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CONVEX IMMERSIONS INTO POSITIVELY-CURVED MANIFOLDS

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1. Introduction

1.1 - Let N be a Riemannian manifold. We say that $K \subset N$ is *strongly convex* if for any pair of points $p,q \in K$ there exists a unique minimal geodesic γ of N connecting p to q and γ is contained in K. We say that $K \subset N$ is *convex*, if for each point p of the closure $\bar{K}$ of K there exists a number $0<r(p)\leq c(p)$ such that $K \cap B_{r(p)}(p)$ is strongly convex; here $c(p)$ is the convexity radius and $B_{r(p)}(p)$ denotes the open ball with center in p and radius $r(p)$. We say that K is *totally convex* if whenever $p,q \in K$ and γ is a geodesic segment from p to q, then γ is contained in K. If K is convex and its interior, int K, is non empty we say that K is a *convex body*. The fundamental properties about convex sets can be found in [5].

1.2 - We will represent by $<\ ,\ >$ and $\bar{\nabla}$ the Riemannian metric and Riemannian connexion of N, respectively. We will denote by $K_N(X,Y)_p$ the sectional curvature of N at the point p relative to the plane generated by the vectors X and Y of the tangent space T_pN of N. When clear from the context, we will only use K_N.

Let $x: M \to N$ be a isometric immersion of a Riemannian manifold M into N. We will identify a vector V of T_pM with $dx_p(V)$ of $T_{x(p)}N$, and for V, W in T_pM we will identify $K_N(V,W)_{x(p)}$ with $K_N(dx_p(V), dx_p(W))_{x(p)}$. The notation

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$K_M > K_N$ will express that for every point $p \in M$ and for every pair of linearly independent vectors $V,W \in T_pM$ we have that $K_M(V,W)_p > K_N(V,W)_{x(p)}$.

1.3 - M and N will indicate orientable complete and connected C^∞-Riemannian manifold with dimensions n and $n+1$ $(n \geq 2)$, respectively.

Our main result is as follows

1.4 Theorem. *Let $x: M \to N$ be a isometric immersion. Suppose that N is noncompact and that there exist a constant K such that $K \geq K_N > 0$. Suppose further that it is possible to choose a unit normal vector field ξ in M so that each eigenvalue λ of the second fundamental form of x with respect to ξ satisfies $\lambda \geq 2\sqrt{K}$. Then x is a embedding, and $x(M)$ is the boundary of a convex body in N. In particular, M is diffeomorphic to a sphere.*

This theorem is a result of our Doctoral Thesis ([13, p. 43], announced in [14] as Theorem D).

1.5 Remark. Our theorem generalizes a series of results that have appeared in the literature: [6], [12], [11] and [4]. It should be specially compared with a result of S. Alexander [1] where N is simply-connected and has nonpositive sectional curvature.

The proof of Theorem 1.4 will be presented in Section 3 after some preliminary facts which will be proved in the next section.

2. Some general basic results

We will use the following property of convex bodies in a Riemannian manifold.

2.1 Lemma. *Let A be a convex body of a Riemannian manifold L such that its boundary S is a submanifold of L. If $\gamma(t)$ is a geodesic of L tangent to S in $p = \gamma(0)$, there exists $\delta > 0$ such that $\gamma(t) \in L\text{-}A$ for all $t \in (-\delta,\delta)$.*

Proof. Let ξ_p be the unit normal vector of S at p, such that for $s > 0$ and sufficiently small $\exp_p(s\xi_p) \in L\text{-}A$. Suppose that for all $\delta > 0$, there exists $t \in (-\delta,\delta)$ such that $\gamma(t) \in A$. Since A is a convex body of L, there exists a number $r = r(p) > 0$ such that $C = B_r(p) \cap A$ is open and strongly convex. Let $\gamma(t_0)$ be a point of γ inside C. Since C is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(\gamma(t_0)) \subset C$. By continuity, there exists a vector v in the 2-plane generated by the vectors ξ_p and $\gamma'(0)$ such that $\langle v,\xi_p\rangle > 0$, and the geodesic $\sigma(t) = \exp_p tv$ has a point $q_1 = \sigma(t_1)$ in the ball $B_\varepsilon(\gamma(t_0))$. By construction, σ is transverse to S in p. Therefore, there exists a neighborhood $(-\tau,\tau)$ of $0 \in R$, such that $\sigma(0,\tau)$ is outside C, and $\sigma(-\tau,0)$ is inside C. In particular if $t_2 \in (-\tau,0)$, the point $q_2 = \sigma(t_2) \in C$. Then σ connects q_1 to q_2 of C, but it is not contained in C. This contradicts the fact that C is strongly convex, and completes the proof.

