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DIFFERENTIAL RINGS AND ORE EXTENSIONS: BROWN-McCOY RINGS

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Abstract. We consider here a ring K, a derivation D of K and the differential polynomial ring $R = K[X;D]$. The ring K is said to be a Brown-McCoy ring if the prime radical coincides with the Brown-McCoy radical in every homomorphic image of K. A D-Brown-McCoy ring is defined in a similar way. We prove the following conditions are equivalent: (i) K is a D-Brown-McCoy ring; (ii) R is a Brown-McCoy ring and for every maximal ideal M of R, $K/(M \cap K)$ is a D-simple ring with 1. In addition, we give some applications and examples on the study of the transfer of the property of being a Brown-McCoy ring between K and R.

Further, we study the relation between the prime and the D-prime ideals of a differential intermediate extension of a liberal extension.

Introduction. A ring K is a Brown-McCoy ring (abbr. BMCR) if the prime radical coincides with the Brown-McCoy radical in every homomorphic image of K. It is known that the polynomial ring $K[X]$ is a BMCR if and only if K is a BMCR [10]. In the case of $S = K[X;\alpha]$, a skew polynomial ring of automorphism type, the question of whether S is a BMCR whenever K is a BMCR was considered in [5]. On the other hand, in [2], we studied the conditions under which a differential polynomial ring $K[X;D]$

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(called also a skew polynomial ring of derivation type in some papers) is a Jacobson ring whenever K is a Jacobson ring.

The central purpose of this paper is to study the transfer of the property of being a Brown-McCoy ring between K and $R = K[X;D]$, where D is a derivation of K. A differential polynomial ring R over a BMCR K need not be a BMCR, even if K is a right Noetherian ring (Example 5.1). Stronger finiteness conditions than being right Noetherian are needed. We shall obtain here sufficient conditions for R to be a BMCR when K is a BMCR. Further, we shall give some classes of differential rings which satisfy these conditions.

In §1, we consider D-radicals and we define D-Brown-McCoy rings. In §2, we shall prove that K is a D-BMCR if and only if $R = K[X;D]$ is a BMCR and for every maximal ideal M of R, $K/M \cap K$ is a D-simple ring with 1.

A natural question to arise is whether K is a D-BMCR if and only if K is a BMCR, under some finiteness conditions. This question is studied in §3.

In §4 we consider a differential ring (S,D) when S is a liberal extension of K and D is a K-derivation of S, and an intermediate extension $K \subseteq T \subseteq S$ with $D(T) \subseteq T$. For the differential ring (T,D) we shall prove that there is a one-to-one correspondence between the set of all the prime ideals of T and the set of all the D-prime ideals. Namely, if P is a prime ideal $M(P)$, the maximum D-subideal of P, is D-prime and every D-prime is of this form. In this case we can apply the former results and $T[X;D]$ is a BMCR if K is a BMCR.

Some remarks and examples are given in section 5.

Throughout this paper we shall use the same notation and terminology used in [2]. In particular, $L(K)$ denotes the prime radical of K and $\mathcal{D}(K)$ denotes the D-prime radical of the differential ring (K,D) ([2], Theorem 1.1).

1. D-Brown-McCoy Rings

Let (K,D) be a differential ring. Following ([1], p. 116), for every $a \in K$ we denote by $G(a)$ the ideal $\{ar + r + \Sigma_i(x_i a y_i + x_i y_i) : r, x_i, y_i \in K\}$ of K and we put $G_D(a) = \Sigma_{i\geq 0} D^i(G(a))$, the smallest D-ideal of K containing $G(a)$. We say that a is a DG-regular element of K if $G_D(a) = K$. A D-ideal I of K is a DG-regular ideal if every element in I is DG-regular. The union $G_D(K)$ of all the DG-regular ideals of K is called the D-Brown-McCoy radical of K. In a similar way to [1] it can easily be verified that $G_D(K)$ is a DG-regular ideal and contains every DG-regular ideal of K. Further, $K/G_D(K)$ is DG-semi-simple ($G_D(K/G_D(K)) = 0$). Then a standard argument proves that $G_D(K)$ is equal to the intersection of all the D-ideals I of K such that K/I is D-simple and DG-semi-simple ([1], Lemma 66). Also, it is not hard to verify that a D-simple differential ring (K,D) is DG-semi-simple if and only if K has an identity element ([1], Lemma 67). Then we have the following (see [1], Theorem 43).

