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SPLITTING VECTOR BUNDLES UP TO COBORDISM

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1. Introduction

J. Milnor in his article [2] obtained a necessary and sufficient condition for a given manifold M^n to be cobordant to a product $N^n \times N^n$. Namely M is cobordant to a product $N \times N$ if and only if all Stiefel-Whitney numbers of M, having odd dimensional Stiefel-Whitney classes as a factor, vanish. Here we have the notion of cobordism for vector bundles over a manifold, and taking Milnor's result as a motivation we ask the following question:

Under what condition a $k{\cdot}m$-dimensional vector bundle ξ^{mk} is cobordant to a Whitney sum of k copies of a m-dimensional vector bundle η^m?

In the following, we shall obtain a necessary and sufficient condition for a given vector bundle to be cobordant to a Whitney sum.

2. Preliminaries and Statement of Results

All manifolds and maps are C^∞. Let $\xi^m \to M^n$ and $\eta^m \to N^n$ be two vector bundles over closed manifolds M and N. We say that ξ is cobordant to η, if there exists a vector bundle $\theta^m \to W^{n+1}$, over a compact manifold with boundary W, such that $\partial W = M \cup N$ and θ restricted to ∂W is equal to the disjoint union $\xi \cup \eta$. This is, obviously, an equivalence relation, and the disjoint union defines a group structure in the set of equivalence classes.

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It turns out that this group is nothing but $\eta_n(BO(m))$, the n-th dimensional bordism group of the classifying space $BO(m)$. A class $[\xi^m \to M^n]$ represents the zero element in $\eta_n(BO(m))$ if and only if all Whitney numbers $\langle w_{i_1} \dots w_{i_r} \bar{v}_{j_1} \bar{v}_{j_2} \dots \bar{v}_{j_s}, [M] \rangle$ are zero, where w_i are the Stiefel-Whitney classes of M, and $\bar{v}_j$ are the Stiefel-Whitney classes of ξ, $i_1 + \dots + i_r + j_1 + \dots + j_s = n$.

Let $\gamma^m \to BO(m)$ and $\gamma^{mk} \to BO(mk)$ be the universal bundles and $f: BO(m) \to BO(mk)$ the classifying map for $\gamma^m + \gamma^m + \dots + \gamma^m \to BO(m)$, the Whitney sum of k copies of γ^m. We have then the induced maps

$$f_*: \eta_n(BO(m)) \to \eta_n(BO(mk)),$$

and

$$f^*: H^*(BO(mk); Z_2) \to H^*(BO(m); Z_2).$$

We now state our main results.

Theorem 1. Let $\xi^{mk} \to V_1^n$ be a mk-dimensional vector bundle over a closed n-dimensional manifold V_1^n. Then ξ is cobordant to a vector bundle of the form

$$\eta^m + \dots + \eta^m \to V_2^n \quad \text{if and only if } [\xi \to V_1^n],$$

viewed as an element of $\eta_n(BO(mk))$, lies in the image of f_*.

The following result is proved in [1, pag. 58]. Let $f: X \to Y$ be a map between finite CW complexes. The necessary and sufficient condition that $[B^n, \phi] \in \eta_n(Y)$ lie in the image of $f_*: \eta_n(X) \to \eta_n(Y)$ is that every characteristic number of $[B^n, \phi]$ associated with an element in the Kernel of $f^*: H^*(Y, Z_2) \to H^*(X, Z_2)$ must vanish.

Let us denote by $\phi: V_1^n \to BO(mk)$ a classifying map for ξ^{mk}.

Corollary 1: A vector bundle $\xi^{mk} \to V_1^n$ is cobordant to a vector bundle of the form $k \cdot \eta^m \to V_2^n$ if and only if every Whitney number of ξ, associated to an element of $\phi^*(\text{Ker } f^*)$, is zero.

Proof: Let us take p big enough such that the inclusions

$$i: G_m(R^p) \to BO(m) \quad \text{and}$$
$$j: G_{mk}(R^p) \to BO(mk) \quad \text{induce isomorphisms:}$$

$$i_*: \eta_n(G_m(R^p)) \to \eta_n(BO(m))$$
$$j_*: \eta_n(G_{mk}(R^p)) \to \eta_n(BO(mk))$$

$$\left.\begin{array}{l} i^*: H^r(BO(m), Z_2) \to H^r(G_m(R^p); Z_2) \\ j^*: H^r(BO(mk), Z_2) \to H^r(G_{mk}(R^p); Z_2) \end{array}\right\} r \leq n$$

