INTERVAL EXCHANGE TRANSFORMATION AND FOLIATION

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Introduction

Let M be a compact, connected, orientable C^∞ two-manifold. Let us consider a foliation F on M, with isolated singularities. If we remove a point on a regular leaf of F we obtain two semileaves starting at x. A semileaf is recurrent if its topology is not the topology induced by M. The proof of the following fact is similar to the corresponding proof for oriented foliations [2]: if x belongs to a recurrent semileaf then there is a simple two-sided closed curve C, containing x, and transversal to the leaves of F. If C is such a curve, let U be a cylinder which is a neighbourhood of C where $F|_U$ can be oriented. Let U_1 and U_{-1} be the connected components of $U-C$. For each subset $Y \subset C$, we set

$$Y^\delta = Y \times \{\delta\}, \qquad \delta \in \{-1,1\}.$$

Given $(x,\delta) \in C^\delta$, $\delta \in \{-1,1\}$, we define a map $T: C^{-1} \cup C^1 \to C^{-1} \cup C^1$ by

$$T(x,\delta) = (Y,\theta), \quad \theta \in \{-1,1\},$$

if the oriented leaf, which crosses C at x from $U_{-\delta}$ to U_δ, comes back to C firstly at y, crossing if from $U_{-\theta}$ to U_θ,

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$\theta \in \{-1,1\}$. The map T, defined above, is called Return Transformation associated to C and induced by F. An injective continuous map $E: C^{-1} \cup C^{1} \to C^{-1} \cup C^{1}$ defined everywhere, except possibly at finitely many points or closed intervals, satisfies the condition C if for each connected component I^{δ} of Dom E, $\delta \in \{-1,1\}$, such that $T(I^{\delta}) = J^{\theta}$, $\theta \in \{-1,1\}$, then $J^{-\theta}$ is a connected component of Dom E and $T(J^{-\theta}) = I^{-\delta}$. Moreover, if $T(x,\delta) = (y,\theta)$, then $T(y,-\theta) = (x,-\delta)$. A return transformation T has the following proprieties:

P_1 - T is defined on $C^{-1} \cup C^{1}$ except at finitely many points or on finitely many closed intervals.

P_2 - T is continuous, injective and has the same class of differentiability as F.

P_3 - T satisfies condition C.

The positive (resp. negative) T-semiorbit of $(x,\delta) \in C^{\delta}$, $\delta \in \{-1,1\}$, is the set $O^{+}(x,\delta) = \{T^{n}(x,\delta),\ n \in \mathbb{Z},\ n \geq 0$, and T^{n} is defined at $(x,\delta)\}$ (resp. $O^{-}(x,\delta) = \{T^{n}(x,\delta),\ n \in \mathbb{Z}$, $n \leq 0$, and T^{n} is defined at $(x,\delta)\}$). The T-orbit of $(x,\delta) \in C^{\delta}$, $\delta \in \{-1,1\}$, is the set $O^{+}(x,\delta) \cup O^{-}(x,\delta)$. Two return transformations

$$T : C^{-1} \cup C^{1} \longrightarrow C^{-1} \cup C^{1}$$

and

$$S : \tilde{C}^{-1} \cup \tilde{C}^{1} \longrightarrow \tilde{C}^{-1} \cup \tilde{C}^{1}$$

are topologically conjugate if there is a homeomorphism $h: C^{-1} \cup C^{1} \to \tilde{C}^{-1} \cup \tilde{C}^{1}$ such that $h \circ t = S \circ h$. We say that a foliation is minimal if every leaf is dense.

The purpose of this paper is to study the relation between foliations with nontrivial recurrences and interval exchange transformations. We say that an injective differentiable map E defined on $[0,1) \times \{-1,1\}$ except possibly at finitely many points is an interval exchange transformation if $|E'(x,\delta)| = 1$, for every (x,δ) in Dom E. If Y is a subset of $[0,1)$, we set

$$Y^{\delta} = \{(x,\delta),\ x \in Y\},\quad \delta = \{-1,1\}.$$

We assume that the foliations have only finitely many singularities, all of them are n-saddles, $n \in \mathbb{N}$; $n \notin \{1,2\}$; that is, singularities with exactly n separatrices, $n \notin \{1,2\}$, and a hyperbolic sector between two consecutive separatrices. We observe that $n = 1$ corresponds to a thorn and $n = 2$ to a regular point. We devote §1 to the study of interval exchange transformations having dense semiorbits, satisfying the condition C and we establish the relation between the interval exchange transformations and the return transformations induced by foliations with nontrivial recurrences. More precisely, given an interval exchange transformation E as in §1, we construct a foliation F on a manifold M having dense semileaves such that, to some circle $C \subset M$ transverse to F, the return map

$$T: C^{-1} \cup C^{1} \longrightarrow C^{-1} \cup C^{1}$$

is topologically conjugate to E. In §3 we show that the construction given in §2 can be extended in a simple way to give all possible foliations with nontrivial recurrences on two manifolds.