Theorem 1.1. The D-Brown-McCoy radical $G_D(K)$ is equal to the intersection of all the D-ideals I of K such that K/I is a D-simple ring with an identity element.

We denote by $G(K)$ the Brown-McCoy radical of K and by $M(G(K))$ the maximum D-subideal of $G(K)$ ([3], p. 11).

Lemma 1.2. $M(G(K)) \subseteq G_D(K)$.

Proof. If $x \in M(G(K))$ and I is a D-ideal of K such that K/I is a D-simple ring with 1, consider an ideal A of K such that A/I is a maximal ideal of K/I. Then $D^i(x) \in A$ for $i \geq 0$. Hence $x \in M(A) = I$.

In general, $M(G(K)) \neq G_D(K)$. In fact, if (K,D) is the differential ring given in ([2], Example 5.1), we have $G(K) = M(G(K)) = L(B)[Y]$ and $G_D(K) = G(B)[Y]$.

We know that $\mathcal{D}(K) \subseteq M(L(K)) \subseteq M(J(K)) \subseteq M(G(K)) \subseteq G_D(K)$. *A differential ring* (K,D) *is said to be a D-Brown-McCoy ring if* $G_D(K/Q) = 0$ *for every D-prime ideal* Q *of* K. Hereafter, Brown-McCoy ring (resp. D-Brown-McCoy ring) is often abbreviated BMCR (resp. D-BMCR). If is clear that K is a D-BMCR if and only if $G_D(K/I) = \mathcal{D}(K/I)$ for every D-ideal I of K.

In [2], we said that a differential ring (K,D) is a quasi-finite differential ring (QFDR for short) if $M(L(K/Q)) = 0$ for every D-prime ideal Q of K, and is said to be a D-Jacobson ring if $M(J(K/Q)) = 0$ for every D-prime ideal Q of K. If K is a D-BMCR, then it is a D-Jacobson ring. Then a D-BMCR is also a QFDR.

The following is easy to prove (see [2], Proposition 1.2).

Proposition 1.3. (K,D) is a D-BMCR if and only if (K^*,D^*) is a D-BMCR, where (K^*,D^*) is the usual extension by the ring of integers.

2. The Main Theorem

If $S = K[X;\alpha]$ is a skew polynomial ring, where α is an automorphism of K, the question of whether S is a BMCR was considered in ([5], section 3). The purpose of this section is to prove the following corresponding result.

Theorem 2.1. Let (K,D) be a differential ring and put $R = [X;D]$. Then K is a D-BMCR if and only if the following conditions hold.

(I) R is a BMCR.

(II) For every maximal ideal M of R, $K/(M \cap K)$ is a D-simple ring with an identity element.

To prove the theorem we need some lemmas. We begin with the following

Lemma 2.2. If (K,D) is a D-simple differential ring with an identity 1, then $G(R) = 0$.

Proof. This can be proved in a similar way to ([10], Lemma 1).

If I is an ideal of K, $I[X]$ denotes the left ideal of R of all polynomials $\Sigma_i X^i b_i$, $b_i \in I$. On the other hand, if L is an ideal of R, $\tau(L)$ denotes the D-ideal of K consisting of 0 and the leading coefficients of non-zero elements of L of least degree.

Lemma 2.3. Let L be a non-zero ideal of R with $L \cap K = 0$. If I is a proper ideal of K such that K/I is a ring with an identity 1 and $I + \tau(L) = K$, then $I[X] + L \neq R$.

Proof. Let $e \in K$ be an element such that $e + I$ is the identity of K/I and let $0 \neq c \in \tau(L)$ with $e - c \in I$. Suppose that $g = X^n c + \dots + c_o \in L$, is a polynomial of minimal degree in L $(n \geq 1)$. If $R = I[X] + L$, then $e = f + h$ with $f \in I[X]$ and $h \in L$. Moreover $c^v h = gq$ for some $q \in R$ and $v \leq \deg(h) - n + 1$ by ([2], Lemma 3.2, (ii)). Hence $c^v e = c^v f + c^v h = c^v f + gq$ and so $e - c^v e = (e - gq) - c^v f$. Since $e - c^v e \equiv 0 \pmod{I}$ we have $e - gq \in I[X]$. From among all the polynomials $q \in R$ with $e - gq \in I[X]$, choose one of minimal degree, k say, and let d be the leading coefficient of q. Put $s = cq - x^k cd$. The polynomial s has degree less than k and $e - gs \in I[X]$, a contradiction.

