TOTALLY GEODESIC FOLIATIONS WITH INTEGRABLE NORMAL BUNDLES

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1. Introduction

In this paper we give a characterization of geodesic flows and foliations on complete Riemannian manifolds satisfying some conditions of positiveness for the curvature tensor, Precisely, we prove the following:

Theorem. "Let F and $F^{\perp}$ be two orthogonal foliations of complementary dimensions over a complete Riemannian manifold M.

If the following three conditions hold:

1) $F^{\perp}$ is totally geodesic

2) $\sum_i R(h, e_i, e_i, h) \geq 0$ where h is the mean curvature vector of F, $\{e_1, e_2, \ldots, e_n\}$ is any orthonormal frame tangent to F, $1 \leq i \leq n$, and R is the curvature tensor of M.

3) If $p \in M$ and $h(p) = 0$ then the matrices (K_{ij}^{α}) have non negative trace for every α, the matrices (K_{ij}^{α}) being defined by:

$$K_{ij}^{\alpha} = R(e_\alpha, e_i, e_i, e_\alpha) \quad \text{where} \quad 1 \leq i, j \leq n \quad \text{and}$$

e_α is a unit vector tangent to $F^{\perp}$.

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Then F is totally geodesic. Consequently, M is locally a Riemannian product of leaves of F and $F^\perp$."

We get then the following:

Corollary 1. "Let M be a complete Riemannian manifold equipped with a pair of orthogonal foliations of complementary dimensions F and $F^\perp$ such that $F^\perp$ is totally geodesic.

Assume that the matrix (K^α_{ij}) has non negative trace for each direction e_α tangent to $F^\perp$ at each point of M. Then F is also totally geodesic and M is locally a Riemannian product of leaves of F and $F^\perp$."

Corollary 2. "Let M be a complete Riemannian manifold of non negative Ricci curvature. If F is a codimension-one foliation over M and the normal flow of F, say $F^\perp$, is geodesic, then F is totally geodesic, $F^\perp$ is parallel and M is locally a Riemannian product of leaves of F and $F^\perp$."

Remarks. 1) K. Abe [1] had already proved the same result stated in corollary 1 using the additional hypothesis of local symmetry of the ambient space M. We removed it.

2) G. Oshikiri [3] proved that if M is a closed Riemannian manifold with non-negative Ricci tensor and F is a minimal codimension-one foliation over M, then the normal flow of F is parallel and F is totally geodesic. Corollary 2 gives a converse of this result in the sense that we suppose the normal flow to be geodesic and get geodesibility (hence minimality) of F.

3) One of the authors [2] proved the following:

Let M be a closed Riemannian manifold with non-negative sectional curvature. Let F and $F^\perp$ be a pair of orthogonal foliations of complementary dimensions over M. If F is minimal, codim $F = 2$ and the Euler class of $F^\perp$ vanishes then M is

locally a Riemannian product of the leaves of F and F.

Corollary 1 is another version of that: there are no assumptions on the topology or on the codimension of the foliation, weaker assumptions on the curvature and on the topology of M, but we assume F to be totally geodesic.

2. Proof of the Theorem

Let $x \in M$ and $\{e_1,\dots,e_n,e_{n+1},\dots,e_{n+p}\}$ be a normal orthonormal adapted frame, (i.e. $e_1,\dots,e_n$ are tangent to F and $e_{n+1},\dots,e_{n+p}$ are tangent to $F^\perp$) in a neighborhood of x.

Let us take $\alpha \geq n+1$ and suppose $e_1,\dots,e_n$ to diagonalize the second fundamental form H^α_F of F in the direction of e_α at a point x.

We have the following lemma:

Lemma. "$e_\alpha < h, e_\alpha > - |H^\alpha_F| - \sum_{i=1}^n R(e_\alpha,e_i,e_i,e_\alpha) = <h,\nabla_{e_\alpha} e_\alpha>$

where ∇, $<\ ,\ >$ and R denote respectively the Riemannian connexion, the scalar product and the curvature tensor of M."

Proof. $e_\alpha < \sum_{i=1}^n \nabla_{e_i} e_i, e_\alpha > - |H^\alpha_F|^2 - \sum_{i=1}^n R(e_\alpha,e_i,e_i,e_\alpha) =$

$$(1)\qquad = \sum_{i=1}^n (<\nabla_{e_\alpha}\nabla_{e_i} e_i, e_\alpha> + <\nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha>) - |H^\alpha_F|^2$$

$$+ \sum_{i=1}^n (<\nabla_{[e_\alpha,e_i]} e_i, e_\alpha> + <\nabla_{e_i}\nabla_{e_\alpha} e_i, e_\alpha> - <\nabla_{e_\alpha}\nabla_{e_i} e_i, e_\alpha>).$$

Here, ∇, $<\ ,\ >$ and R denote respectively the Riemannian connection, the scalar product and the curvature tensor of M.

Since $F^\perp$ is totally geodesic, then $\nabla_{e_\alpha} e_\alpha$ is orthogonal to F and

$$(2) \qquad \sum_i \langle \nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha \rangle = \langle h, \nabla_{e_\alpha} e_\alpha \rangle$$

where h is the mean curvature vector of F.