1 - Interval Exchange Transformations versus Return Transformations

Let $X^{\delta} = [0,1) \times \{\delta\}$, $\delta \in \{-1,1\}$, be two copies of the half-open interval $[0,1)$ and let $X = X^{-1} \cup X^{1}$ be the disjoint union of X^{-1} and X^{1}. Assume that X^{1} has the positive orientation: from 0 to 1, and X^{-1} has the opposite one: from 1 to 0. Let τ be a permutation of the symbols $1,2,\ldots,q$. If $\alpha = ((\alpha_1,\ldots,\alpha_n),(\alpha_{n+1},\ldots,\alpha_q))$ and, for $1 \leq u \leq q$,

$\alpha^{\tau} = ((\alpha_{\tau^{-1}(1)},\ldots,\alpha_{\tau^{-1}(u)}),(\alpha_{\tau^{-1}(u+1)},\ldots,\alpha_{\tau^{-1}(q)}))$ are two pairs of probability vectors then we set $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \alpha_j$ (mod. 1), for $1 \leq i \leq q$, $X_i = [\beta_{i-1},\beta_i)^{-1} \subset X^{-1}$, $1 \leq i \leq n$, and $X_i = [\beta_{i-1}, \beta_i)^1 \subset X^1$, $n+1 \le i \le q$. Let $\beta_i^\tau = \sum_{j=1}^{i} \alpha_{\tau^{-1}(j)}$ (mod. 1), $i = 1,\ldots,q$,

$$X_i^\tau = [\beta_{i-1}^\tau, \beta_i^\tau)^{-1}, \quad 1 \le i \le u, \quad X_i^\tau = [\beta_{i-1}^\tau, \beta_i)^1 \subset X^1,$$

for $u+1 \le i \le q$. Let us assume that there exists a permutation σ of the symbols $\{1,2,\ldots,q\}$ such that (α,τ,σ) satisfies the following properties:

i) $\alpha_i = \alpha_{\sigma i}$ and $\beta_{\sigma i} = \beta^\tau_{\tau(i+1)-1}$, for $i=1,2,\ldots,q$.

ii) $\sigma^2 i = i$, $i=1,2,\ldots,q$.

iii) if $i \le n$ and $\tau i \le u$ then $\sigma i > n$ and $\tau\sigma i > u$;

if $i \le n$ and $\tau i > u$ then $\sigma i \le n$ and $\tau\sigma i > u$.

Each (α,τ,σ) satisfying (i), (ii) and (iii) determines an interval exchange transformation satisfying condition C, by setting:

$$T(x,\delta) = (x-\beta_{i-1} + \beta^\tau_{\tau i-1}, \delta^\tau_{\tau i-1}), \quad \text{for} \quad (x,\delta) \in X_i$$

$1 \le i \le = q$, $\delta_j^\tau = 1$ if $q > j > u$ and $\delta_j^\tau = -1$ if $0 \le j \le u$.

T maps each X_i isometrically onto $X^\tau_{\tau i}$ and we can see that each interval exchange transformation satisfying condition C determines only two pairs of probability vectors and permutations τ and σ of $\{1,2,\ldots,q\}$ satisfying the conditions (i), (ii) and (iii) above.

Let T be the set of interval exchange transformations satisfying condition C.

Definition 1.1 - For $(x,\delta) \in X$, $\delta \in \{-1,1\}$ we define the positive orbit of (x,δ):

$$0^+(x,\delta) = \{T^n(x,\delta), \ n \ge 0, \ n \in \mathbb{Z} \ \text{ and } \ T^n \text{ is defined at } (x,\delta)\},$$

the negative orbit of (x,δ):

$$0^-(x,\delta) = \{T^n(x,\delta), \ n < 0, \ n \in \mathbb{Z} \ \text{ and } \ T^n \text{ is defined at } (x,\delta)\}$$

and the orbit of (x,δ):

$$0(x) = 0^+(x) \cup 0^-(x).$$

Definition 1.2. We say that $T = (\alpha,\tau,\sigma) \in T$ satisfies the minimality condition if

$M1$ - T is aperiodic

$M2$ - If F is a finite union of half-open intervals whose endpoints all bellong to the countable set

$$D^\infty = \bigcup_{i=0}^{q-1} 0(\beta_i,\delta_i) \cup \{(1,-1),(1,1)\}, \ \delta_i \in \{-1,1\}$$

$(1,-1) \in X^{-1}$ and $(1,1) \in X^1$, then $TF = F$ implies that either $F = X$, or $F = X^1$, or else $F = X^{-1}$

The following result was proved by Keane [4], for $T = (\alpha,\tau,\sigma)$ and $\tau(\{1,2,\ldots,n\}) = \{1,2,\ldots,n\}$.

