BOL. SOC. BRAS. MAT., VOL. 17 Nº 1 (1986), 1-19

CLOSED PRINCIPAL LINES AND BIFURCATION

C. Gutierrez and J. Sotomayor (1)

Abstract. The simplest pattern through which the closed principal lines (cycles) of a one parameter family of immersed surfaces bifurcate, while being apart from umbilical points, is studied in this paper.

1. Introduction. Let M be a compact connected, oriented, two dimensional smooth manifold. An immersion α of M into $\mathbb{R}^3$ is a map such that $D\alpha_p : TM_p \to \mathbb{R}^3$ is one to one, for every $p \in M$. Denote by $J^r = J^r(M, \mathbb{R}^3)$ the set of C^r-immersions of M into $\mathbb{R}$. When endowed with the C^s-topology, $s \leq r$, this set is denoted by $J^{r,s} = J^{r,s}(M, \mathbb{R})$.

Associated to every $\alpha \in J^r$ is defined the normal map $N_\alpha : M \to S^2$:

$$N_\alpha(p) = \frac{\alpha_u \wedge \alpha_v}{\|\alpha_u \wedge \alpha_v\|}$$

where $(u,v) : (M,p) \to (\mathbb{R}^2, 0)$ is a positive chart of M around p, $\wedge$ denotes the exterior product of vectors in $\mathbb{R}^3$, determined by a once for all fixed orientation of $\mathbb{R}^3$, $\alpha_u = \frac{\partial \alpha}{\partial u}$, $\alpha_v = \frac{\partial \alpha}{\partial v}$ and $\| \; \| = < \; , \; >^{\frac{1}{2}}$ is the Euclidean norm in $\mathbb{R}^3$.

Since $DN_\alpha(p)$ has its image contained in the image of $D\alpha(p)$ the endomorphism $\omega_\alpha : TM \to TM$ is well defined by

$$D\alpha . \omega_\alpha = DN_\alpha$$

It is well known that ω_α is a self adjoint endomorphism, when TM is endowed with the metric $< \, , \, >_\alpha$ induced by α from the metric in $\mathbb{R}^3$. Clearly N_α is well defined and of class C^{r-1} in M.

(1) Partially supported by the J.S. Guggenheim Foundation.

Recebido em 06/12/85.

Let $K_\alpha = \det(\omega_\alpha)$ and $H_\alpha = -\frac{1}{2}$ trace (ω_α) be the *Gaussian and Mean curvatures* of the immersion α.

A point $p \in M$ is called an *umbilical point* of α if $(H_\alpha(p))^2 - K_\alpha(p) = 0$. This means that the eigenvalues of ω_α are equal at p. The set of umbilical points of α will be denoted by U_α.

Outside U_α the eigenvalues of ω_α are distinct. Their opposite values given by $K_\alpha = H_\alpha + \{(H_\alpha)^2 - K_\alpha\}^{\frac{1}{2}}$ and $k_\alpha = H_\alpha - \{(H_\alpha)^2 - K_\alpha\}^{\frac{1}{2}}$ are called respectively *maximal* and *minimal principal curvatures* of α. The eigenspaces associated to the principal curvatures define two C^{r-2} line fields L_α and ℓ_α mutually orthogonal in TM (with the metric $\langle\ ,\ \rangle_\alpha$), called *the principal line fields of* α. They are characterized by Rodrigues equations [St].

$$L_\alpha = \{v \in TM;\ \omega_\alpha v + K_\alpha v = 0\}$$

$$\ell_\alpha = \{v \in TM;\ \omega_\alpha v + k_\alpha v = 0\}.$$

The integral curves of L_α (resp. ℓ_α) are called *lines of maximal* (resp. *minimal*) principal curvature. The family of such curves i.e. the integral foliation of L_α (resp. ℓ_α) in $M - U_\alpha$ will be denoted by F_α (resp. $\mathcal{f}_\alpha$) and called the *maximal* (resp. *minimal*) *principal foliation of* α.

The triple $P_\alpha = (U_\alpha, F_\alpha, \mathcal{f}_\alpha)$ will be called the *principal configuration of* α.

The global structure of principal configurations is known only for very rare classical surfaces: surfaces of revolution and surfaces which belong to a triply orthogonal system of surfaces ([St], [Ch. Theorem 6.3 of Chapter 3]). In the first case the principal foliations are contained in the parallel and meridian curves and the umbilical points form meridian curves.