2.2 Proposition. *Assume that M is submanifold of N and that M separates N in two connected components. Assume further that the eigenvalues of the second fundamental form of M do not change sign. Then M is the boundary of a convex body in N.*

Proof: Let A and B be the connected components of $N\text{-}M$. We can choose an unit normal vector field in M such that the second fundamental form is semidefinite positive. By [2], M is locally convex. This means that for every $p \in M$ there exists a neighborhood V_p of the origin in T_pN such that $\exp_p(V_p \cap T_pM)$ is contained in the closure of one of the two connected components of $N-M$, (here $\exp_p$ denotes the exponential map of N). Let us assume that this connected component is B. In this case, we will show that $\bar{A}$ is a convex body of N. In fact, it is enough to show that $\bar{A}$ is convex.

The argument to be used is an adaptation of the method used by E. Schmidt to show that the simple locally convex curves of the plane are boundaries of convex bodies.

If $\bar{A}$ is not convex, then there exists a point $p \in \bar{A}$ such that, for every $\varepsilon > 0$ $\bar{A} \cap B_\varepsilon(p)$ is not strontly convex. It is clear that such p must be in M. Let $\varepsilon_0 > 0$ be such that $B_{\varepsilon_0}(p)$ is strongly convex and that $C = \bar{A} \cap B_{\varepsilon_0}(p)$ is connected. Then there are points $\bar{p}$ and $\bar{q}$ in C that cannot be connected by a minimal geodesic contained in C. Since int $C \neq \emptyset$, there exists distinct points $p_1 = \bar{p}, p_2, \ldots, p_m = \bar{q}$ in int C and there exists a unique minimal geodesic joining p_i to p_{i+1} which is contained in C. However, there exists an index k such that for $i \leq k$, p_1 can be joined to p_i by a minimal geodesic contained in int C but p_1 cannot be joined to p_{k+1} by a minimal geodesic contained in int C. Let $g(t)$ be the minimal geodesic joining $p_k = g(0)$ to $p_{k+1} = g(\ell)$, and let $\gamma_t(s)$ be the minimal geodesic joining p_1 to $g(t)$. Set $L = \{t \in [0,\ell] \mid \gamma_t(s)$ is contained in int $C\}$. Since L is bounded and nonempty, there exists t_0 such that $t_0 = \sup L$. The geodesic $\gamma_0 = \gamma_{t_0}$ connecting p_1 to $g(t_0)$ is contained in $\bar{C}$, because γ_0 is limit of geodesics contained in int C. Furthermore, γ_0 is tangent to M. In fact, since $t_0 = \sup L$, γ_0 has a point in common with the boundary ∂C of C. Since $B_{\varepsilon_0}(p)$ is strongly convex and γ_0 has points in int $B_{\varepsilon_0}(p)$, by Lemma 2.1, cannot be tangent to $\partial B_{\varepsilon_0}(p)$. Therefore γ_0 is tangent to M. Let $q = \gamma_0(s_1)$ be the first point of M where γ_0, issuing from p_1 is tangent M. Then the geodesic $\sigma(s) = \gamma_0(s_1 - s)$ that starts at q and passes through p_1 is contained in A, for $0 < s \leq s_1$. This contradicts the fact that M is locally convex.

Therefore $\bar{A}$ is a convex body. This completes the proof of Proposition 2.2.

2.3 - Let V be an open ball of the origin of T_pN such that the restriction $\exp_p|V$ is a diffeomorphism. We will call the set $\exp_p(V)$ a *normal neighborhood of* p.

2.4 Proposition. *Let* A *be a convex body in* N. *Suppose that the boundary* $M = \partial A$ *of* A *is a compact and connected submanifold of* N. *If* M *is contained in a normal neighborhood of an interior point of* A, *then* M *is diffeomorphic to a sphere.*

Proof. Let u be a normal neighborhood of a point $p \in \text{int } A$, such that $M \subset u$. Then any geodesic that issues from p leaves u, hence $\bar{A}$. Since M is the boundary of a convex body, by Lemma 2.1, the geodesics that issue from p must meet M transversely. On the other hand, since u is a normal neighborhood of the point p, the geodesics that issue from p do not meet in u. Thus, we can define a map

$$\phi: M \to S^n \subset T_pN$$

by

$$\phi(q) = \frac{\exp_p^{-1}(q)}{|\exp_p^{-1}(q)|}.$$

Clearly ϕ is a diffeomorphism, and this concludes the proof.

Proposition 2.4 has the following consequence which is interesting in its own right.

2.5 Corollary. *Suppose that* N *is simply connected and* $K_N \leq 0$. *If* M *is a compact hypersurface of* N *such that* $K_M > K_N$ *then,* *there exists a point* $p \in M$ *and orthonormal vectors* V *and* W *in* T_pM *such that* $K_M(V,W)_p > 0$.