Theorem 1.3. $T \in T$ satisfies the minimality condition if and only if $0(x,\delta)$ is dense in X^δ, for each $(x,\delta) \in X^\delta$, $\delta \in \{-1,1\}$.

Furthermore, Keane [4] gave a sufficient criterium for the minimality condition to be satisfied. We call a permutation τ irreducible if $\tau(\{1,2,\ldots,n\}) = \{1,2,\ldots,n\}$ and $(\{1,2,\ldots,j\}) \neq \{1,2,\ldots,j\}$, for each $1 \le j \le n-1$.

Lemma 1.4. If τ is irreducible and the orbits of the points of $D = \{(\beta_1,-1),\ldots,(\beta_{n-1},-1)\}$ are infinite and distinct, then the minimal condition is satisfied.

Proof. See [4].

We can generalize those results for our case if we glue the intervals X^1 and X^{-1} by their endpoints $(1,-1) \in X^{-1}$ and $(0,1) \in X^{-1}$ and consider the interval exchange transformation defined on the union $X^1 \cup X^{-1}$.

This result gives a good connection between the definitions of minimal foliations and minimal interval exchange transformations.

We will study the relation between interval exchange transformations and foliations with nontrivial recurrences, having finitely many singularities, which are n-saddles, $n \in \mathbb{N}$. To each interval exchange transformation $T: X^{-1} \cup X^1 \to X^{-1} \cup X^1$ satisfying the condition C, there is a pair or probability vectors $((\alpha_1,\ldots,\alpha_n),(\alpha_{n+1},\ldots,\alpha_q))$ and permutations τ, σ of the symbols $\{1,\ldots,q\}$, such that for each connected component X_i of Dom T we have $TX_i = X^{\tau}_{\tau i}$ and $\alpha_i = \alpha_{\sigma i}$, σ^2 being the identity. Let us identify X^{δ} with $\mathbb{R}/\mathbb{Z} \times \{\delta\}$, $\delta \in \{-1,1\}$, and we recall that X^1 and X^{-1} have opposite orientations.

Proposition 1.5. Given an interval exchange transformation $T = (\alpha,\tau,\sigma)$ in $\mathcal{T}$, there exists a topological manifold N with an oriented foliation σ, called suspension of T, satisfying:

i) $\overline{\bigcup_{t \in \mathbb{R}} |\phi_t(X^1) \cup \phi_t(X^{-1})|} = N$.

ii) The return transformation associated to $X^{-1} \cup X^1$ induced by ϕ is exactly $T\Big|_{\bigcup_{i=1}^{q} \text{int } X_i}$

Furthermore, there is a continuous foliation (M,F) defined on a topological two-manifold M, two times covered by (N,ϕ), whose return transformation associated to $X^{-1} \cup X^1$ coincides with $T\Big|_{\bigcup_{i=1}^{q} \text{int } X_i}$.

Before to prove this result we can observe that T is defined by the fact that p and $T(p)$ correspond to endpoints of a leaf of $F\Big|_{M-(X^1 \cup X^{-1})}$.

Proof. To each pair of intervals $X_j \subset \text{Dom } T$ and $TX_j = X^{\tau}_{\tau j} \subset \text{Im } T$ we consider a rectangle $\mathbb{R}_j$ having X_j and $TX_j = X^{\tau}_{\tau j}$ for vertical edges, and lines segment of length 2 for the horizontal ones. Let us consider a continuous horizontal flow with one singularity on each middle point of the two horizontal edges. Next, we glue the upper horizontal left semi-edge of $\mathbb{R}_i$ to the lower horizontal left semi-edge of $\mathbb{R}_{i+1}$, $1 \le i \le q$, by an isometry making to coincide the respective middle points. We repeat this process with the right semi-edges, that is, we glue the right upper horizontal semi-edge of $\mathbb{R}_i$ with the right lower horizontal one of $\mathbb{R}_{\tau^{-1}(\tau i+1)}$, $1 \le i \le q$, by an isometry starting at the respective middle points. With this construction we obtain a topological manifold N provided with a continuous flow ϕ, whose singularities are all $2p$-saddles, (exactly the identified middle points).

