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POLYNOMIAL VECTOR FIELDS ON THE TORUS

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Abstract: In this paper it is shown that the structurally stable polynomial vector fields on the torus T^2, with singularities, are open and dense in the set of such vector fields. Many kinds of distinct dynamical phenomena are also presented by a list of examples including the Cherry flows. The above result works for analytical vector fields on T^2 with the same proof.

Introduction: We first define what we mean by a polynomial vector field on T^2. We initially consider the vector fields on $\mathbb{R}^2$ of the form:

$$X_K(x,y) = (P_K(x,y),\ Q_K(x,y)),$$

where,

$$P_K(x,y) = \sum_{j=0}^{K} \Big[\sum_{m+n=j} (a_{mn}\cos(mx)\cos(ny) + b_{mn}\sin(mx)\cos(ny) +$$

$$c_{mn}\cos(mx)\sin(ny) + d_{mn}\sin(mx)\sin(ny)) \Big],$$

and,

$$Q_K(x,y) = \sum_{j=0}^{K} \Big[\sum_{m+n=j} (a'_{mn}\cos(mx)\cos(ny) + b'_{mn}\sin(mx)\cos(ny) +$$

$$+ c'_{mn}\cos(mx)\sin(ny) + d'_{mn}\sin(mx)\sin(ny)) \Big].$$

Let $\Pi : \mathbb{R}^2 \to T^2$ given by $\Pi(x,y) = (e^{ix}, e^{iy})$ be the natural covering map. We denote $\Pi(X_k)$ by $\bar{X}_k$. The set of degree k polynomial vector fields on T^2 consists of the vector fields

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$\tilde{X}_k$ and will be indicated by P_k. We often identify T^2 with the square $[0,2\pi]\times[0,2\pi]$ as usually. It is clear that P_k is a finite dimensional vector space. It will be always considered with the coeficient topology. We have the following.

Theorem: Let $A_k \subset P_k$ be the set of the degree k polynomial vector fields with at least one singularity. Then the set of Morse-Smale vector fields in A_k is open and dense for all $k = 1,2,3,\ldots$.

A good and perhaps difficult question is to show the same result for vector fields in $P_k - A_k$. The first section is dedicated to a list of examples presenting different kind of dynamical phenomena including Cherry flows, even in P_1 (*). The second section is dedicated to the proof of the Theorem.

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1. The phase space of some polynomial vector fields on T^2.

Our first examples will be projection on T^2 of gradients of real functions defined on $\mathbb{R}^2$, of the orthogonal of such gradients or of their perturbations.

The following remarks will be useful in dealing with this kind of examples.

- If $X = \nabla f$ is a gradient, then $\tilde{X}$ has no closed orbits which bound discs on T^2.

(*) These Cherry flows examples were a result of a joint work with Alcides Lins.

- If Y is orthogonal to a gradient ∇f then the orbits of Y are contained in level curves of f. In many cases it is easy to recognize such curves.

- Notice that in general, $\tilde{\nabla f}$, $f:\mathbb{R}^2\to\mathbb{R}$, is not a gradient of a real function defined on T^2.

Example 1.1. Let $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = ax + b\sin x + cy + d\sin y$. Then $\nabla f(x,y) = (a + b\cos x,\ c + d\cos y)$.

If $|\frac{a}{b}| < 1$ and $|\frac{c}{d}| < 1$, the phase space of $\tilde{\nabla f}$ is given in Fig. 1, where F, P, S_1, S_2 are hyperbolic singularities. Here we have a Morse Smale vector field with singularities and without limit cycles.

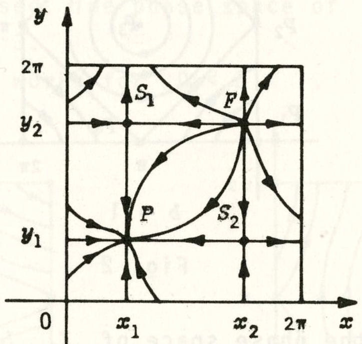


Fig. 1

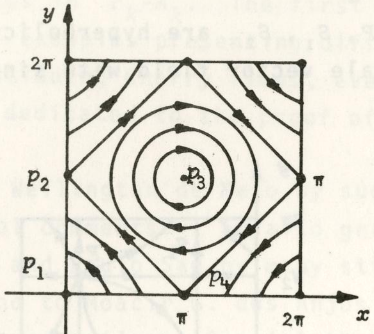
Example 1.2. Let Y_b be the orthogonal of ∇f where $f(x,y) = -b\cos x - \cos y$, $b > 0$. Then $Y_b(x,y) = (-\sin y,\ b\sin x)$. We are going to describe the phase space of Y_b when $b = 1$.