Proof: Since $K_M > K_N$, the eigenvalues of the second fundamental form do not change sign. Since N is simply connected and M is a compact hypersurface of N, M separates N in two connected components. (An argument to show this fact can be found in e.g. [8 p. 72].) By Proposition 2.2, M is the boundary of a convex body and by Proposition 2.4, M is diffeomorphic to a sphere. If $K_M < 0$, there M is covered by $\mathbb{R}^n$, which is a contradiction.

2.6 - Let L be an orientable $(n+1)$-dimensional Riemannian manifold and let $f: L \to \mathbb{R}$ be a differentiable functions without critical points. We will denote by $S_t = f^{-1}(t)$ the level hypersurface of f at t. We will denote by η_t a unit normal vector field of S_t, and by $\mu_t(p)$ the greatest eigenvalue of the second fundamental form of S_t at p along η_t. Let H be an orientable n-dimensional Riemannian manifold, and let $x: H \to L$ be an isometric immersion. We will denote by ξ a unit normal vector field of H, and by λ_p the smallest eigenvalue of the second fundamental form of x at p along ξ.

2.7 Proposition. *With the above notation, assume that at each critical point* p *of* $f\circ x$

$$\lambda_p > \mu_{x(p)}.$$

Then, $f\circ x$ *is a Morse function that has no saddle points.*

Proof. We denote by $h = f\circ x$ the restriction of f to $x(H)$. If h has no critical points the result is trivial. Assume that $p_0 \in H$ is a critical point of h. Let S_{t_0} be the level hypersurface of h which passes through $x(p_0)$. We must show that p_0 is a nondegenerate critical point of h and that p_0 is not a saddle point of h.

By Nash's Theorem [10], we may assume that L is isometrically embedded in $\mathbb{R}^r$, for r large. We consider the orthogonal decomposition of $\mathbb{R}^r$ given by

$$\mathbb{R}^r = T_{x(p_0)}L \oplus (T_{x(p_0)}L)^{\perp}$$

and let $P: \mathbb{R}^r \to T_{x(p_0)}L$ be the corresponding orthogonal projection. Because the result is local, we can restrict ourselves to a neighborhood V of $x(p_0)$ in L where the restriction $P|_V$ is a diffeomorphism onto $P(V)$. To simplify the notation, we will assume that x is an embedding and we will identify H with $x(H)$. We will also denote $H = H \cap V$ and $S_{t_0} = S_{t_0} \cap V$.

By projecting orthogonally V onto $T_{p_0}L$ by P, we will obtain submanifolds $\tilde{H} = P(u)$ and $\tilde{S}_{t_0} = P(W)$ in $T_{p_0}L$, where u and W are, respectively, neighborhoods of p_0 in H and S_{t_0}, with the property that the restrictions $P|_u$ and $P|_W$ are embeddings. Since p_0 is a critical point of h, $T_{p_0}H = T_{p_0}S_{t_0}$. Thus is clear that $\tilde{H}$ and $\tilde{S}_{t_0}$ are contained in $T_{p_0}H \oplus \{t\xi_{p_0} \mid t \in \mathbb{R}\}$.

Denote by $\tilde{\lambda}_{p_0}$ the smallest eigenvalue of the second fundamental form of $\tilde{H}$ at p_0 along ξ_{p_0}, and by $\tilde{\mu}_{p_0}$ the greatest eigenvalue of $\tilde{S}_{t_0}$ at p_0, with respect to ξ_0. Since $\lambda_{p_0} > \mu_{X(p_0)}$, we have that $\tilde{\lambda}_{p_0} > \tilde{\mu}_{p_0}$.

Consider the function $F = f\circ P^{-1}: P(V) \to \mathbb{R}$. It is clear that F is differentiable. Moreover, the level hypersurfaces of F are manifolds $\tilde{S}_t = P(V \cap S_t)$.

Claim 1. *If* $X \in T_{p_0}H$, *then* $d^2f_{p_0}(X,X) = d^2F_{p_0}(X,X)$.