To construct M we identify each rectangle $\mathbb{R}_i$ with $\mathbb{R}_{\sigma i}$ gluing the edge X_i to $X^{\tau}_{\tau\sigma i}$, $1 \le i \le q$, by an orientation preserving isometry, extending it to all horizontal trajectories of $\mathbb{R}_i$ and $\mathbb{R}_{\sigma i}$, $1 \le i \le q$. We note that the foliation F obtained by this procedure is not oriented. Thus, X^1 lies identified with X^{-1} and we get a continuous foliation (M,F) two times covered by (N,ϕ), whose return transformation associated to $X^{-1} \cup X^1$ coincides with $T\Big|_{\bigcup_{i=1}^{q} \text{int } X_i}$. Notice that this construction is possible by the condition C. □

The following proposition is proved in [1].

Proposition 1.6. Let T, T_1: $[0,1) \to [0,1)$ be continuous and injective maps defined everywhere except possibly at finitely many points, provided with a dense positive semiorbit and $T = T_1^{-1}$. Then, T is topologically conjugate to an interval exchange transformation $E:\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

Proposition 1.7. If $T, T_1 : X^{-1} \cup X^{1} \to X^{-1} \cup X^{1}$ are maps as in 1.6 and satisfy condition C, then T is topologically conjugate to an interval exchange transformation $E \in T$.

Proof. Let us consider X^{-1} identified with $[0,1)$ and X^{1} with $[1,2)$. Let $H:[0,2) \to [0,2)$ be a homeomorphism given by 1.6, such that $E = H \cdot T \cdot H^{-1}$. It is easy to see that E satisfies condition C. To prove that $H([0,1)) = [0,1)$ and $H([1,2))=[1,2)$ it is sufficient to note that the condition C implies that, for each interval $H(X^{\tau}_{\tau i}) \subset H([1,2))$ of length λ_i, there is another interval $H(X_{\sigma i}) \subset H([0,1))$ having the same length λ_i. □

2. The Main Theorem

Before stablishing the main theorem we give some definitions and results.

Definition 2.1. Let M be a C^{∞} two-manifold with boundary and corner (not necessarily compact). A C^{r} foliation F, $r \geq 0$, defined on M is of Type (a) if, for every p in M, there is a neighbourhood $V_p \subset M$ of p such that, in terms of local coordinates : $p = (0,0)$, $V_p = I_i \times I_j$, $i,j = 1,2$, $I_1 = (-1,1)$, $I_2 = [0,1)$, and all the leaves of F in V_p are horizontal lines.

Definition 2.2. Let J_{δ} be a subset of $[-1,1]\times\{\delta\}$, $\delta \in \{-1,1\}$, having at most one point. Let $M = [-1,1] \times [-1,1] - (J_{-1} \cup J_1)$ and denote by $F = (M,F)$ the smooth foliation of type (a) defined on M, induced by the vector field $(1,0)$, having J_1 as its

singular set. Let $I \subset \{1,2\}$, which can be empty, and let $\{c_i\}_{i\in I}$ (resp. $\{\tilde{c}_i\}_{i\in I}$) be the connected components of $[-1,1] \times \{1\} - J_1$ (resp. $[-1,1] \times \{-1\} - J_{-1}$) such that, there exists an orientation preserving homeomorphism $h_i : c_i \to \tilde{c}_i$, for each $i \in I$. Denote by $\tilde{M}$ the quociente $M/\bigcup_{i\in I} h_i$ and by $\tilde{F}$ the foliation of type (a) on $\tilde{M}$ induced by F. A continuous foliation of type (a), $F_1 = (M_1,F_1)$, is said to be a simple canonical region if it is topologically equivalent to a foliation $(\tilde{M},\tilde{F})$ as above.

Lemma 2.3. Let E be a minimal interval exchange transformation. Let $I \subset \mathbb{N}$ (I may be empty), and $\{O(p_i,\delta_i)\}_{i\in I,\ \delta_i \in \{-1,1\}}$ be a family of E-orbits. There are a continuous injective map $T : \mathbb{R}/\mathbb{Z} \times \{-1\} \cup \mathbb{R}/\mathbb{Z} \times \{1\} \to \mathbb{R}/\mathbb{Z} \times \{-1\} \cup \mathbb{R}/\mathbb{Z} \times \{-1\}$, and continuous maps $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and $H : \mathbb{R}/\mathbb{Z} \times \{-1\} \cup \mathbb{R}/\mathbb{Z} \times \{1\} \to \mathbb{R}/\mathbb{Z} \times \{-1\} \cup \mathbb{R}/\mathbb{Z} \times \{1\}$ such that

a) $H(x,\delta) = (hx,\delta)$, $\delta \in \{-1,1\}$ and if we identify $\mathbb{R}/\mathbb{Z}$ with $[0,1)$ in the canonical way, then h is a monotone non-decreasing surjective continuous map;