In the case of triply orthogonal systems of surfaces, the principal foliations of a surface of one of the systems are obtained intersecting the surface with the elements of the other two systems. This result can be used to visualize the principal configuration of the ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1, \quad 0<a<b<c.$$

This is done by considering this ellipsoid $E(0)$, as an element of the triply orthogonal family of "confocal quadrics" $E(\lambda)$, $H_1(\lambda)$, $H_2(\lambda)$, defined by

$$\frac{x^2}{(a^2-\lambda)^2} + \frac{x^2}{(b^2-\lambda)^2} + \frac{x^2}{(c^2-\lambda)^2} = 1,$$

with $\lambda < a^2$ for $E(\lambda)$ (ellipsoids), $a^2 < \lambda < b^2$ for $H_1(\lambda)$ (hyperboloids of one sheet) and $b^2 < \lambda < c^2$ for $H_2(\lambda)$ (hyperboloids of two sheets).

This shows that, except for four principal lines which join the four umbilical points of $E(0)$, all other principal lines of $E(0)$ are closed (principal cycles).

The possible principal configurations of immersions for which the mean curvature $H_\alpha = \frac{1}{2}(K_\alpha + k_\alpha)$ is a constant, have been characterized by Gutierrez and Sotomayor in [G-S.4], where local analytical models for the principal configurations around umbilical points as well as natural transversal measures, invariant under F_α and $\mathcal{f}_\alpha$, have been found. As a consequence it was shown that the principal cycles for these immersions always appear packed in open cylinders.

The study of isolated principal cycles seems to have been considered for the first time by Gutierrez and Sotomayor in [G-S.1] and [G-S.2], where the structural stability and genericity properties of principal configurations were established. These properties will be reviewed below because they constitute the starting point for the present work, whose main concern will be the study of the simplest patterns through which isolated principal cycles approach each other, loose their structural stability and bifurcate as the immersion changes along one parameter families of immersions α_t. The present study is restricted to the bifurcations that occur away from umbilical points. Other patterns of the bifurcations of lines of principal curvature and umbilical points have been studied in [G-S.3], [G-S.5], [G-S.6].

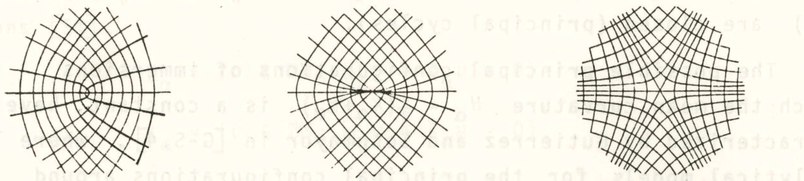
2. Formulation of the main results

2.1. Umbilical points

Let $(u,v):(M,p) \to (\mathbb{R}^2,0)$ be a chart on M with $p \in U_\alpha$ and Γ be an isometry of $\mathbb{R}^3$ with $\Gamma(\alpha(p)) = 0$, such that $\Gamma \circ \alpha(u,v) = (u,v,h(u,v))$, with 3-jet at 0 given by

$$J_0^3 h(u,v) = (k/2)(u^2+v^2) + (a/6)u^3 + (b/2)uv^2 + (c/6)v^3.$$

Below are defined three different types of umbilical points and their local principal configurations are illustrated. These three points, denoted D_1, D_2 and D_3, are called *Darbouxian* or of type D [G-S.1].



D_1: $b(b-a)\neq 0$, $a/b>(c/2b)^2+2$ D_2: $b(b-a)\neq 0$, $1<a/b<(c/2b)^2+2$ D_3: $b(b-a)\neq 0$, $1>a/b$

Fig. 2.1

The index $i=1,2,3$ of D_i denotes the number of *umbilical separatrices* of p. These are principal lines which approach the umbilical point p and which separate regions of different patterns of approach to p.

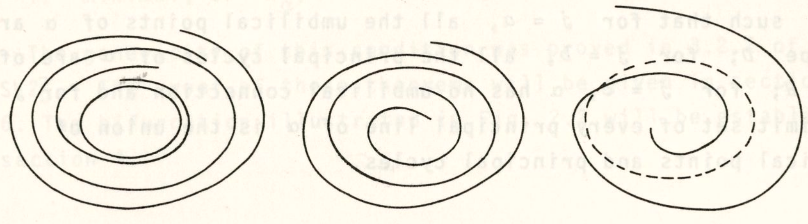
2.2. Principal cycles

A compact line c of F_α (resp. f_α) is called *maximal* (resp. *minimal*) *principal cycle* of α.

Call $\pi = \pi_c$ the Poincaré first return map (holonomy) defined by the lines of the foliation to which c belongs, defined on a segment of a line of the orthogonal foliation through 0 in c.

A cycle is called of *type H* or *hyperbolic* if $\pi_0' \neq 1$ and of *type SH* or *semi-hyperbolic* if $\pi_0' = 1$ and $\pi_0'' \neq 0$.

The bifurcation of a semi-hyperbolic cycle is illustrated in Figure 2.2.