In fact, by the definition of F,

$$dF_{p_0}(X) = df_{P^{-1}(p_0)} \cdot dP^{-1}_{p_0}(X)$$

and

$$d^2F_{p_0}(X,X) = d^2f_{P^{-1}(p_0)}(dP^{-1}_{p_0}(X), dP^{-1}_{p}(X)) + df_{P^{-1}(p_0)}\, d^2P^{-1}_{p_0}(X,X).$$

Since p_0 is a critical point of h, $dh_{p_0}(v) = $ $= df_{x(p_0)}dx_{p_0}(v) = 0$ for every vector $v \in T_{p_0}H$. But $x(p_0) = P^{-1}(p_0) = p_0$. Then $df_{p_0}(w) = 0$ for every $w \in T_{p_0}H$. Therefore,

$$d^2F_{p_0}(X,X) = d^2f_{p_0}(X,X).$$

Claim 2. $p_0 = P(p_0)$ *is a nondegenerate critical point of* $F|_{\tilde{H}}$, *which is not saddle point.*

Since p_0 is a critical point of h, $T_{p_0}\tilde{H} = T_{p_0}\tilde{S}_{t_0}$. We may assume that $\tilde{H}$ and $\vec{S}_t$ are graphs of functions α and β defined in $T_p\tilde{H}$, respectively. Thus,

$$\tilde{H} = \{(x_1,\ldots,x_n,x_{n+1}) \mid x_{n+1} = \alpha(x_1,\ldots,x_n)\}$$

$$\tilde{S}_{t_0} = \{(x_1,\ldots,x_n,x_{n+1}) \mid x_{n+1} = \beta(x_1,\ldots,x_n)\}.$$

Now, we will express the second derivative of F at the point p_0, by computing $\frac{\partial^2 F}{\partial x^2}$ with respect to $\tilde{H}$ and $\tilde{S}_{t_0}$.

Along $\tilde{H}$, we obtain:

$$\frac{\partial^2}{\partial x_i^2}F(x_1,\ldots,x_n,(x_1,\ldots,x_n)) = \frac{\partial^2 F}{\partial x_i^2} + \frac{\partial^2 F}{\partial x_{n+1}\partial x_i}\cdot\frac{\partial\alpha}{\partial x_i} + \frac{\partial F}{\partial x_{n+1}}\cdot\frac{\partial^2\alpha}{\partial x_i^2}.$$

But, at p_0, $\frac{\partial\alpha}{\partial x_i} = 0$. Therefore

$$\frac{\partial^2}{\partial x_i^2}F(x_1,\ldots,x_n,\alpha(x_1,\ldots,x_n)) = \frac{\partial^2 F}{\partial x_i^2} + \frac{\partial F}{\partial x_{n+1}}\frac{\partial^2\alpha}{\partial x_i^2} \quad (1)$$

Similarly, along $\tilde{S}_t$, we have

$$\frac{\partial^2}{\partial x_i^2}F(x_1,\ldots,x_n,\beta(x_1,\ldots,x_n)) = \frac{\partial^2 F}{\partial x_i^2} + \frac{\partial F}{\partial x_{n+1}}\frac{\partial^2\beta}{\partial x_i^2} \quad (2)$$

Since $F(\tilde{S}_{t_0})$ is constant, because $\tilde{S}_{t_0}$ is a level hypersurface of F, $\frac{\partial^2}{\partial x_i}F(x_1,\ldots,x_n,\beta(x,\ldots,x_n)) = 0$. Thus, (2) becomes

$$\frac{\partial^2 F}{\partial x_i^2} + \frac{\partial F}{\partial x_{n+1}}\frac{\partial^2\beta}{\partial x_i^2} = 0. \quad (3)$$

It follows from (1) and (3), that, at the point p_0,

$$\frac{\partial^2 F}{\partial x_i^2} = \frac{\partial F}{\partial x_{n+1}}\left(\frac{\partial^2}{\partial x_i^2}(\alpha-\beta)\right).$$

Since f has no critical point in V, F has no critical point in $P(V)$. Since $\frac{\partial F}{\partial x_i}(p_0) = 0$, for $i = 1,2,\ldots,n$, we have that $\frac{\partial F}{\partial x_{n+1}}(p_0) \neq 0$.

Now, observe that

$$\frac{\partial^2\alpha}{\partial x_i^2} = B^1\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i}\right)_{p_0}$$

and

$$\frac{\partial^2\beta}{\partial x_i^2} = B^2\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i}\right)_{p_0}$$

where $B^1\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i}\right)_{p_0}$ (resp. $B^2\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i}\right)_{p_0}$) denotes the value for the pair $\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i}\right)$ of the second fundamental form of $\tilde{H}$ (resp. $\tilde{S}_{t_0}$) at p_0, along ξ_{p_0} (resp. η_{p_0}).

Since $\tilde{\lambda}_{p_0} > \tilde{\mu}_{p_0}$, $\frac{\partial^2}{\partial x_i^2}(\alpha-\beta) > 0$.

This completes the proof of Proposition 2.7.

3. The proof of the Theorem 1.4

Suppose that M and N are as Theorem 1.4.