b) $h^{-1}(p)$ is a one point set for each p such that $(p,\delta) \in \mathbb{R}/\mathbb{Z}\times\{\delta\} - \bigcup_{i\in I} O(p_i,\delta_i)$, $\delta \in \{-1,1\}$, and $h^{-1}(q)$ is a closed interval not reduced to a point for every $q \in \mathbb{R}/\mathbb{Z}$ such that $(q,\delta) \in \bigcup_{i\in I} O(p_i,\delta_i)$;

c) $\{(q,\delta) \in \mathbb{R}/\mathbb{Z}\times\{\delta\},\ \delta \in \{-1,1\},\ /H(q,\delta) \in \bigcup_{i\in I} O(p_i,\delta_i)\}$ is dense in $\mathbb{R}/\mathbb{Z} \times \{\delta\}$;

d) T may be not defined everywhere. If I^{δ} is an interval which is a connected component of Dom E then $H^{-1}(I^{\delta})$ is an open interval, which is a connected component of Dom T.

e) The following diagram is commutative

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z}\times\{-1\}\cup\mathbb{R}/\mathbb{Z}\times\{1\} & \xrightarrow{\;T\;} & \mathbb{R}/\mathbb{Z}\times\{-1\}\cup\mathbb{R}/\mathbb{Z}\times\{1\} \\ \downarrow H & & \downarrow H \\ \mathbb{R}/\mathbb{Z}\times\{-1\}\cup\mathbb{R}/\mathbb{Z}\times\{1\} & \xrightarrow{\;E\;} & \mathbb{R}/\mathbb{Z}\times\{-1\}\cup\mathbb{R}/\mathbb{Z}\times\{1\}; \end{array}$$

f) T is unique (mod. topological conjugacy).

Proof. For each $(q,\delta)\in\bigcup_{i\in I} O(p_i,\delta_i)$, we choose a real positive number $\mu(q)$ such that $\sum_{(q,\delta)\in\bigcup_{i\in I} O(p_i,\delta_i)}\mu(q)=1$.

Identifying $\mathbb{R}/\mathbb{Z}\times\{1\}$ and $\mathbb{R}/\mathbb{Z}\times\{-1\}$ with $[0,1)$ in the usual way, we define $g:[0,1)\times[0,1)$ by

$$\begin{cases} g(0)=0 \\ g(x)=\displaystyle\sum_{\substack{(q,\delta)\in\bigcup_{i\in I} O(p_i,\delta_i)\\ q\in 0,x)}}\mu(q), & \text{if } x\neq 0 \end{cases}$$

We can see that g is strictly monotone and g is continuous at q if and only if $(q,\delta)\in\mathbb{R}/\mathbb{Z}\times\{\delta\}-\bigcup_{i\in I} O(p_i,\delta_i)$, $\delta\in\{-1,1\}$.

Next we define a continuous surjective non-decreasing map

$h:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ by setting

$$h(y)=\inf\{x, g(x)>y\}.$$

We notice that

i) h is constant on each open interval $(a,b)\subset\mathbb{R}/\mathbb{Z}-\operatorname{Im} g$;

ii) $h^{-1}(p)$ is a one point set x for all $p=g(x)$ such that $(p,\delta)\notin\bigcup_{i\in I} O(p_i,\delta_i)$, $\delta\in\{-1,1\}$;

iii) $h\,g(x)=x$, $\forall x\in\mathbb{R}/\mathbb{Z}$;

iv) the set $D=\{q\in\mathbb{R}/\mathbb{Z};\ (h(q),\delta)\in\bigcup_{i\in I} O(p_i,\delta_i)\}$, is dense.

If we set $H(x,\delta)=(hx,\delta)$ for all $x\in\operatorname{Im} g$ and $ghx=x$, we define a map T by the following composite function:

$$(x,\delta)\xrightarrow{\;H\;}(hx,\delta)\xrightarrow{\;E\;}E(hx,\delta)=(\mu(x,\delta),\xi)\longrightarrow(g(\mu(x,\delta)),\xi)=T(x,\delta).$$

For the intervals I_k where h is constant, $I_k=[a_k,b_k]$, and $gha_k=a_k=ghb_k$, we define T by choosing some orientation preserving homeomorphism $T_k^j: I_k\to I_j$, where I_j is an interval $[a_j,b_j]$, such that $(a_j,\xi)=T(a_k,\xi_k)$. It is trivial to see that T is a one-to-one continuous map and T, H satisfy the required conditions. □

Definition 2.4. Let E, T, H be as in 2.3. Then we say that T has been obtained by blowing-up E-orbits via H.

If p and q are points on the same leaf of a foliation F then we will denote by $\widehat{(p,\delta)(q,\xi)}$ the closed arc of trajectory beginning at (p,δ) and ending at (q,ξ), δ, $\xi\in\{-1,1\}$.