Two H cycles an SH cycle appears no cycles

Fig. 2.2

Is is proved that hyperbolicity of c is equivalent to [G-S.1, Proposition 4.1]

$$\int_c \frac{dk_\alpha}{K_\alpha - k_\alpha} = \int_c \frac{dK_\alpha}{K_\alpha - k_\alpha} \neq 0.$$

A more complex integral formula, envolving the geodesic curvature of c and the derivative, in the normal direction to c, of K_α and k_α, will be found to characterize the semi-hyperbolicity of c. See Proposition 3.1, formula 2.

2.3. The main results

Let C be a subset of J^r. An element α belonging to C is said to be *C^s-structurally stable relative to* C, (resp. *C^s-structurally stable along* C) if there is a neighborhood V of α in $J^{r,s}$ such that for every $\beta \in V \cap C$ (resp. β in the connected component of α in $V \cap C$), there is a homeomorphism $h = h_\beta$ of M which maps U_α onto U_β and the lines of F_α and f_α respectively onto those of F_β and f_β. When C is the whole J^r, α is called simply *C^s-structurally stable*.

A principal line z which is a separatriz of two different umbilical points p, q of α or twice a separatrix of the same umbilical point p of α is called an *umbilical connection* of α; in the second case z is also called an *umbilical loop*.

Call $S^r(j)$, $j=a,b,c,d$, respectively, the set of $\alpha \in J^r$, $r \geq 4$ such that for $j = a$, all the umbilical points of α are of type D; for $j = b$, all the principal cycles of α are of type H; for $j = c$, α has no umbilical connection and for $j = d$, the limit set of every principal line of α is the union of umbilical points and principal cycles.

Theorem ([G-S.1], [G-S.2])

$S^r = \cap S^r(j)$, $j=a,b,c,d$, $r \geq 4$, is open in $J^{r,3}$ and dense in $J^{r,2}$. Every $\alpha \in S^r$ is C^3-structurally stable.

Call $S_1^r(b)$, the set of $\alpha \in S^r(a) \cap S^r(c) \cap S^r(d)$, $r \geq 4$, such that all of its principal cycles are of type H except one, which is of type SH.

Call $\tilde{S}_1^r(b)$ the subset of immersions in $S_1^r(b)$ for which the SH principal cycle is not the limit simultaneously, from both sides, of a pair umbilical separatrices or of any single principal line.

Let $J_1^{r,s}$ (resp. J_1^r) be the subspace (resp. the subset) $J^r - S^r$ of $J^{r,s}$ (resp. J^r).

Theorem 1. (Stability and smooth structure). Let $r \geq 4$.

a) $S_1^r(b)$ (resp. $\tilde{S}_1^r(b)$) is a one to one immersed Banach submanifold (resp. embedded submanifold) of codimension one and class C^{r-3} of $J^{r,r}$.

b) $\tilde{S}_1^r(b)$ is an open subset of $J_1^{r,4}$.

c) Every $\alpha \in S_1^r(b)$ (resp. $\alpha \in \tilde{S}_1^r(b)$) is C^4-structurally stable along $S_1^r(b)$ (resp. relative to J_1^r).

Theorem 2 (Density). The set $\tilde{S}_1^r(b)$ is dense in the subspace $J_1^r(b,*)$ of $J_1^{r,2}$, where $J_1^r(b,*)$ denotes the subset in J^r of immersions α having some non-hyperbolic principal cycle on which k_α, if minimal, or K_α, if maximal, is not constant.

The genericity of this condition was proved in 3.2.2 of [G-S.2]. The proof of these theorems will be given in sections 3 to 6. The bifurcation illustrated in Fig. 2.2 will be established in section 4.

3. An integral expression for *SH* cycles

A characterization of semi-hyperbolic principal cycles involving the geometric invariants of the immersion is found in Proposition 3.1 below.

Given $\alpha \in J^r$ and (u,v)-coordinates, in what follows $(\partial/\partial u)\alpha$, $(\partial^2/\partial u \partial v)\alpha$, ... will be denoted by α_u, α_{uv},... respectively. Moreover the following notation will be used:

$$E = \langle\alpha_u, \alpha_u\rangle \qquad e = \langle\alpha_u \wedge \alpha_v, \alpha_{uu}\rangle$$

$$F = \langle\alpha_u, \alpha_v\rangle \qquad f = \langle\alpha_u \wedge \alpha_v, \alpha_{uv}\rangle$$

$$G = \langle\alpha_v, \alpha_v\rangle \qquad g = \langle\alpha_u \wedge \alpha_v, \alpha_{vv}\rangle$$

Here E, F, G and $e/\|\alpha_u \wedge \alpha_v\|$, $f/\|\alpha_u \wedge \alpha_v\|$, $g/\|\alpha_u \wedge \alpha_v\|$ are respectively the coefficients of the first and second fundamental forms of α, expressed in the coordinates (u,v).