Let $f: BO(m) \to BO(mk)$ be a cellular classifying map for $\gamma^m + \dots + \gamma^m$ (k times). Since $f(G_m(R^p)) \subset G_{mk}(R^p)$, the composition $G_m(R^p) \xrightarrow{i} BO(m) \xrightarrow{f} BO(mk)$ defines a map $h: G_m(R^p) \to G_{mk}(R^p)$ in such way that the following diagrams are commutative:

$$\begin{array}{ccc} \eta_n(BO(m)) & \xrightarrow{f_*} & \eta_n(BO(mk)) \\ \uparrow \cong i_* & & \uparrow \cong j_* \\ \eta_n(G_m(R^p)) & \xrightarrow{h_*} & \eta_n(G_{mk}(R^p)) \end{array}$$

$$\begin{array}{ccc} H^r(BO(m)) & \xleftarrow{f^*} & H^r(BO(mk)) \\ \downarrow \cong i^* & & \downarrow \cong j^* \\ H^r(G_m(R^p)) & \xleftarrow{h^*} & H^r(G_{mk}(R^p)) \end{array} \quad r \leq n$$

Since $G_m(R^p)$ and $G_{mk}(R^p)$ are finite C-W complexes, the theorem proved in [1, pag. 58] holds for the map h. So it follows that it also holds for f.

Then this fact and Theorem 1 proves the corollary 1.

We would like to have a computable splitting condition in terms of Whitney numbers. But in general, it is not easy to compute ker f^*. We were able to do this in some cases and, before we state the other results let us introduce some terminology.

Let $\xi^k \to V^n$ be a k-dimensional vector bundle over V^n, w_i the Stiefel-Shitney classes of V, v_i the Stiefel-Whitney classes of ξ and $\binom{k}{e}$ the combinatorial number. We denote by

$$A = \{v_e \mid \binom{k}{e} \text{ is odd}\}$$

$$B_1 = \{V_1^r + V_2^r, \text{ where } V_i^r \text{ is a monomial of degree } r \text{ in the classes } v_i \in A\}.$$

$$B_2 = \{v_i \mid \binom{k}{i} \text{ is even}\}.$$

Theorem 2: Let $\xi^k \to V^n$ be a k-dimensional vector bundle over a closed manifold V. There exists a closed manifold W^n and a one-dimensional vector bundle $\eta^1 \to W^n$ such that $(k.\eta^1) \to W^n$ is cobordant to ξ if and only if every Whitney number $\langle w_{i_1} \dots w_{i_r} v_{j_1} \dots v_{j_s} \bar{V}^\ell, [V^n]\rangle$ such that $i_1 + \dots + i_r + j_1 + \dots + j_s + \ell = n$, $\bar{V}^\ell \in B_1 \cup B_2$, vanishes.

Finally, we have

Theorem 3: Let $\xi^{2^r m} \to V^n$ be a $2^r m$-dimensional vector bundle over a closed manifold V^n. There exists a m-dimensional vector bundle $\eta^m \to V_2^n$ over a closed manifold V_2^n such that ξ is cobordant to $2^r\eta$ if and only if every Whitney number $\langle w_{i_1} \dots w_{i_r} v_{j_1} v_{j_2} \dots v_{j_s}, [V_1]\rangle$ such that $i_1 + \dots + i_r + j_1 + \dots + j_s = n$ and some j_t not of the form $j \cdot 2^r$, $j = 1,2,\dots,m$, vanishes.

3 - Proofs of The Theorems

Proof of Theorem 1:

Suppose there exists a m-vector bundle $\eta^m \to V_2^n$ such that $[\xi^{mk} \to V_1] = f_*[k\eta^m \to V_2]$. If $\Phi: V_1 \to BO(mk)$ and $\psi: V_2 \to BO(m)$ are the classifying maps for ξ and η, respectively, we have $[V_1, \Phi] = f_*[V_2, \psi] = [V_2, f \circ \psi]$. It follows that
$[\xi \to V_1] = [(f \circ \psi)^*(\nu^{mk}) \to V_2] = [\psi^*(f^*(\nu^{mk}) \to BO(m))] =$
$= [\psi^*(\nu^m \oplus \dots \oplus \nu^m \to BO(m))] = [\psi^*(\nu^m) \oplus \dots \oplus \psi^*(\nu^m) \to V]$.
Conversely, suppose that

$$[\xi^{mk} \to V_1] = [\eta^m \oplus \dots \oplus \eta^m \to V_2], \text{ and}$$

consider $g: V_1 \to BO(mk)$, $h: V_2 \to BO(Mk)$ and $\psi: V_2 \to BO(m)$ classifying maps for ξ^{mk}, $\eta^m \oplus \dots \oplus \eta^m$ and η^m respectively. By hypothesis