Definition 2.5. A continuous foliation (M,F) of type (a) is said to be a canonical recurrent region with generator $T: C^{-1}\cup C^{1}\to C^{-1}\cup C^{1}$, $C^{\delta}=C\times\{\delta\}$, $\delta\in\{-1,1\}$, if the following conditions are satisfied.

(i) C is a circle in M transverse to F, C^{-1} and C^{1} have opposite orientations and the return map associated to C induced by F is precisely $\tilde{T}$.

(ii) Either Dom T is $\mathbb{R}/\mathbb{Z}\times\{-1\}\cup\mathbb{R}/\mathbb{Z}\times\{-1\}$ or the connected components of Dom T are finitely many intervals $X_1,\dots,X_n \subset \mathbb{R}/\mathbb{Z}\times\{-1\}$ and $X_{n+1},\dots,X_q \subset \mathbb{R}/\mathbb{Z}\times\{1\}$. Moreover, the restriction of T to $\bigcup_{j=1}^{q} X_j$ - {endpoints of X_j} is a continuous map obtained by blowing up orbits of a minimal interval exchange transformation satisfying the condition C, modulo topological conjugacy.

iii) If we consider X_j and $T(X_j)$, $j=1,\dots,q$, to be disjoint sets and denote by $S(X_j)$ the set $\{\widehat{(x,\delta)\tilde{T}(x,\delta)},\ (x,\delta)\in I_j\}$ then $(S(X_j), F|_{S(X_j)})$ is a simple canonical region.

iv) $S(X_j)-(X_j \cup T(X_j))$ and $S(X_k)-(X_k \cup T(X_k))$, $j \neq k$, interesect in at most finitely many trajectories of their boundaries.

v) If we identify $S(X_j)$ with $S(X_{\sigma j})$, for each $j=1,\dots,q$, firstly gluing X_j to $T(X_{\sigma j})$ and TX_j to $X_{\sigma j}$ by an orientation preserving isometry, and then, gluing $\widehat{(x,\delta)T(x,\delta)}$ with $\widehat{T(y,-\xi),(y,-\xi)}$ for each $(x,\delta)\in X_j$, $(y,\xi) = T(x,\delta)$, we obtain

$$M = \bigcup_{j=1}^{q} S(X_j).$$

Lemma 2.6. Let M be a compact connected two manifold provided with a foliation F, having at most finitely many singularities and a recurrent leaf f. Then:

(a) for $x \in f$, there is a circle $\tilde{C}$ transverse to F and containing x.

(b) If C is a circle transverse to F then the domain of definition of the return transformation $T: C^{-1}\cup C^{1} \to C^{-1}\cup C^{1}$, induced by F, is a finite union of open intervals (a,b) whose extremes a and b go by F to the singularities of F.

(c) Let $T: C^{-1}\cup C^{1} \to C^{-1}\cup C^{1}$ be as in (b), (S,δ) be an interval contained in Dom T and $S^{\delta}(T)$ be the set $\{\widehat{(x,\delta)T(x,\delta)}\}$ - singularities of $F\}$ identified with $\{\widehat{T^{-1}(x,-\delta)(x,-\delta)}\}$ - singularities of $F\}$. If we consider (S,δ) and $T(S,\delta)$ to be disjoint sets, the pair $((S,\delta)(T), F|_{(S,\delta)(T)})$ is two times covered by a simple canonical region.

Proof. The proof of item (a) is similar to the one for oriented foliations, [2]. Applying [3, Lemma 3.3] to the oriented foliation induced by F on the double covering constructed in (1.3) we obtain itens (b) and (c).

Lemma 2.7. Let M be a compact connected two-manifold whose boundary has finitely many simple closed curves. Let $\gamma_1,\dots,\gamma_n \subset M$ be pairwise disjoint simple closed curves, bounding no discs. If n is large, two of then will enclose a cylinder of M.

Proof. See [5, Lemma 1].

Lemma 2.8. Let M, F be as in (2.6). There is a circle C transverse to F such that, if $T: C^{-1}\cup C^{1} \to C^{-1}\cup C^{1}$ denotes the return transformation induced by F, then

1) There are at most finitely many connected components $A_1,\dots,A_s \subset C^{-1}-\{f\}$, $A_{s+1}\dots A_r \subset C^{1}-\{f\}$ which are not contained in Dom T and $W_1,\dots,W_p \subset C^{-1}-\{f\}$, $W_{p+1},\dots,W_\ell \subset C^{1}-\{f\}$ which are not contained in Im T;

2) Let U be a connected component of $C^{-1} \cup C^{1} - \{f\}$. If for some $m \in \mathbb{Z}$, T^{m} is not defined in U, we have only the following two possibilities:

2a) There exists $k \in \mathbb{Z}$, $0 \leq k$, such $T^{k}(U) \in \{A_1, \ldots, A_r\}$ and for each $j \leq k$ $(j \in \mathbb{Z})$, T^{j} is defined on U;

2b) There exists $k \in \mathbb{Z}$, $k \leq 0$, such that $T^{k}(U) \in \{W_1, \ldots, W_\ell\}$ and for each $j \geq k$ $(j \in \mathbb{Z})$, T^{j} is defined on U.