3.1. Proposition

Let c be a minimal non-hyperbolic principal cycle of $\alpha \in J^r$. Give (u,v)-coordinates defined around c which satisfy (Cf. [G-S,1])

$$(1) \qquad \alpha(u,v) = \alpha\circ c(u) + vN_\alpha\circ c(u)\wedge T_\alpha(u) + v^2\Big[\frac{K_\alpha\circ c}{2}(u) + A(u,v)\Big]N_\alpha\circ c(u),$$

where $c:\mathbb{R}\to M^2$ is the minimal principal cycle s-periodic on u and parametrized by arc length, $T_\alpha(u) = (\alpha\circ c)'(u)$, N_α is the positive normal of α and K_α is the maximal principal curvature. Then, the return map π of c satisfies

$$(2) \qquad \pi''(0) = \int_0^s \left(\left(\exp\Big[\int_0^u \frac{K'}{K-k}ds\Big]\right)\left[\frac{2k_g(K-k)H_u + 2k'H_v}{(K-k)^2}\right]\right)du.$$

Where $K = K_\alpha\circ c$ is the minimal principal curvature, $H = H_\alpha\circ c$ is the mean curvature and k_g is the geodesic curvature of c in $\alpha(M)$.

Proof. The differential equation of lines of principal curvature is given by [St]:

$$(3) \qquad (dv)^2(Fg-Gf) + (du)(dv)(Eg-Ge) + (du)^2(Ef-eF) = 0.$$

Call $v = v(u,v_0)$ the solution of (3) with $v(0,v_0) = v_0$. Clearly the return map π of c is given by $\pi(v_0) = v(s,v_0)$.

Call $\eta(u) = \frac{\partial v}{\partial v_0}(u,0)$, $\nu = \frac{\partial^2 v}{\partial v_0^2}(u,0)$.

By substituting $v = v(u,v_0)$ into (3) and differentiating twice, follows that η' and ν' satisfy the following system of linear differential equations:

$$(4) \qquad \eta'[Eg-Ge] + \nu[Ef-Fe]_v = 0,$$

$$(5) \qquad \nu'[Eg-Ge] + \nu[Ef-Fe]_v + 2(\eta')^2[Fg-Gf] + 2\eta'\eta[Eg-Ge]_v + \eta^2[Ef-Fe]_{vv} = 0$$

It will be seen in (23) that $[Fg-Gf](u,0) \equiv 0$. Therefore, integrating, follows

$$(6) \qquad \eta(u) = \exp\Big[-\int_0^u \frac{[Ef-Fe]_v}{[Eg-Ge]}du\Big],$$

$$(7) \qquad \nu(s) = -\eta(s)\int_0^s \eta(u)\cdot\left[\frac{-2[Ef-Fe]_v[Eg-Ge]_v + [Eg-Ge][Ef-Fe]_{vv}}{[Eg-Ge]^2}\right]du.$$

To find a more explicit expression for $\pi'(0) = \eta(s)$, $\pi''(0) = \nu(s)$, write

$$A = A(u,v), \qquad N = N_\alpha\circ c, \qquad T = T_\alpha.$$

Therefore, (1) adopts the following simplified form

$$(8) \qquad \alpha(u,v) = \alpha\circ c(u) + vN(u)\wedge T(u) + v^2\Big[\frac{K(u)}{2} + A(u,v)\Big]N(u).$$

The Frenet equations for $\alpha\circ c$ in $\alpha(M)$ can be written as

$$(9) \qquad \begin{aligned} T'(u) &= k_g N\wedge T + kN \\ (N\wedge T)'(u) &= -k_g T \\ N'(u) &= -kT. \end{aligned}$$

Differentiating (8) and using (9), it is obtained:

$$(10) \qquad \alpha_u = \Big(1 - k_g v - \frac{kK}{2}v^2 - kv^2A\Big)T(u) + \Big(\frac{K'v^2}{2} + A_u v^2\Big)N(u)$$

$$(11) \qquad \alpha_v = N\wedge T + \{A_v v^2 + Kv + 2Av\}N(u)$$

$$(12) \qquad \begin{aligned} \alpha_{uu} = &\Big(k_g'v - \frac{k'K}{2}v^2 - k'Av^2 - kK'v^2 - 2kA_u v^2\Big)T(u) \\ &+ \Big(k - kk_g v - \frac{k^2K}{2}v^2 - k^2Av^2 + \frac{K''}{2}v^2 + A_{uu}v^2\Big)N(u) \\ &+ \Big(k_g - k_g^2 v - \frac{kKk_g}{2}v^2 - kk_g Av^2\Big)N\wedge T \end{aligned}$$

$$\alpha_{uv} = \{-k_g - kKv - 2kAv - kA_v v^2\}T(u) + \tag{13}$$
$$+ \{K'v + 2A_u v + A_{uv}v^2\}N(u)$$

$$\alpha_{vu} = \{-k_g - kKv - 2kAv - kA_v v^2\}T(u) \tag{14}$$
$$+ \{v^2 A_{uv} + K'v + 2vA_u\}N(u)$$

$$\alpha_{vv} = \{4vA_v + v^2 A_{vv} + K + 2A\}N(u). \tag{15}$$

From (10) and (11), it results

$$\alpha_u \wedge \alpha_v = \left[1 - k_g v - \frac{kK}{2}v^2 - kv^2 A\right]N - \left[\frac{K'v^2}{2} + A_u v^2\right]T \tag{16}$$
$$- \left[1 - k_g v - \frac{kK}{2}v^2 - kv^2 A\right]\left[A_v v^2 + Kv + 2Av\right]N \wedge T$$