$$[V_1, g] = [V_2, h] \quad \text{in } \eta_n(BO(mk)).$$

Hence

$$(f \circ \psi)^*(\nu^{mk}) = \psi^*(\nu^m \oplus \dots \oplus \nu^m) = \eta^m \oplus \dots \oplus \eta^m$$

and $f \circ \psi: V_2 \to BO(mk)$ also is a classifying map for $\eta^m \oplus \dots \oplus \eta^m$. Then $f \circ \psi$ and h are homotopic maps, and this implies that

$$[V_1^n, g] = [V_2, h] = [V_2, f \circ \psi] = f_*[V_2, \psi]$$

Observation: We should mention that theorem 1 also gives a solution to the problem with degree of stability t; namely, to decide when $\xi^{mk} \oplus \varepsilon^t$ is cobordant to $\eta^m \oplus \dots \oplus \eta^m \oplus \varepsilon^t$ where ε^t denotes the t-dimensional trivial bundle. In fact if $\phi: BO(n) \to BO(n+t)$ is the classifying map for $\nu^n \oplus \varepsilon^t$, then the induced map $\phi_*: \eta_r(BO(n)) \to \eta_r(BO(n+t))$ is a monomorphism.

Proof of Theorem 2

Let $\phi: V \to BO(k)$, $f: BO(1) \to BO(k)$ be the classifying maps for ξ and $k\nu^1$, respectively, $U = 1+u$ and $\bar{V} = 1+\bar{v}_1 + \ldots + \bar{v}_k$ be the total Stiefel-Whitney classes of ν^1 and ν^k, respectively, and $\bar{B}_1$, $\bar{B}_2$ defined in the same manner as B_1 and B_2 in the classes $\bar{v}_i$. Then if $V_1^r + V_2^r \in \bar{B}_1$, $\bar{V}_r \in \bar{B}_2$, we have $f^*(V_1^r + V_2^r) = u^r + u^r = 0$ and $f^*(\bar{v}^r) = \binom{k}{r}u^r = 0$, and so, Ker $f^* \subset$ the ideal generated by $\bar{B}_1 \cup \bar{B}_2$. Now take $S \in \ker f^*$. S can be written as $S = (S_1 + \ldots + S_m) + (T_1 + \ldots + T_e)$ where S_i are monomials in the variables $\bar{v}_j$ with $\binom{k}{j}$ odd and T_i monomials in the variables $\bar{v}_e$ with at least one having $\binom{k}{e}$ even. It follows that

$$0 = f^*(S) = \sum_{i=1}^{m} f^*(S_i) + \sum_{i=1}^{e} f^*(T_i) = \sum_{i=1}^{m} f^*(S_i) = u^{n_1} + \ldots + u^{n_m}.$$

The last sum must then have an even number of terms of the form:

$$(u^{r_1} + u^{r_1}) + (u^{r_2} + u^{r_2}) + \ldots + (u^{r_t} + u^{r_t}).$$

This implies that

$$\sum_{i=1}^{m} S_i = (S_{i_1} + S'_{i_1}) + \ldots + (S_{i_t} + S'_{i_t})$$

where degree S_{i_p} = degree S'_{i_p}, and so $\sum_{i=1}^{m} S_i$ belongs to the ideal generated by $\bar{B}_1$; since $\sum_{i=1}^{e} T_i$ belongs to the ideal generated by $\bar{B}_2$, we conclude that $\ker f^*$ = the ideal generated by $\bar{B}_1 \cup \bar{B}_2$

Proof of Theorem 3

Let $f: BO(m) \to BO(2^r m)$, $\phi: V^n \to BO(2^r m)$ be classifying maps for $2^r\nu^m$ and $\xi^{2^r m}$ respectively, $V = 1+v_1+\ldots+v_{2^r m}$ the

total Stiefel-Whitney class of ξ, $U = 1 + u_1 + \ldots + u_m$ the total Stiefel-Whitney class of η^m, $B = \{v_p, 0 \leq p \leq 2^r m$, p not of the form $2^r j$, $j = 1,2,\ldots,m\}$ and $\bar{B} \subset H^*(BO(2^r m), Z_2)$ defined in the same manner as B, in the classes $\bar{v}_p$, where $\bar{v}_p$ are the Stiefel-Whitney classes of the classifying bundle $\nu^{2^r m} \to BO(2^r m)$. We must show that Ker f^* = the ideal generated by $\bar{B}$.