3) $\overline{\{f\} \cap C}$ is either a Cantor set or is equal to C.

Proof. Applying (2.6) we obtain a circle $\tilde{C}$ transverse to F satisfying (1) and (2) such that, for U as in (2) we have the following possibility besides (2.a) and (2.b):

2c) There exist $n, k \in \mathbb{Z}$, $n \leq 0 \leq k$, such that

$$T^{k}(U) \in \{A_1, \ldots, A_r\},$$

$$T^{n}(U) \in \{W_1, \ldots, W_\ell\}$$

and for each $n \leq j \leq k$ $(j \in \mathbb{Z})$, T^{j} is defined on U. Choose an interval $[a,b] \subset C$ such that $[a,b]^{-1} \cup [a,b]^{1}$ does not intersect any connected component of $C^{-1} \cup C^{1} - \{f\}$ that has finite orbit. Using the fact that $[a,b]$ is a transversal section and crosses f infinitely many times we construct the circle C as done in (2.6). By the choice of $[a,b]$ we see that the return transformation associated to C satisfies either (2.a) or (2.b) and never (2.c). It is not hard to prove (3). □

The main result of this paper is given by the following theorem which generalizes its similar one for oriented foliations, proved in [3, Theorem B].

Theorem 2.9. Let M be a connected compact C^{∞} two-manifold

and let F be a continuous foliation in M with at most finitely many singularities. Then, $\tilde{M} = M - \{\text{singularities of } F\}$ is the union of finitely many submanifolds $M_1, \ldots, M_k$ (not necessarily F - invariants) such that

(a) for each i, j, M_i and M_j intersect in a subset (possibly empty) of their boundaries. Moreover, F induces on M_i a foliation denoted by $F|_{M_i}$;

(b) $(M_1, F/M_1), \ldots, (M_{k-1}, F|_{M_{k-1}})$ are canonical recurrent regions. Furthermore, $(M_k, F|_{M_k})$ is a canonical recurrent region if and only if it contains a point of a recurrent leaf;

(c) the decomposition of $\tilde{M}$ satisfying (a) and (b) above is unique, modulo topological equivalence and permutation of the suffixes.

Proof. For a recurrent leaf $f_1 \in F$, let C be a circle as in (2.8). Assume that $\overline{\{f\} \cap C}$ is a Cantor set. Let h be a Cantor map([3]). Then $h: C^{-1} \cup C^{1} \to C^{-1} \cup C^{1}$ is constant in some interval $S^{\delta} \subset C^{\delta}$, $\delta \in \{-1, 1\}$, if and only if S^{δ} is contained in the closure of a connected component of $(C - \{f\})^{\delta}$. The maps T and T^{-1} induce continuous injective maps, $E, \tilde{E}: C^{-1} \cup C^{1} \to C^{-1} \cup C^{1}$ such that

(a) the following diagrams commute:

$$\begin{array}{ccc} C^{-1} \cup C^{1} & \xrightarrow{T} & C^{-1} \cup C^{1} \\ \downarrow h & & \downarrow h \\ C^{-1} \cup C^{1} & \xrightarrow{E} & C^{-1} \cup C^{1} \end{array} \quad \text{and} \quad \begin{array}{ccc} C^{-1} \cup C^{1} & \xrightarrow{T^{-1}} & C^{-1} \cup C^{1} \\ \downarrow h & & \downarrow h \\ C^{-1} \cup C^{1} & \xrightarrow{\tilde{E}} & C^{-1} \cup C^{1} \end{array}$$

(b) E (resp. $\tilde{E}$) is defined everywhere except at $\{h(\bar{A}_1), \ldots, h(\bar{A}_r)\}$ (resp. $\{h(\bar{W}_1), \ldots, h(\bar{W}_p)\}$ which is either empty or a finite set of points;

(c) $E^{-1} = \tilde{E}$;

(d) E has a dense positive semileaf.

We can assume that $C = \mathbb{R}/\mathbb{Z}$ and E is a minimal interval exchange transformation. We notice that T can be obtained by blowing up E-orbits via h.