Lemma 2.4. Let K be a D-prime ring with $G_D(K) = 0$. Suppose that $P \neq 0$ is an ideal of R such that $P \cap K = 0$ and $I \supseteq P$ is the ideal of R such that $I/P = G(R/P)$. Then $I \cap K = 0$.

Proof. Let M be a D-ideal of K such that K/M is a D-simple ring with 1. If $\tau(P) \nsubseteq M$, then $M[X;D] + P \neq R$ by Lemma 2.3. Hence $0 = G_D(K) = (I \cap K)\tau(P)$, by the same way as in ([9], Lemma 4). Thus, $I \cap K = 0$.

Lemma 2.5. Let K be a D-prime ring with $G_D(K) = 0$ and let M be a maximal ideal of R with $M \cap K = 0$. Then K is a D-simple ring with 1.

Proof. Since $G_D(K) = 0$ there exists a D-ideal I of K such that K/I is a D-simple ring with 1. If $M \neq 0$, choose I such that $\tau(M) \not\subseteq I$. Hence $\tau(M) + I = K$ and so $I[X;D] \subseteq M$, by Lemma 2.3. Then $I = 0$, as required. The same result is clear if $M = 0$.

Now we are ready to prove the theorem.

Proof of Theorem 2.1. Suppose that K is a D-BMCR and let P be a prime ideal of R. By factoring out $P \cap K$ from K we may assume that K is D-prime, $P \cap K = 0$, and $G_D(K) = 0$. If $P \neq 0$, then $G(R/P) = 0$ by Lemma 2.4 and ([2], Lemma 3.3). If $P = 0$, for every D-ideal I of K such that K/I is D-simple with an identity we have $G((K/I)[X;D]) = 0$ by Lemma 2.2. Then $G(R) \subseteq I[X;D]$ and so $G(R/P) = G(R) \subseteq G_D(K)[X;D] = 0$. Therefore, R is a BMCR.

Now, let M be a maximal ideal of R. Put $\bar{K} = K/(M \cap K)$ $\bar{R} = \bar{K}[X;D]$, and $\bar{M}$ the image of M in $\bar{K}$. Then (II) follows from Lemma 2.5.

Conversely, suppose that the conditions (I) and (II) hold and let Q be a D-prime ideal of K. Then $Q[X;D]$ is a prime ideal of R and so $Q[X;D] = \cap\{M: M \supseteq Q[X;D]$ and R/M is a simple ring with $1\} = \cap\{M: M \supseteq Q[X;D]$ is a maximal ideal of $R\}$. Hence, Q is equal to the intersection of the ideals $M \cap K$ and then $G_D(K/Q) = 0$.

3. Some Assumptions on (K,D)

Hereafter, we shall suppose that every ring has an identity element and if $K \subset S$ is a ring extension, then K and S share the identity 1. We shall consider the following condition.

(C) *For every D-prime ideal Q of K such that $M(L(K/Q))=0$, then $L(K/Q)$ is the maximum ideal among all the ideals I of K/Q with $M(I) = 0$.*

If the condition (C) holds and M is a maximal ideal of K, then $M/M(M) = L(K/M(M))$. Hence $K/M(M)$ has a unique maximal ideal $M/M(M)$ and so it is D-simple. Consequently, $M(M)$ is a D-maximal ideal.

On the other hand, if (K,D) is a QFDR and the condition (C) is satisfied, for every D-prime ideal Q of K, $L(K/Q)$ is the unique prime ideal of K/Q with $M(L(K/Q)) = 0$. Therefore, for every D-prime ideal Q of K there exists a unique prime P with $M(P) = Q$, where P is the ideal such that $P/Q=L(K/Q)$. Moreover, Q is a D-maximal ideal if and only if P is a maximal ideal.

Theorem 3.1. Assume that the differential ring (K,D) satisfies the condition (C). Then K is a D-BMCR if and only if K is a BMCR and (K,D) is a QFDR.

Proof. Suppose that K is a D-BMCR. Then (K,D) is a QFDR. Further, if P is a prime ideal of K, put $Q = M(P)$. Then $M(G(K/Q)) = G_D(K/Q) = 0$ and $M(P/Q) = 0$. Hence, by (C), $G(K/Q) = P/Q = L(K/Q)$ and it follows that $G(K/P) = G((K/Q)/(P/Q)) = 0$. Therefore, K is a BMCR.