Also, the coefficients of the first fundamental form are

$$E(u,v) = \left[1 - k_g v - \frac{kK}{2}v^2 - kv^2 A\right]^2 + \left[\frac{K'v^2}{2} + A_u v^2\right]^2 \tag{17}$$

$$F(u,v) = \left[\frac{K'v^2}{2} + A_u v^2\right]\{A_v v^2 + Kv + 2Av\}$$

$$G(u,v) = 1 + \{A_v v^2 + Kv + 2Av\}^2 .$$

Moreover,

$$e(u,v) = k - 2kk_g v - k_g Kv - k^2 Kv^2 + kk_g^2 v^2 + \tag{18}$$
$$+ \frac{K''}{2}v^2 + 2k_g^2 Kv^2 - k_g v^2 A_v - 2k_g vA + 0(v^3)$$

$$f(u,v) = K'v - \frac{k_g K'}{2}v^2 + v^2 A_{uv} + 2vA_u + 0(v^3)$$

$$g(u,v) = K - Kk_g v + 2A + 4vA_v - 4v^2 k_g A_v +$$
$$+ v^2 A_{vv} - \frac{kK^2}{2}v^2 - 2vk_g A + 0(v^3).$$

Therefore, it follows that

$$E(u,0) = 1, \quad E_v(u,0) = -2k_g, \quad E_u(u,0) = 0 \tag{19}$$

$$E_{uv}(u,0) = -2k_g', \quad E_{vv}(u,0) = 2(k_g^2 - kK)$$

$$F(u,0) = F_v(u,0) = F_u(u,0) = F_{vv}(u,0) = F_{uv}(u,0) = 0$$

$$G(u,0) = 1, \quad G_v(u,0) = G_{uv}(u,0) = G_u(u,0) = 0.$$

$$e(u,0) = k, \quad e_u(u,0) = k'. \tag{20}$$

$$e_v(u,0) = -2kk_g - k_g K, \quad e_{uv}(u,0) = -2(kk_g)' - (k_g K)',$$

$$e_{vv}(u,0) = 2(-k^2 K + kk_g^2 + \frac{K''}{2} + 2k_g^2 K - 3k_g A_v),$$

$$f(u,0) = 0, \quad f_v(u,0) = K' \tag{21}$$

$$f_{vv}(u,0) = 2(-\frac{k_g K'}{2} + 3A_{uv}),$$

$$g(u,0) = K, \quad g_u(u,0) = K' \tag{22}$$

$$g_v(u,0) = (-Kk_g + 6A_v), \quad g_{uv}(u,0) = -(Kk_g)' + 6A_{uv}.$$

Also, from (19)-(22) follows that

$$[Fg-Gf](u,0) = 0 \tag{23}$$

$$[Eg-Ge]_v(u,0) = -2k_g(K-k) + 6A_v$$

$$[Eg-Ge](u,0) = K-k$$

$$[Ef-Fe]_v(u,0) = K'$$

$$[Ef-Fe]_{vv}(u,0) = -5k_g K' + 6A_{uv}.$$

Substituing (23) into (7) leads to

$$\pi''(0) = \eta(s)\int_0^s \eta(u)\left\{\frac{[-2K'][-2k_g(K-k) + 6A_v] + [-5k_g K' + 6A_{uv}](K-k)}{(K-k)^2}\right\}du . \tag{24}$$

Integration by parts gives

(25) $$\int_0^s \frac{6\eta(u)A_{uv}}{K-k}du = \int_0^s \frac{6\eta(u)A_v(2K'-k')}{(K-k)^2}du.$$

Therefore, (24) can be written as

(26) $$\pi''(0) = -\eta(s)\int_0^s \eta(u)\frac{k_gK'(K-k)+6k'A_v}{(K-k)^2}du.$$

From $H=\frac{K+k}{2}$, differentiating $2H(EG-F^2)^{\frac{3}{2}} = eG-2fF+gE$, follows from (19) that

(27) $$2H_v - 6Hk_g = 6A_v - 2kk_g - 4k_gK = 6A_v - 4k_gH - 2k_gK$$

$$2H_v = -k_g(K-k)+6A_v$$

$$6A_v = 2H_v + k_g(K-k).$$

Substituing $6A_v$ into (26), it follows

(28) $$\pi''(0) = \eta(s)\int_0^s \eta(u)\left[\frac{2k_g(K-k)H_u+2k'H_v}{(K-k)^2}\right]du.$$