3.1 Lemma. $K_M > 4K$.

Proof: Let p be a point of M, and X, Y a pair of orthonormal vectors of T_pM. Then, by the Gauss equation,

$$K_M(X,Y)_p - K_N(X,Y)_p = \langle\bar{\nabla}_X\xi, X\rangle_p \langle\bar{\nabla}_X\xi, Y\rangle_p - \langle\bar{\nabla}_X\xi, Y\rangle_p^2. \quad (1)$$

It p is an umbilical point, or if X and Y are eigenvectors of the second fundamental form at p, it is clear that

$$K_M(X,Y)_p - K_N(X,Y)_p > 4K.$$

If p is not umbilical, let $E_1, E_2, \dots, E_n$ be eigenvectors of the second fundamental form at p, with eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively. We can write $X = \sum x_iE_i$ and $Y = \sum y_iE_i$, where $\sum x_i^2 = \sum y_i^2 = 1$ and $\sum x_iy_i = 0$.

By using the above values of X and Y in (1) we obtain

$$K_M(X,Y)-K_N(X,Y) = (\sum_i x_i^2\lambda_i)(\sum_j y_j^2\lambda_j) - (\sum_i x_iy_i\lambda_i)^2 =$$

$$= \sum_{i,j} x_i^2y_j^2\lambda_i\lambda_j - \sum_{i,j} x_ix_jy_iy_j\lambda_i\lambda_j =$$

$$= \sum_i x_i^2y_i^2\lambda_i^2 + \sum_{i<j} (x_i^2y_j^2+x_j^2y_i^2)\lambda_i\lambda_j -$$

$$-(\sum_i x_i^2y_i^2\lambda_i^2 + \sum_{i<j} 2x_ix_jy_iy_j\lambda_i\lambda_j) =$$

$$= \sum_{i<j} (x_i^2y_j^2+x_j^2y_i^2-2x_ix_jy_iy_j)\lambda_i\lambda_j.$$

Since $\lambda_i \geq 2\sqrt{K}$,

$$K_M(X,Y)-K_N(X,Y)_p \geq 4K \sum_{i<j} (x_i^2y_j^2+x_j^2y_i^2-2x_ix_jy_iy_j). \quad (2)$$

Since $\sum x_i^2 = \sum y_i^2 = 1$,

$$(\sum x_i^2)(\sum y_j^2) = \sum x_iy_i + \sum_{i<j} (x_i^2y_j^2+x_j^2y_i^2) = 1. \quad (3)$$

Since $\sum x_iy_i = 0$,

$$(\sum x_iy_i)^2 = \sum x_i^2y_i^2 + 2\sum_{i<j} x_ix_jy_iy_j = 0. \quad (4)$$

Subtracting (4) from (3) we obtain

$$\sum_{i<j} (x_i^2y_j^2+x_j^2y_i^2-2x_ix_jy_iy_j) = 1. \quad (5)$$

Substituting (5) in (2), we have

$$K_M(X,Y)_p - K_M(X,Y)_p \geq 4K. \quad (6)$$

Finally, since (6) is true for every point $p \in M$ and every pair of orthonormal vectors of T_pM, $K_M > 4K$, thus proving Lemma 3.1.

3.2 - We denote by $i(N)$ the injectivity radius of N, that is to say, $i(N)$ is the largest number $\rho > 0$ such that, for all $p \in N$, the exponential map, $\exp_p$, is an embedding in the open ball of radius ρ in T_pN. In [9], M. Maeda proved that, under the hypothesis of Theorem 1.4, $i(N) \geq \frac{\pi}{\sqrt{K}}$.

Let $\mathcal{D}$ be a compact totally convex set of N, such that

$$\mathcal{D} \supset \bigcup_{p\in M} B_{\frac{\pi}{\sqrt{K}}}(x(p)).$$

(The proof of existence of such sets can be found in [5, p. 137]).

Set

$$a = \inf\{K_N(X,Y)_p \mid p \in \mathcal{D};\ X,Y \in T_pN \text{ and } \langle X,Y\rangle = 0\}.$$

Since $K_N > 0$ and $\mathcal{D}$ is compact, $a > 0$.

Now, we will make use of the following fact, whose proof can be found in [7, p. 397].

3.3 Lemma. *Let* $\gamma(t)$ *a geodesic in* int $\mathcal{D}$ *with* $|\gamma'(t)| = 1$, *and let* $Y(t)$ *be a Jacobi field along* γ, *such that* $Y(0) = 0$ *and* $\langle Y(t), \gamma'(t)\rangle = 0$. *Then, for all* $0 \leq t < \frac{\pi}{\sqrt{K}}$ *one has:*

$$\sqrt{a}\,\frac{\cos\sqrt{at}}{\sin\sqrt{at}} \geq \frac{|Y(t)|'}{|Y(t)|} \geq \sqrt{K}\,\frac{\cos\sqrt{kt}}{\sin\sqrt{kt}}.$$

3.4 We will denote by $B(p)$ the open ball of N with center at p and radius equal to $\frac{\pi}{2\sqrt{K}}$, and by $S(p)$ the geodesic sphere which is the boundary of $B(p)$.