Conversely, if Q is a D-prime ideal, then $Q = \cap\{M(P)$: P prime$\}$, because (K,D) is a QFDR. Since every prime is an intersection of maximal ideals we have $Q = \cap\{M(M): M$ maximal$\}$. Then Q is the intersection of the D-maximal ideals $M(M)$.

The following is clear.

Corollary 3.2. Suppose that (K,D) is a QFDR and the condition (C) is satisfied. Then the following statements are equivalent.

(i) K is a BMCR.

(ii) K is a D-BMCR.

(iii) $R = K[X;D]$ is a BMCR and for every maximal ideal M of R, $M \cap K$ is a D-maximal ideal of K.

Now we shall consider some particular cases in which the condition (C) is satisfied. In [2], we said that a differential ring (K,D) is a FDR if D satisfies (F) on K, where (F) is the condition given in ([3], section 4). We say here that (K,D) is a *strong finite differential ring* (abbr. SFDR) if the following condition (SF) is satisfied:

(SF) *For every* $a \in K$ *there exists a positive integer* $m = m(a)$ *such that* $D^m(b)$ *is contained in the ideal of* K *generated by* $b, D(b), \ldots, D^{m-1}(b)$, *for each* $b \in KaK$.

Lemma 3.3. Let (K,D) be a differential ring. Suppose that either K satisfies the descending chain condition on two sided ideals or (K,D) is a SFDR. Then the condition (C) is satisfied.

Proof. Let Q be a D-prime ideal of K and put $\bar{K} = K/Q$. Suppose that I is an ideal of $\bar{K}$ with $I \supseteq L(\bar{K})$ and $M(I) = 0$. For each $a \in I$ we have $\bigcap_{i=0}^{\infty} D^{-i}(\bar{K}a\bar{K}) = M(\bar{K}a\bar{K}) = 0$. Then, from the assumption, there is an integer m such that $\bigcap_{i=0}^{m-1} D^{-i}(\bar{K}a\bar{K}) = 0$. Hence $(\bar{K}a\bar{K})^m = 0$ and so $a \in L(\bar{K})$. Therefore, $I = L(\bar{K})$.

When K satisfies the descending chain condition on two sided ideals, it is not clear whether (K,D) must be a QFDR. We have

Lemma 3.4. Assume that K is a D-prime differential ring which satisfies the descending chain condition on two sided ideals. Then the following are equivalent.

(i) $L(K)$ is a nilpotent ideal.

(ii) $L(K)$ is the union of all the nilpotent ideals of K.

(iii) $M(L(K)) = 0$.

Proof. (i) → (ii). It is clear.

(ii) → (iii). Let $a \in M(L(K))$ be. Put $I_j = \sum_{i=0}^{j} KD^i(a)K$ and denote by $\ell(I_j)$ the left annihilator of I_j in K. Then I_j is a nilpotent ideal and $\ell(I_0) \supseteq \ell(I_1) \supseteq \ldots \supseteq \ell(I_n) \supseteq \ldots$. There is an integer, m say, such that $\ell(I_m) = \ell(I_s)$ for $s \geq m$. Then if $x \in \ell(I_m)$ we have $xI_j = 0$ for all j, and so $x\sum_{i=0}^{\infty} KD^i(a)K = 0$. Therefore, either $a = 0$ or $\ell(I_m) = 0$. If $\ell(I_m) = 0$, since I_m is nilpotent it follows easily that $I_m = 0$. Then $a = 0$.

(iii) → (i). Since $\bigcap_{i=0}^{\infty} D^{-i}(L(K)) = M(L(K)) = 0$, there is an integer, m say, such that $\bigcap_{i=0}^{m-1} D^{-i}(L(K)) = 0$. Then $L(K)^m = 0$.

Corollary 3.5. Let (K,D) be a differential ring and suppose that one of the following holds.

(a) (K,D) is a SFDR.

(b) K satisfies the descending chain condition on two sided ideals and for every D-prime ideal Q of K, $L(K/Q)$ coincides with the union of all the nilpotent ideals of K/Q.

Then the following are equivalent.

(i) K is a BMCR.

(ii) K is a D-BMCR.