Substituing (23) into (6) and then into (28), results that

(29) $$\pi''(0) = \left\{\exp\left[-\int_0^s \frac{dK}{K-k}\right]\right\}\left\{\int_0^s \exp\left[\int_0^u \frac{K'}{K-k}ds\right] \cdot \left[\frac{2k_g(K-k)H_u+2k'H_v}{(K-k)^2}\right]du\right\}.$$

For non-hyperbolic principal cycles,

(30) $$\pi'(0)=\eta(s)=\exp\left[-\int_0^s\frac{dK}{K-k}\right]=1.$$

Therefore,

(31) $$\pi''(0) = \int_0^s \left\{\left(\exp\left[\int_0^u \frac{K'}{K-k}ds\right]\right)\left[\frac{2k_g(K-k)H_u + 2k'H_v}{(K-k)^2}\right]\right\}du.$$

This proves the proposition.

3.2. Remark

If c is a maximal principal cycle of $\alpha \in J^r$, Proposition 3.1 implies that:

$$\pi''(0) = \int_0^s\left\{\left(\exp\left[\int_0^u \frac{-k'}{K-k}ds\right]\right)\left[\frac{2k_g(K-k)H_u+2K'H_v}{(K-k)^2}\right]\right\}du.$$

4. Smoothness of $S_1^r(b)$ and bifurcations

It is clear now how to define $S_1^r(b)$ implicitly near $\alpha_0 \in S_1^r(b)$. In fact, $S_1^r(b) = B^{-1}(0)$, where B is the differentiable function

(1) $$B(\alpha)=\pi_\alpha(x(\alpha)) - x(\alpha).$$

The point $x=x(\alpha)$ is defined implicitly by the condition $\pi_\alpha'(x(\alpha))=1$.

Clearly, the derivative of B at α_0 on the direction $\tilde\alpha$ is

(2) $$DB_{\alpha_0}(\tilde\alpha) = D_\alpha(\pi_{\alpha_0}(0))(\tilde\alpha)$$

To show that $DB_{\alpha_0}\neq 0$, the following lemma will be nedded.

4.1. Lemma [G-S.2, Lemma 4.3]

Let $\alpha\in J^s$, $\infty \geq s \geq 3$, and $p\in M$ be such that $dk|_{\ell_\alpha(p)}\neq 0$.

Let $(u,v):M\to 2I\times 2I$ be coordinates such that

$(u(p), v(p)) = (0,0)$, $\ell_\alpha = \mathbb{R}(\partial/\partial u)$ and $L_\alpha = \mathbb{R}(\partial/\partial v)$, where $I = [-1,1]$. Then given any $\varepsilon > 0$ and any sequence of C^r-norms $\|\ \|_r$, $r = 2,3,\ldots$, on J^s, $s \geq r+1$, there are numbers $\delta = \delta(\varepsilon, \|\ \|_2) > 0$ and $\tau = \tau(\varepsilon, \|\ \|_2) > 0$ such that for any $\rho \in (0,\delta]$ and any $p_0 \in u^{-1}(-1) \cap v^{-1}((1-2\delta)I)$ it is possible to construct a C^{s-1} family $\{\alpha_\mu\}$, $\mu \in [-1,1]$, of C^{s-1} immersions which satisfy the following conditions:

i) The support of $\alpha_\mu - \alpha$ is contained in $D = (u,v)^{-1}([-1,1] \times \{v(p_0) + 2\rho I\})$ and $\alpha_0 = \alpha$.

ii) For all $\mu \in [-1,1]$, $\|\alpha_\mu - \alpha\|_2 < \varepsilon$

iii) The minimal principal arc of $\alpha_\mu|_D$ which passes through p_0 meets the segment $u^{-1}(1)$ in a point denoted by $\xi_\mu(p_0)$. The range of the map $\mu \to v(\xi_\mu(p_0))$, $\mu \in [0,1]$, contains the interval $[v(p_0), v(p_0)+\rho\tau]$. See fig 4.1. Moreover $\frac{\partial}{\partial\mu}(\xi_\mu(p_0)) \neq 0$.

iv) There exists $\xi_0 = \mu_0(r) > 0$ such that $\|\alpha_{\mu_0} - \alpha\|_r < \varepsilon$ and $v(\xi_{\mu_0}(p_0)) > v(p_0)$.

Under the conditions of Lemma 4.1 applied to α_0 and the point $x(\alpha_0) = p$ (appropiately chosen), denote by π_μ the Poincaré return map induced by the minimal foliation of α_μ. Write $\pi_\mu = \tilde{\pi} \circ \xi_\mu$, where $\tilde{\pi}: u^{-1}(1) \to u^{-1}(-1)$. By iii) of Lemma 4.1, $\frac{\partial}{\partial\mu}(\tilde{\pi}\, \xi_\mu(p))\Big|_{\mu=0} \neq 0$. Therefore $DB_{\alpha_0}(\beta) \neq 0$, where $\beta = \frac{\partial}{\partial\mu}(\alpha_\mu)\Big|_{\mu=0}$.