3.5 Lemma. *We can choose a unit normal vector field* η *in* $S(p)$, *such that each eigenvalues* μ *of the second fundamental form of* $S(p)$ *with respect to* η *satisfies*

$$\sqrt{K} > \mu \geq 0.$$

Proof: We can consider $\mathcal{D}$ sufficiently large, so that $S(p) \subset$ $\subset$ int $\mathcal{D}$. Let X be a differentiable unit tangent vector field in $S(p)$ defined in a neighborhood of a point q. Let $\alpha\colon (-\varepsilon,\varepsilon) \to S(p)$ be the solution of X such that $\alpha(0) = q$ and $\alpha'(0) = X_q$.

Let $\sigma\colon (-\varepsilon,\varepsilon) \times [0, \frac{\pi}{2\sqrt{K}}] \to N$ be the variation defined by

$$\sigma(s,t) = \exp_p t\,\tilde{\alpha}(s) \quad \text{where} \quad \tilde{\alpha}(s) = \frac{\exp_p^{-1}(\alpha(s))}{|\exp_p^{-1}(\alpha(s))|}$$

Since $B(p)$ is contained in a normal neighborhood, $\tilde{\alpha}$ is well-defined and σ is differentiable.

Denote by $J(t) = \frac{\partial\sigma}{\partial s}(0,t) = (d\exp_p)_{t\tilde{\alpha}(0)}\, t\tilde{\alpha}'(0)$ the Jacobi field along the geodesic $\sigma(0,t)$. It is clear that $J(0) = 0$ and $J(\frac{\pi}{2\sqrt{K}}) = X_q$. Denote by $Z(t) = \frac{\partial\sigma}{\partial t}(0,t) = (d\exp_p)_{t\tilde{\alpha}(0)}\tilde{\alpha}(0)$ the velocity vector of the geodesic $\sigma(0,t)$.

Choose a unit normal vector field η such that

$$\eta_q = -Z(\tfrac{\pi}{2\sqrt{K}}).$$

Then

$$\mu(q) = \langle\bar{\nabla}_X X, \eta\rangle_q = -\langle\bar{\nabla}_X \eta, X\rangle_q = \langle\bar{\nabla}_X(-\eta), X\rangle_q =$$

$$= \left\langle \frac{\bar{D}}{ds}\frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial s}\right\rangle(0,\tfrac{\pi}{2\sqrt{K}}) = \left\langle \frac{\bar{D}}{dt}\frac{\partial\sigma}{\partial s}, \frac{\partial\sigma}{\partial s}\right\rangle(0,\tfrac{\pi}{2\sqrt{K}}) =$$

$$= \frac{1}{2}\frac{d}{dt}\left\langle\frac{\partial\sigma}{\partial s}, \frac{\partial\sigma}{\partial s}\right\rangle(0,\tfrac{\pi}{2\sqrt{K}}) = \frac{1}{2}\langle J(t), J(t)\rangle'_{\frac{\pi}{2\sqrt{K}}}$$

(where $\bar{D}$ is the covariant derivative of N).

Observe that

$$\frac{|J(t)|'}{|J(t)|} = \frac{\langle J(t), J'(t)\rangle}{\langle J(t), J(t)\rangle} = \frac{1}{2}\frac{\langle J(t), J(t)\rangle'}{\langle J(t), J(t)\rangle},$$

and that in $t = \frac{\pi}{2\sqrt{K}}$, $\langle J(t), J(t)\rangle = 1$. It follows from Lemma 3.3 that

$$\sqrt{a}\,\cot\sqrt{a}\,\frac{\pi}{2\sqrt{K}} \geq \langle\bar{\nabla}_X X, \eta\rangle_q \geq 0, \qquad 0 < a < K.$$

By taking $u = \frac{\sqrt{a}}{\sqrt{K}}\frac{\pi}{2}$, one has

$$\frac{2\sqrt{K}}{\pi}\,u\cot u \geq \langle\bar{\nabla}_X X, \eta\rangle_q \geq 0, \qquad 0 < u < \frac{\pi}{2}.$$

Now, set $f(u) = u\cot u$, $0 < u < \frac{\pi}{2}$. Observe that

i) $1 = \lim_{u\to 0} f(u)$

ii) $f'(u) = \dfrac{\sin 2u - 2u}{2\sin^2 u} < 0$, if $u > 0$.