(iii) R is a BMCR and for every maximal ideal M of R, $M \cap K$ is a D-maximal ideal of K.

Proof. Since (K,D) is a QFDR, it follows easily by Corollary 3.2 and Lemmas 3.3 and 3.4.

Remark 3.6. Corollary 3.5 can be applied when K is a ring which satisfies the ascending and the descending chain conditions on two sided ideals. In fact, the ascending chain condition implies that (K,D) is a QFDR. On the other hand, it can also be applied to obtain a corollary corresponding to ([2], Corollary 3.9). Thus, if T is a Galois extension of a BMCR K of characteristic p with a Galois p-group G and $T_K \oplus > K_K$, then T is also a BMCR.

4. Liberal and Intermediate Extensions

In this section we consider a liberal extension $S = \sum_{i=1}^{n} Ka_i$ of K and a K-derivation D of S, and an intermediate extension T with $D(T) \subseteq T$. *We say that* T *is a differential intermediate extension*. Applying the methods used in [7] and [8] we study the relation between the prime and the D-prime ideals of T. As a consequence we shall see that Corollary 3.2 can be applied in this case.

Firstly, we prove the following.

Lemma 4.1. Let K be a centrally closed prime ring with center C and M a torsion free liberal R-bimodule with a generating set of n centralizing elements. If $\phi: M \to M$ is a K-bimodule homomorphism, then $c_0, c_1, \dots, c_{n-1}$ in C exist such that $\phi^n(x) = \sum_{i=0}^{n-1} c_i \phi^i(x)$, for every $x \in M$.

Proof. We can suppose that $M = \sum_{i=1}^{n} Km_i$ is free over K with the centralizing basis $\{m_i\}$. Then the centralizer of K in M, V say, is equal to $\sum_{i=1}^{n} Cm_i$ and $\phi(V) \subseteq V$. Since C is a field, there exists $t \leq n$ such that $m_1, \phi(m_1), \dots, \phi^{t-1}(m_1)$ are linear independent over C and $\phi^t(m_1) = \sum_{i=0}^{t-1} c_i \phi^i(m_1)$, for $c_i \in C$. Then, $N = \sum_{i=0}^{t-1} K\phi^i(m_1)$ is a K-sub-bimodule and $\phi(N) \subseteq N$. Moreover, $\phi^t(x) = \sum_{i=0}^{t-1} c_i \phi^i(x)$, for all $x \in N$, as it can easily be verified.

On the other hand, we can find a subset of the generators $m_{t+1}, \dots, m_n$ say, such that $E = \{m_1, \phi(m_1), \dots, \phi^{t-1}(m_1), m_{t+1}, \dots, m_n\}$ is a basis of V. Since $M \simeq K \otimes_C V$ ([7], Lemma 2.2), E is a centralizing basis of M. Let $\bar{M} = M/N = \sum_{i=t+1}^{m} K\bar{m}_i$ and $\bar{\phi}: \bar{M} \to \bar{M}$ the map induced by ϕ. Then $\bar{M}$ has a basis $\bar{m}_{t+1}, \dots, \bar{m}_n$.

By induction, $d_{t+1}, \dots, d_n$ in C exist such that $\bar{\phi}^{n-t}(\bar{x}) = \sum_{j=0}^{n-t-1} d_j \bar{\phi}^j(\bar{x})$, for all $\bar{x} \in \bar{M}$. Hence $y = \phi^{n-t}(x) - \sum_{j=0}^{n-t-1} d_j \phi^j(x) \in N$ and so $\phi^t(y) = \sum_{i=0}^{t-1} c_i \phi^i(y)$. It follows that $\phi^n(x) = \sum_j d_j \phi^{t+j}(x) + \sum_i c_i \phi^{i+n-t}(x) - \sum_{i,j} c_i d_j \phi^{i+j}(x)$, for every $x \in M$.

Let $S = \sum_{i=1}^{n} Ka_i$ a liberal extension of K, D a K-derivation of S and T a differential intermediate extension. If S is D-prime, then K is prime and we can consider CK, the central closure of K. Furthermore, CS is a liberal extension of CK and there is a CK-derivation D^* of CS such that $D^*/S = D$, where CS is also a D^*-prime ring (see [2], section 4). Finally, CT is a differential intermediate extension. In the rest of this section we use this notation.

The following improves Theorem 4.3 of [2].