Now take a C^r family $\{\tilde{\alpha}_\mu\}$ of C^r immersions which is C^3 close to $\{\alpha_\mu\}$ and satisfies $\tilde{\alpha}_0 = \alpha_0$. It follows that $DB_{\alpha_0}(\beta) \neq 0$, where $\beta = \frac{\partial}{\partial\mu}(\tilde{\alpha}_\mu)\Big|_{\mu=0}$.

This proves the differentiability of the manifold $S_1^r(b)$. It follows from the analysis of zeroes of B that, by crossing $S_1^r(b)$ transversally, the SH principal cycle splits into two H cycles or dissappear as illustrated in Figure 2.2.

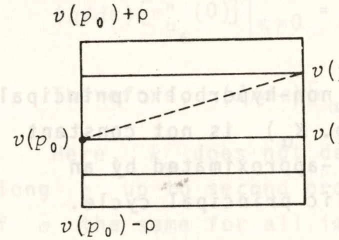


Fig. 4.1

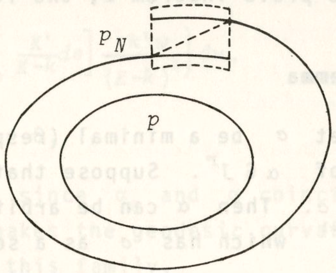


Fig. 5.1

The conclusion that $S_1^r(b)$ (resp. $\tilde{S}_1^r(b)$) is a one-to-one immersed (resp. embedded) submanifold of J^r can be obtained by projecting the embedded submanifold S_1^r of $\mathbb{R}_+ \times J^r$ which consists of pairs (s,α) such that $\alpha \in S_1^r(b)$ and s is the lenght of the SH principal cycle of α. The proof that $\tilde{S}_1^r(b)$ is embedded submanifold is similar to that of the case of vector fields ([So], [A-L]).

5. Proof of the main theorems

5.1. Proof of Theorem 1

a) This has been done in section 4.

b) The proof of this part is similar to that of the case of vector fields ([So], [A-L]). In fact when the SH closed principal cycle is destroyed by a small perturbation of $\alpha \in \tilde{S}_1^r(b)$, the resulting immersion belongs to S^r. This is not the case for $\alpha \in S_1^r(b) - \tilde{S}_1^r(b)$. Actually, there are arbitrarily small perturbations for which the SH principal cycle disappears and have other SH principal cycles or umbilical connections with arbitrarily large lenghts.

c) As in [G-S.1, Section 5], the method of cannonical regions applies to this case.

To prove Theorem 2, the following lemma will be needed.

5.2. Lemma

Let c be a minimal (resp. maximal) non-hyperbolic principal cycle of $\alpha \in J^r$. Suppose that k_α (resp. K_α) is not constant along c. Then α can be arbitrarily C^{r-1}-approximated by an $\tilde{\alpha} \in J^{r-1}$ which has c as a semi-hyperbolic principal cycle.

Proof. Let $p \in c$ such that $dk_\alpha|_{\ell(p)} \neq 0$ and let w be a non-negative smooth function such that $w(p) = 1$. It may be assumed that dk_α is non-zero on ℓ restricted to the support of w. Take coordinates (u,v), s-periodic in u, so that $F(u,0) \equiv 0$ and the minimal principal cycle c of lenght s is given by $v = 0$. For ε small define in these coordinates:

$$\alpha_\varepsilon = \alpha + (\varepsilon/6)v^3 w N_\alpha.$$

It will be seen that, for any $\varepsilon \neq 0$ small α_ε has c as a semi-hyperbolic principal cycle.

Certainly

$$k_{\alpha_\varepsilon} = k_\alpha, \text{ on } c.$$

Moreover, in a small neighborhood of c, for some real valued functions $R_1 = R_1(u,v)$ and $R_2 = R_2(u,v)$, it is satisfied:

$$K_{\alpha_\varepsilon} = K_\alpha + \varepsilon v w + v^2 R_1$$

$$H_{\alpha_\varepsilon} = H_\alpha + (\varepsilon/2)vw + v^2 R_2.$$

Therefore, by (30) of Proposition 3.1 which establishes the integral formula for π' in 2.2, follows that $\pi'_{\alpha_\varepsilon}(0) = \pi'_\alpha(0) = 1$ and moreover by Proposition 3.1,

$$\pi''_{\alpha_\varepsilon}(0) = \int_0^s \left(\left[\exp\left[\int_0^u \frac{K'}{K-k}ds\right]\right]\left[\frac{2k_g(K-k)H_u + 2k'\left[H_v + (\varepsilon/2)w\right]}{(K-k)^2}\right]\right)du.$$

where $(K_\alpha, k_\alpha, H_\alpha) = (K,k,H)$. Thus

$$(d/d\varepsilon)\left[\pi''_{\alpha_\varepsilon}(0)\right]\Big|_{\varepsilon=0} = \int_0^s \left[\exp\left[\int_0^u \frac{K'}{K-k}ds\right]\frac{k'w}{(K-k)^2}\right]du.$$

