2, θ_2 and r_1, θ_1 respectively. We consider tangent frames $\bar{e}'_I = B_{IJ}e'_J$, $\bar{e}''_I = B_{IJ}e''_J$ on M' and M'' respectively as in (3.7).

Let $\tilde{A}^*: \mathbb{R}^n \to O(n)$ be the matrix function associated to M^* with respect to e^*_{n+i-1}, obtained by normal projection of e'_{n+i-1} or e''_{n+i-1} on ν^* (see Remark 2), then $\tilde{A}^*$ is a solution of (1.12).

We conclude the proof observing that for the pair of pseudo-spherical geodesic congruences ℓ_1, ℓ_2, we have $C = A_2A_1^t$, $\bar{e}_I = C_{IJ}e_J$, therefore, if we consider the pairs ℓ_1^{-1}, ℓ_2^* and ℓ_2^{-1}, ℓ_1^*, we obtain by analogy that

$$B = \tilde{A}^*A^t.$$

Hence, it follows from (4.1) that $A^* = \tilde{A}^*$ and therefore A^* is a solution of (1.12).

q.e.d.

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Universidade de Brasília
Departamento de Matemática
70.910 - Brasília-DF