Let $S_1,\dots,S_s \subset C^{-1} - \{A_1,\dots,A_s\}$ and $S_{s+1},\dots,S_s \subset C^1 - \{A_{s+1},\dots,A_r\}$ be the connected components of $C^{-1}-\{A_1,\dots,A_s\}$ and $C^1-\{A_{s+1},\dots,A_r\}$ respectively. We denote by J_i the smallest interval that contains $S_i \cap \{f_1\}$, for each $i=1,2,\dots,r$. Let $\{K_1,\dots,K_s\}$ and $\{K_{s+1},\dots,K_r\}$ be the connected components of $C^{-1} - \{J_1,\dots,J_s\}$ and C^1 $\{J_{s+1},\dots,J_r\}$ respectively. Let $J_i(T)$ be the set $\{\overline{(x,\delta)T(x,\delta)}, (x,\delta) \in J_i$ - {singularities of F}}, $i=1,2,\dots,r$. Identifying $J_i(T)$ with the corresponding set $J_{\sigma(i)}(T)$, $i=1,2,\dots,r$, as in (2.6), and setting

$$M_1 = \bigcup_{i=1}^{r} J_i(T)$$

then $(M_1, F/M_1)$ is a recurrent canonical region. We claim that M_1 has the following properties:

a) each K_i, $i=1,2,\dots,r$ is on ∂M_1.

b) each K_i, $i=1,2,\dots,r$, is entirely contained in the interior of $\operatorname{Im} T$.

In fact, the map h defined above gives a semiconjugation between T and a minimal interval exchange transformation E such that:

i) the semiconjugation h is constant on K_i, for each $i=1,2,\dots,r$;

ii) $h(K_i) \subset \mathbb{R}/\mathbb{Z}$ - Dom E

iii) $E^{-1}(h(K_i)) \in \operatorname{Dom} E$ and $T^{-1}(K_i) \subset \operatorname{Dom} T$.

We know that $\{f_1\}$ accumulates on the extreme points of K_i and $K_i \cap \{f_1\} = \emptyset$, for each $i=1,2,\dots,r$. Hence there is an open interval I_i satisfying: $K_i \subset I_i \subset \operatorname{Im} T$, for $i=1,2,\dots,r$.

We conclude that there exists a rectangle $J_{j(i)}(T)$ of M with side $T(J_{j(i)})$ containing K_i. This implies that $K_i \subset \partial M_1$, and we have (a) and (b).

If $\tilde{C}$ is another curve transverse to $\{f_1\}$, $\tilde{T}:\tilde{C}^{+1} \cup \tilde{C}^{-1} \to \tilde{C}^{+1} \cup C^{-1}$ is the return transformation associated to $\tilde{C}$ and $\tilde{M}$ is the recurrent canonical region generated by T, then the fact that $\tilde{M}$ and M_1 are topologically equivalent results from the following considerations.

Let $\tilde{K} \subset \tilde{C}$ be an interval on $\partial\tilde{M}$ such that:

a) $f_1 \cap \tilde{K} = \emptyset$

b) the extremes of $\tilde{K}$ are accumulation points of $\{f_1\}$.

Let us consider the band $\tilde{F}$ of $\tilde{M}$ obtained by the backward saturation of the interval $\tilde{K}$ by the flow F. We can observe that $\tilde{F}$ accumulates on $\{f_1\}$, too. Let K be the interval of C determined at the first time that $\tilde{F}$ crosses C. Hence, there is an integer $n_0 \geq 0$ for which $T^{n_0}K$ is on ∂M_1 and satisfies (a) and (b) above. Let F be the band like $\tilde{F}$, associated to K_1. Now we can establish an isotopy between F and $\tilde{F}$ which carries K_1 onto $\tilde{K}$, and it is not hard to complete the proof of the topological equivalence between M and $\tilde{M}$. Let $\tilde{y} \in M$-M_1 be a point a recurrent leaf $f_2 \in F$. Since $\{f_2\} \cap M_1 = \emptyset$, there is a recurrent canonical region $(M_2, F/M_2)$ such that $M_1 \cap M_2 = \partial M_1 \cap \partial M_2$. Thus we obtain a sequence $M_1, M_2, \dots$ of recurrent canonical regions such that $M_k \cap M_\ell = \partial M_k \cap \partial M_\ell$. Furthermore, since each generator T_i of M_i is associated to a simple closed curve C_i bounding no disc and each pair c_i, c_j of those curves cannot be homotopic (because $M_i \cap M_j = \partial M_i \cap \partial M_j$), using (2.7) we conclude that the sequence $\{M_i\}$ is finite. The uniqueness results from the fact that two curves transverse to a recurrent leaf $\{f\}$ generates topologically equivalent submanifolds.

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