Hence, $1 \geq u \cot u$, and therefore,

$$\frac{2}{\pi}\sqrt{K} \geq \langle \bar{\nabla}_X X, \eta \rangle_q \geq 0.$$

We finally conclude that

$$\sqrt{K} > \frac{2}{\pi}\sqrt{K} \geq \mu \geq 0,$$

and this completes the proof of Lemma 3.5.

3.6 Lemma. *For all* $p \in N$ *the open ball* $B(p)$ *is strongly convex.*

Proof: Since $i(N) \geq \frac{\pi}{\sqrt{K}}$, $S(p)$ is contained in a normal neighborhood u of p. Furthermore, if q_1 and q_2 are points of $B(p)$ there exists a unique minimal geodesic connecting q_1 to q_2. Since u is simply connected, $S(p)$ separates u into two connected components ([8, p. 72]). By Lemma 3.5, the eigenvalues of the second fundamental form of $S(p)$ do not change sign. By Proposition 2.2, $S(p)$ is then a boundary of a convex body of N.

It is enough to show that the minimal geodesic that joins two points of $B(p)$ is contained in $B(p)$. This follows by using the same adaptation of the E. Schmidt's method used in the proof of Proposition 2.2. This concludes the proof of Lemma 3.6.

Assertion 1. *There exists a Morse function defined in* M *that has only two critical points, one maximum and one minimum.*

Let p_0 be a point of N, and let $\gamma(t)$ be a geodesic of N passing through p_0. Reparametrize γ so that $|\gamma'(t)| = 1$ and $\gamma(\frac{\pi}{\sqrt{K}}) = p_0$.

We will denote by $T_{\gamma(t)}$ the parallel translation of N along γ from $\gamma(0)$ to $\gamma(t)$. Consider the set:

$$\tilde{\Sigma}_\gamma(0) = \{v \in T_{\gamma(0)}N \mid \langle v, \gamma'(0)\rangle > 0 \text{ and } |v| = \frac{\pi}{2\sqrt{K}}\}.$$

Thus, $\Sigma_\gamma(t) = \exp_{\gamma(t)} T_t(\tilde{\Sigma}_\gamma(0))$ is a hemisphere of the geodesic sphere with center in $\gamma(t)$ and radius $\frac{\pi}{2\sqrt{K}}$.

3.7 Lemma. *For* $0 < t < \frac{\pi}{\sqrt{K}}$, *the family* $\{\Sigma_\gamma(t)\}$ *is a foliation of* $B(p_0)$.

Proof. First, we claim that if $0 < t_1 < t_2 < \frac{\pi}{\sqrt{K}}$, then $\Sigma_\gamma(t_1) \cap \Sigma_\gamma(t_2) \cap B(p_0) = \emptyset$. In fact, suppose there exists $q \in \Sigma_\gamma(t_1) \cap \Sigma_\gamma(t_2) \cap B(p_0)$. Then $d(q,\gamma(t_1)) = d(q,\gamma(t_2)) = \frac{\pi}{2\sqrt{K}}$, and $d(q,p_0) < \frac{\pi}{2\sqrt{K}}$.

Consider the open ball $B(q)$ with center in q and radius $\frac{\pi}{2\sqrt{K}}$. By Lemma 3.6, $B(q)$ is strongly convex. It is clear that $p_0 \in B(q)$. Let $\sigma_i(s)$ $(i=1,2)$ be the minimal geodesic connecting $\gamma(t_i)$ $(i=1,2)$ to q. By definition of $\Sigma_\gamma(t)$, $\langle \sigma_i'(0), \gamma'(t_i)\rangle > 0$, hence, γ is transverse at $\gamma(t_i)$ to the geodesic sphere $S(q)$, boundary of $B(q)$, $(i=1,2)$. This implies that there exist disjoint neighborhoods V_1 and V_2 of t_1 and t_2, respectively, such that $\gamma(V_i)$ has points inside $B(q)$ and outside $B(q)$ near $\gamma(t_i)$ $(i=1,2)$. Now, let $\gamma(t_0)$ be a point of $\gamma(v_1) \cap B(q)$. Then $\gamma(t)$, $t_0 \leq t \leq \frac{\pi}{\sqrt{K}}$, is a segment of a minimal geodesic connecting $\gamma(t_0)$ to p_0 inside $B(q)$, and $\gamma(t)$ leaves $B(q)$. This contradicts the fact that $B(q)$ is strongly convex, and proves our claim.

Now, let q be any point of $B(p_0)$. Consider the geodesic sphere $S(q)$. Since p_0 is inside $B(q)$, the geodesic $\gamma(t)$ has points inside $B(q)$. By ([5, p. 152]), γ goes to infinite, hence it leaves the closure $\overline{B(q)}$ of $B(q)$.