Theorem 4.2. For every D-prime ideal Q of T there exists a prime ideal P such that $M(P) = Q$.

Proof. As in ([7], Theorem 3.2) we can see that $Q \cap K$ is prime and there exists a D-prime ideal Q' of S such that $Q' \cap K = Q \cap K$ and $Q' \cap T \subseteq Q$. By factoring out from K, T and S respectively the ideals $Q' \cap K$, $Q' \cap T$ and Q' we may suppose that K is prime, S is a D-prime liberal extension and Q is a D-prime ideal of T with $Q \cap K = 0$.

If K is a centrally closed prime ring, (T,D) is a SFDR by the former lemma. Then, there exists a maximal ideal, P say, with respect to $M(P) = Q$, and P is clearly prime.

In general, consider $CK \subseteq CT \subseteq CS$ and the derivation D^*. Then CQ is a D^*-ideal of CT and $CQ \cap T = Q$. This can easily be verified as in the proof of Theorem 3.3 in [7]. Let Q' be a D^*-maximal ideal of T with respect to $Q' \supseteq CQ$ and $Q' \cap T = Q$. Then Q' is a D^*-prime ideal of CT and $Q' \cap CK = 0$. From the first part there is a prime ideal P' of CT such that $M(P') = Q'$. Therefore, $P = P' \cap T$ is a prime ideal of T and $M(P) = Q$.

The following corollary completes the results of [2] concerning with intermediate extensions (see Corollary 4.5 in [2])

Corollary 4.3. Let K be a Jacobson ring and T a differential intermediate extension. Then $T[X;D]$ is also a Jacobson ring.

Proof. Since (T,D) is a QFDR, we can apply ([2], Corollary 3.6). Then we must prove that T is a Jacobson ring. But this is an easy consequence of going up ([6], Corollary 4.2) and incomparability ([7], Theorem 3.3).

Theorem 4.4. The differential ring (T,D) satisfies the condition (C). Moreover, if Q is a D-prime ideal and I is an ideal with $M(I) = Q$, then $I^n \subseteq Q$.

Proof. By factoring out convenient ideals we may suppose, as in Theorem 4.2, that K is prime, S is a D-prime liberal extension, T is a differential intermediate extension and Q is a D-prime ideal of T such that $Q \cap K = 0$.

If K is a centrally closed prime ring we consider the differential ring $(T/Q,D)$ and we apply Lemma 4.1. If A is an ideal with $M(A) = 0$, as in Lemma 3.3 we have $A^n \subseteq \bigcap_{i=0}^{n-1} D^{-i}(A) = 0$. Then $I^n \subseteq Q$.

In general, consider the central closure CK of K. Then $CK \subseteq CT \subseteq CS$, where CS is a D^*-prime ring and $D^*(CT) \subseteq CT$. Firstly, suppose that P is a prime ideal of T with $M(P) = Q$. We now use the same way used in ([7], Theorem 3.3). Then $CQ \cap T = Q$ and there exists a D^*-prime ideal Q' of CT which is D^*-maximal with respect to $Q' \supseteq CQ$ and $Q' \cap T = Q$. Also there exists a prime ideal P' of CT which is maximal with respect to $P' \supseteq CP + Q'$ and $P' \cap T = P$. Further, $P' \cap CK = 0$ and $M(P') \supseteq Q'$. By Theorem 4.2, $Q' = M(H)$ for a prime ideal H of CT. Then $H^n \subseteq Q' \subseteq P'$ and hence $H \subseteq P'$. It follows that $H = P'$, by ([7], Theorem 3.3). Therefore, $P^n \subseteq P'^n \cap T \subseteq Q' \cap T = Q$.

Finally, suppose that $M(I) = Q$. Then $I \cap K = 0$ and let P be an ideal of T which is maximal with respect to $P \supseteq I$ and $P \cap K = 0$. Hence P is prime. Furthermore, $Q = M(H)$ for a prime ideal H and we can see that $H = P$ as above. It follows that $I^n \subseteq P^n \subseteq Q$ and the proof has been completed.

Combining the above results we have the following.

Corollary 4.5. For every D-prime ideal Q of T there is a unique prime ideal P of T such that $M(P) = Q$. Moreover, P is the ideal of T such that $P/Q = L(T/Q)$, and $P^n \subseteq Q$. Finally, Q is D-maximal if and only if P is maximal.