Observing that

$$f^*(v_s) = \begin{cases} u_j^{2^r} & \text{if } s = j\cdot 2^r,\ j = 1,\ldots,m \\ 0 & \text{otherwise} \end{cases}$$

it follows imediately that the ideal generated by $\bar{B} \subset \ker f^*$.

Suppose now that V_1, V_2 are monomials of $H^*(BO(2^r m), Z_2)$ only involving classes of the form $V_{j\cdot 2^r}$, let us say

$$V_1 = v_{j_1\cdot 2^r}^{s_1} \cdot v_{j_2\cdot 2^r}^{s_2} \ldots v_{j_t\cdot 2^r}^{s_t}$$

and

$$V_2 = v_{i_1\cdot 2^r}^{q_1} \ldots v_{i_e\cdot 2^r}^{q_e}, \quad \text{where } j_a \neq j_b,$$

$i_c \neq i_d$ if $a \neq b$, $c \neq d$.

If $f^*(V_1) = f^*(V_2)$ then we have

$u_{i_1}^{q_1\cdot 2^r} \ldots u_{i_e}^{q_e\cdot 2^r} = u_{j_1}^{s_1\cdot 2^r} \ldots u_{j_t}^{s_t\cdot 2^r}$, and so by the structure of $H^*(BO(m), Z_2)$ we conclude that $e = t$ and up to a permutation of the indices

$$i_1 = j_1, \ldots, i_e = j_e,$$

and

$q_1 2^r = s_1\cdot 2^r \ldots q_e\cdot 2^r = s_e\cdot 2^r$; then we have $V_1 = V_2$.

Let now $P \in H^*(BO(2^r\cdot m), Z_2)$ be a polynomial such that

$f^*(P) = 0$. We can suppose that $P = V_1 + \dots + V_s + \bar{V}_1 + \dots + \bar{V}_q$, where each V_i is a monomial only involving classes of the type $v_{j\cdot 2^r}$ while each $\bar{V}_i$ is a monomial involving at least one class v_p, p not of the form $j\cdot 2^r$. We have then

$$0 = f^*(P) = f^*(V_1) + \dots + f^*(V_s).$$

Each $f^*(V_i)$ is a monomial, and the summation above must have an even number of terms of the form:

$$(f^*(V_1) + f^*(V_1')) + \dots + (f^*(V_u) + f^*(V_u'))$$

where $f^*(V_i) = f^*(V_i')$, $i = 1,\dots,u$. Then it follows that $V_i = V_i'$, $i = 1,\dots,u$ and so $P = \bar{V}_1 + \dots + \bar{V}_q$. But this means that $\ker f^*$ = the ideal generated by $\{v_p,\ 0 \le p \le 2^r m,\ p$ not of form $2^r\cdot j,\ j = 1,\dots,m\}$.

As an application we can show that every 4-dimensional vector bundle $\eta^4 \to S^2 \times S^2$ is cobordant to a vector bundle of the form $4\cdot\nu^1 \to V^4$.

In fact, let $\alpha \in H^2(S^2, Z_2)$ be the non zero element. Then,

$$H^2(S^2 \times S^2, Z_2) = Z_2(\alpha_1) \oplus Z_2(\alpha_2)$$

and

$$H^4(S^2 \times S^2, Z_2) = Z_2(\alpha_1 \cdot \alpha_2)$$

where $\alpha_i = \pi_i^*(\alpha)$, $\pi_i : S^2 \to S^2 \to S^2$ are the projections.

In this case we have $A = \{v_4\}$, $B_2 = \{v_1, v_2, v_3\}$ where $V = 1 + v_1 + \dots + v_4$ is the total Stiefel-Whitney class of η^4. Then we have $B = \{v_4^i + v_4^i,\ i 0,1,\dots\} = \{0\}$.

Since $v_1 = v_3 = 0$ and $W(S^2 \times S^2) = 1$, η^4 is cobordant to $4\nu^1$ if and only if $\langle v_2^2, [S^2 \times S^2]\rangle = 0$.

But we can conclude trivially that $\langle v_2^2, [S^2 \times S^2]\rangle = 0$, so the result follows.

By using the same kind of arguments we can show that every vector bundle $\eta^{2^r} \to S^{2^{r-1}} \times S^{2^{r-1}}$ is cobordant to a vector bundle of the form

$$2^r\,\nu^1 \to V^{2^r}$$

References

[1] Conner, P.E. - *Differentiable Periodic Maps* - Lectures notes in Mathematics, Vol. 738 - Springer-Verlag.

[2] Milnor, J.W. - *On the Stiefel* - Whitney numbers of Complex manifolds and of spin manifolds; Topology, Vol. 3, 1965, 223-230.

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