Let $\gamma(t_1)$ be the point where γ enters $B(q)$ for first time before passing through p_0. Then, $q \in \sum_\gamma(t_1)$. In fact, by construction, $d(q,\gamma(t_1)) = \frac{\pi}{2\sqrt{K}}$. Furthermore, since γ is transverse to $S(q)$ at $\gamma(t_1)$, if $\sigma(s)$ is the minimal geodesic joining $\gamma(t_1)$ to q, then $<\sigma'(0),\gamma'(t_1)> > 0$. This fact completes the proof of Lemma 3.7.

Let $f_\gamma : B(p_0) \to \mathbb{R}$ be the function defined by

$$f_\gamma(q) = t \text{ if only if } q \in \sum_\gamma(t).$$

By Lemma 3.7, f_γ is well-defined and by definition of the family $\{\sum_\gamma(t)\}$ f_γ is differentiable.

Since $K_M > 4K > 0$, by Bonnet-Myers' Theorem, M is compact and $\operatorname{diam} M \leq \frac{\pi}{2\sqrt{K}}$ (diam M denotes the diameter of M). Since $K_M > K_N$, no curve of $x(M)$ can be a geodesic in N, and so

$$\operatorname{diam} x(M) < \operatorname{diam} M \leq \frac{\pi}{2\sqrt{K}},$$

then, for every point $p \in M$, $x(M) \subset B(x(p))$. Now, by fixing $p \in M$ and a geodesic γ in N passing through $x(p)$; we can construct a function f_γ as above. Therefore, we can define the function $h_\gamma : M \to R$ by $h_\gamma = f_\gamma \circ x$.

3.8 Lemma. *h_γ is a Morse function that has two critical points, one maximum and one minimum.*

Proof: It is clear that h_γ is well-defined and is differentiable. Observe now, that f_γ has no critical points in $B(x(p))$. On the other hand, the maximum eigenvalues μ_t of the second fundamental form of each level surface $\sum_\gamma(t)$, with respect to the unit normal vector field as in Lemma 3.5, is strictly less that the minimum eigenvalue of the second fundamental form of x with respect to ξ according to Lemma 3.5. By Proposition 2.7, h_γ is a Morse function without saddle points. Since M is compact, h_γ has only two critical points, one maximum and one minimum ([3, p. 174]). This completes the proof of the Lemma 3.8 and of the Assertion 1.

Assertion 2. *x is a embedding.*

Proof of Assertion 2: Suppose, by contradiction, that x is not an embedding. Then, there exists distinct points p and q of M, such that $x(q) = x(p)$.

Consider the geodesic $\gamma(t)$ that passes through $x(p) = \gamma(\frac{\pi}{\sqrt{K}})$ and that $\gamma'(\frac{\pi}{\sqrt{K}}) = \xi_p$ is the unit normal vector field ξ of M at p.

Now, consider the function $h_\gamma = f_\gamma \circ x$. By Lemma 3.8 h_γ is a Morse function that has only two critical points, one maximum and one minimum.

By construction of h_γ, p is a critical point of h_γ, which we assume to be a point of minimum, with $h_\gamma(p) = t_0$. (The case where p is a point of maximum can be treated similarly).

Let u and v be disjoint neighborhoods of p and q, respectively, such that x restricted to u or to v is an embedding. We will consider two cases:

1st case. $x(u)$ is not transverse to $x(v)$ at $x(p)$. In this case, q is also critical point of h_γ and so, is a point of maximum. Further, $h_\gamma(q) = h_\gamma(p) = t_0$. Since q is a point of maximum of h_γ, there exists a neighborhood v_1 of q in M such that if $r \in v_1$ and $r \neq q$, then $h_\gamma(r) < t_0$. This implies that there exists a point of minimum if h_γ in M distinct of p. This contradicts Lemma 3,8.

2nd case. $x(u)$ is transverse to $x(v)$ at $x(p)$. In this case, there exist points of $x(V)$ contained in the level below $x(p)$. This implies that there exists another point of minimum distinct from p. This contradicts Lemma 3.8.

Then, x is embedding, thereby proving Assertion 2.

3,11 - Now, since $B(x(p))$ is simply connected and x is an embedding, $x(M)$ separates $B(x(p))$ in two connected components ([8, p. 72]). Since the eigenvalues of the second fundamental form do not change sign, by Proposition 2.2, $x(M)$ is the boundary of a convex body of N. Since $x(M)$ is contained in a normal neighborhood of p_0, by Proposition 2.4, $x(M)$ is diffeomorphic to a sphere. Therefore M is diffeomorphic to a sphere. This completes the proof of Theorem 1.4.

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