Corollary 4.6. The following conditions are equivalent

(i) K is a BMCR.

(ii) T is a BMCR.

(iii) T is a D-BMCR.

(iv) $R = T[X;D]$ is a BMCR and for every maximal ideal M of R, $M \cap T$ is a D-maximal ideal of T.

Proof. (i) $\longleftrightarrow$ (ii) It can easily be proved using going up and incomparability ([6], Corollary 4.2 and [7], Theorem 3,3).

(ii) $\longleftrightarrow$ (iii) $\longleftrightarrow$ (iv) It is a direct consequence of Corollary 3.2, and Theorems 4.2 and 4.4.

5. Remarks and Examples

If K is a right Noetherian Jacobson ring and D is a derivation of K, then $K[X;D]$ is also a Jacobson ring ([2] and [4]). The same result is not true for Brown-McCoy rings.

Example 5.1. Let $K = \mathbb{Q}[Y]$ be a polynomial ring over the field $\mathbb{Q}$ of the rational numbers and let D be the $\mathbb{Q}$-derivation of K defined by $D(Y) = Y$. Then K is a BMCR and (K,D) is a QFDR. We can easily see that $G_D(K) = (Y)$, the ideal generated by Y, and $D(K) = 0$. Hence K is not a D-BMCR. If M is a maximal ideal of $R = K[X;D]$, then $M \cap K = (Y)$ is D-maximal. Therefore, R is not a BMCR, by Theorem 2.1.

There is an alternative definition of a D-BMCR. In [2], we said that K is a D-Jacobson ring if $M(J(K/Q)) = 0$, for every D-prime ideal Q of K. We say here that K is a weakly D-Brown-McCoy ring (abbr. wD-BMCR) if $M(G(K/Q)) = 0$, for every D-prime ideal Q of K. Thus, a D-BMCR is a wD-BMCR, but the converse is not true. In fact, the differential ring given in example 5.1 is a wD-BMCR. This example also shows that if K is a wD-BMCR, $R = K[X;D]$ need not be a BMCR.

If K is a right Noetherian ring, then K satisfies the ascending and the descending chain conditions on two sided ideals and also (K,D) is a FDR. Hence, if Q is a D-prime ideal of K, then there exists a prime P with $M(P) = Q$ and $P/Q = L(K/Q)$ by Lemma 3.3. Therefore, the prime radical $L(K/Q)$ is prime (see also [4], Theorem 2.2). This result is similar to that in Corollary 4.5.

Now, if the abelian group $(K,+)$ is torsion free we can easily see that $M(L(K)) = L(K)$. Then, if we suppose in addition that K is an algebra over the field $\mathbb{Q}$ we have $0 = M(P/Q) = M(L(K/Q)) = L(K/Q) = P/Q$ and so $P = Q$. Finally, if P is a given prime we put $Q = M(P)$ and as above we have $P = M(P)$ is a D-prime ideal. Therefore the set of all prime ideals coincides with the set of all D-prime ideals in this case.

More generally, suppose that the condition (C) is satisfied and let P be a prime such that $(K/M(P),+)$ is torsion free. As above we get $P = M(P)$ is a D-ideal of K. Thus if K is an algebra over $\mathbb{Q}$ and the condition (C) holds, every prime ideal of K is a D-prime ideal.

Suppose in addition that (K,D) is a QFDR. Then for every D-prime ideal Q there is a prime P with $Q = M(P) = P$. Therefore, we again have that the set of all prime ideals coincides with the set of all D-prime ideals in this case. Thus we have

Remark 5.2. Suppose that T is an algebra over the field of rational numbers and D is a derivation of T, and assume that one of the following conditions is satisfied.

(i) The ascending and the descending chain conditions on two sided ideals of T.

(ii) (T,D) is a SFDR.

(iii) T is a differential intermediate extension of K and $D/K = 0$.

Then the set of all the D-prime ideals of T coincides with the set of all the prime ideals.

Finally, the following examples are easy to verify.

Example 5.3. Let (K,D) be the differential ring given in ([2], section 2). Then K is a BMCR, $R = K[X;D]$ is not a BMCR, and K is not a wD-BMCR. Further, the condition (II) in Theorem 2.1 also holds in this case.

Example 5.4. Let (K,D) be the differential ring given in ([2], Example 5.2). Then K is not a BMCR, but it is a D-BMCR and $R = K[X;D]$ is a BMCR.

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