Therefore, $(d/d\varepsilon)\left[\pi_{\alpha_\varepsilon}''(0)\right]\Big|_{\varepsilon=0} \neq 0.$

Here, k_g does not depend on ε since α_ε and α coincide along c up to second order, which makes the geodesic curvature of c the same for all immersions of this family.

5.3. Proof of Theorem 2

Let $\alpha \in J^r(b,*)$. Consider only the case in which α has a minimal principal cycle on which k_α in not constant. By Lemma 5.2. There exists $\alpha_1 \in J^{r-1}$ which is C^2-close to α and has c as a semi-hyperbolic minimal principal cycle. The same argument used in section 4 to prove that $S_1^r(b)$ is a codimension one submanifold of $J^{r,r}$, shows that if V and $V \subset J^{r-1,r-1}$ are small open neighborhoods of c and α_1, respectively; then the set of $\beta \in V$ having a semi-hyperbolic minimal principal cycle contained in V is a codimension one submanifold of $J^{r-1,r-1}$. Therefore, an immersion $\tilde{\alpha} \in J^\infty$, C^2-close to α_1, and having a semi-hyperbolic minimal principal cycle in V can be found as the transversal intersection of this submanifold with an appropriate curve of immersions of J^∞.

Now it will be seen that, by a small C^2-perturbation of $\tilde{\alpha}$ away of its semi-hyperbolic principal cycle $\tilde{c}$, it may be assumed that there is a very small open cylinder containing $\tilde{c}$ and bounded by minimal principal cycles. In fact. Let $\{p_n\}$ be a sequence of consecutive intersections of a minimal principal line approaching $\tilde{c}$ with a maximal principal line crossing $\tilde{c}$. Certainly $\lim_n p_n = q \in \tilde{c}$. Assume that $dk_{\tilde{\alpha}}(q)\ell_{\tilde{\alpha}}(q)$ is not zero and consider (u,v)-coordinates as in Lemma 4.1 such that $(u(q),v(q)) = 0$. As the derivative of the first Poincaré return map at q (induced by the minimal foliation of $\tilde{\alpha}$) is 1, $v(p_n)/v(p_{n+1})$ goes to 1 as n goes to infinity. Therefore, by Lemma 4.1, by a small C^2-perturbation away from $\tilde{c}$ and for N large enough, it may be assumed that there is a minimal principal cycle passing through p_N. See Figure 5.1.

Under these conditions, the proof that α can be arbitrarily C^2-approximated by an element $\beta \in \tilde{S}^r$ which has unique semi-hyperbolic minimal principal cycle is similar to that of [G-S.2, Theorem 3.1].

6. Acknowledgement

The authors are grateful for the support received from California Institute of Technology (C. Gutierrez) and Brown University (J. Sotomayor) during the preparation of this work.

References

[A-L] Andronov A; Leontovich E. et al., *Theory of Bifurcations of Dynamical Systems on the plane*, John Wiley, New York, 1973.

[Ch] Chuan-Chih Hsiung, *A first course in differential geometry*, John Wiley and Sons, N.Y. 1981.

[G-S.1] Gutierrez C., Sotomayor J. *Structurally stable configurations of lines of principal curvature.* Asterisque 98-99 (1982.

[G-S.2] Gutierrez C. Sotomayor J. *An approximation theorem for immersions with stable configurations of lines of principal curvature.* Springer Lectures Notes in Math. 1007, (1983).

[G-S.3] Gutierrez C., Sotomayor J., *Stability and bifurcations of configurations of principal lines.* Preprint IMPA, Aport. Mat., 1 (1985), Soc. Mat. Mex.

[G-S.4] Gutierrez C., Sotomayor J., *Principal lines on surfaces immersed with constant mean curvature.* Trans. Amer. Math. Soc., 239 (1986).

[G-S.5] Gutierrez C., Sotomayor J., *Bifurcations of umbilical points and related principal cycles.* In preparation.

[G-S.6] Gutierrez C., Sotomayor J., *Bifurcations of principal lines connecting umbilical points.* Preprint IMPA, 1986.

[So] Sotomayor J., *Generic one-parameter families of vector fields on two-dimensional manifolds.* Publ. Math. IHES, 43 (1974).

[St] Struik D., *Lectures on classical differential geometry*, Addison Wesley, 1950.

Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina 110
Jardim Botânico
22.460 Rio de Janeiro-RJ