Also

$$(3) \qquad \langle \nabla_{e_\alpha} e_i, e_\alpha \rangle = -\langle e_i, \nabla_{e_\alpha} e_\alpha \rangle = 0.$$

Since

$$\nabla_{e_\alpha} e_i \Big|_x = \sum_{j=1}^{n} b_{ij} e_j \Big|_x \quad \text{(total geodesibility of } F^\perp\text{)},$$

$$\nabla_{e_i} e_\alpha \Big|_x = \lambda_i^\alpha e_i + \sum_{\beta=n+1}^{n+p} c_\beta e_\beta \Big|_x$$

and $b_{ii} = 0$, then

$$(4) \qquad \langle \nabla_{e_i} \nabla_{e_\alpha} e_i, e_\alpha \rangle = -\langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle = -\lambda_i^\alpha b_{ii} \langle \nabla_{e_\alpha} e_i, e_i \rangle = 0.$$

On the other hand,

$$(5) \qquad \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle = \langle \nabla_{\nabla_{e_\alpha} e_i} e_i, e_\alpha \rangle - \langle \nabla_{\nabla_{e_i} e_\alpha} e_i, e_\alpha \rangle.$$

But, according to the notation used till now,

$$(6) \qquad \begin{aligned} \langle \nabla_{\nabla_{e_i} e_\alpha} e_i, e_\alpha \rangle \Big|_x &= \lambda_i^\alpha \langle \nabla_{e_i} e_i, e_\alpha \rangle + \\ &+ \sum_{\beta=n+1}^{n+p} c_\beta \langle \nabla_{e_\beta} e_i, e_\alpha \rangle \Big|_x = -(\lambda_i^\alpha)^2 \end{aligned}$$

and

$$(7) \qquad \begin{aligned} \langle \nabla_{\nabla_{e_\alpha} e_i} e_i, e_\alpha \rangle \Big|_x &= \sum_{j=1}^{n} b_{ij} \langle \nabla_{e_j} e_i, e_\alpha \rangle \Big|_x = \\ &= -\sum_{j=1}^{n} b_{ij} \langle e_i, \nabla_{e_j} e_\alpha \rangle \Big|_x = -b_{ii} \lambda_i^\alpha. \end{aligned}$$

But $b_{ii} = \langle \nabla_{e_\alpha} e_i, e_i \rangle = 0$, so by (5), (6) and (7), it follows that

$$(8) \qquad \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle = (\lambda_i^\alpha)^2.$$

Now, by (1), (2), (4) and (8), it follows that

$$(9) \qquad e_\alpha \langle h, e_\alpha \rangle - |H_F^\alpha|^2 - \sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha) = \langle h, \nabla_{e_\alpha} e_\alpha \rangle.$$

Now, we are going to prove that F is minimal, i.e. that h vanishes identically.

Suppose by absurd that it does not. Then there exists a point x of M such that $h(x) \neq 0$. Let us take an adapted frame $\{e_1, \ldots, e_{n+p}\}$ in a neighbourhood of x such that $e_{n+1} = \dfrac{h}{|h|}$ is parallel to h and $\{e_1, \ldots, e_n\}$ diagonalizes the second fundamental form of F in the direction of e_{n+1} at x. By (9) we get

$$(10) \qquad e_{n+1} \langle h, e_{n+1} \rangle - |H_F^{n+1}|^2 - \sum_{i=1}^{n} R(e_{n+1}, e_i, e_i, e_{n+1}) = 0.$$

The sum

$$\sum_{i=1}^{n} R(e_{n+1}, e_i, e_i, e_{n+1}) = \frac{1}{|h|^2} \sum_{i=1}^{n} R(h, e_i, e_i, h)$$

on the part $\{e_1, \ldots, e_n\}$ of the frame, tangent to F and is non-negative because of hypothesis 2) of the theorem.

Suppose that $\gamma: [0, b[$ is a maximal integral curve of h such that $\gamma(0) = x$, $\gamma'(0) \neq 0$.

Let's reparametrize γ by the arc length parameter s so that $\gamma'(s) = 1$, $\forall s$. By equation (10) we see that $|h|$ increases along the orbit γ so γ has infinite length and s is defined on $[0, +\infty.[$

Now, set $|h|(s) = |h|(\gamma(s))$. From equation (10), we get:

$$|h|'(s) - |H_F^{n+1}(\gamma(s))|^2 - \sum_{i=1}^{n} R(e_{n+1}, e_i, e_i, e_{n+1})(\gamma(s)) = 0 \quad \forall s > 0.$$

On the other hand, the inequality $\left(\sum_{i=1}^{n} \lambda_i\right)^2 \leqq n^2 \sum_{i=1}^{n} \lambda^2$ implies:

$$|H_F^{n+1}(\gamma(s))|^2 \geq \frac{|h|^2(s)}{n^2}.$$

So

(11) $$|h|'(s) - \frac{|h|^2(s)}{n^2} \geq 0.$$

Equation (11) implies that $|h|$ is not defined for every $s > 0$. This is a contradiction because $|h|$ is globally defined on M.

Therefore $h \equiv 0$ on M. Finally equation (9) shows that

$$-|H_F^{\alpha}|^2 - \sum_{i=1}^{n} K(e_i, e_\alpha) = 0 \quad \text{for every } \alpha \geq n$$

Thus $H \equiv 0$ on M and F is totally geodesic.

q.e.d.

References

[1] Abe, K., *Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions*, Tôhoku Math. J. 25 (1973), 425-444.

[2] Brito, F., *Minimal foliations of codimension two*, Tôhoku Math. J., 36 (1984).

[3] Oshikiri, G., *A remark on minimal foliations*, Tôhoku Math. J. 33 (1981), 133-137.

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