The singularities of Y_1 $\bmod(2\pi\mathbb{Z}\times 2\pi\mathbb{Z})$ are $p_1 = (0,0)$, $p_2 = (0,\pi)$, $p_3 = (\pi,\pi)$ and $p_4 = (\pi,0)$. We have that p_i is a center for $DY_1(p_i)$, $i=1,3$, and p_2, p_4 are hyperbolic saddles of Y_1.

Since $f(p_2+(2K\pi,\ 2m\pi)) = f(p_4+(2K\pi,\ 2m\pi)) = 0$ and $f(x,y)=0$ if and only if $y \mp x = \pi \pmod{2K\pi}$, we have that the lines $y = \pm x + \pi + 2K\pi$ are made by orbits of Y_1.

Let Q be the square of vertices $(0,\pi)$, $(\pi,0)$, $(\pi,2\pi)$ and $(2\pi,\pi)$. It is easy to see that Q is Y_1-invariant and that $f(Q) = [-2,0]$.

We have that -2 and 0 are the unique singular levels of $f|_Q$, $(f|_Q)^{-1}(-2) = p_3$, $(f|_Q)^{-1}(0)$ is the boundary of Q and the connected components of $(f|_Q)^{-1}(\alpha)$ are diffeomorphic to circles for all $\alpha \in (-2,0)$. Then the phase space of Y_1 is the one described in Fig. 2.



$b = 1$

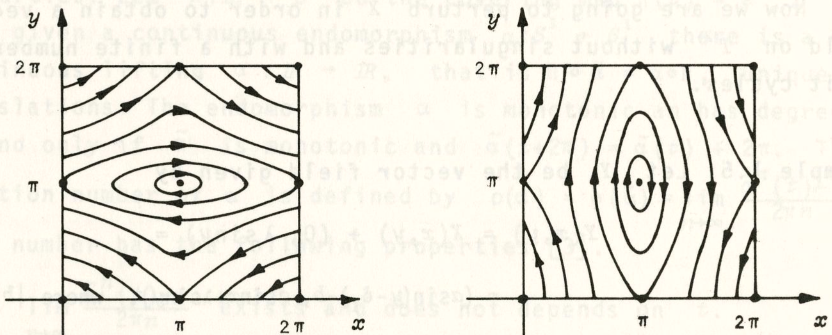
Fig. 2

The study of the phase space of Y_b, $b > 1$ can be done as follows. The singularities of Y_b mod$(2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$ are $p_1 = (0,0)$, $p_2 = (0,\pi)$, $p_3 = (\pi,\pi)$ and $p_4 = (\pi,0)$. We have that p_i is a center for $DY_b(p_i)$ $i=1,3$, and p_2, p_4 are hyperbolic saddles of Y_b. It is clear that $p_3\ (p_1)$ is the point of maximum (minimum) of the Morse function $f(x,y) = -b\cos x - \cos y$, and then the orbits of Y_b near $p_3\ (p_1)$ being the level curves of f, they are closed of Y_b, and thus $p_3\ (p_1)$ are in fact a center for Y_b itself. If one looks the slope of Y_b along the square with vertices p_4, p_2, $p_4+(0,2\pi)$, $p_2+(2\pi,0)$ and along the segment $[p_4, p_4+(0,2\pi)]$, then by the Poincaré Bendixon

Theorem and because $f\big|_{[p_4,\,p_4+(0,2\pi)]}$ has (π,π) as the point of absolute maximum and $p_4, p_4+(0,2\pi)$ as the points of absolute minimum, and f is monotonic on the intervals $[p_4,(\pi,\pi)]$, $[(\pi,\pi),\ p_4+(0,2\pi)]$, one can conclude that the saddle p_4 is doubly connected with the saddle $p_4+(0,2\pi)$. Thus we have a graph bounding a disc containing p_3. By the analycity of Y_b all the orbits in the interior of such disc except p_3 are closed. Similar analysis works for p_1 and p_2.

We have that $-Y_b(x,\ \pi+\alpha)$ is the reflection of $Y_b(x,\pi-\alpha)$ with respect to the horizontal line $Y = \pi$. Thus the projection of the complement of the discs bounded by the graphs of Y_b in $\mathbb{R}^2$ mod$(2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$ are made up by cilinders on T^2. Now, $f(x,0) = f(x,2\pi)$ and f is strictly monotonic in the intervals $[0,\pi]$ and $[\pi,2\pi]$. Therefore by the Poincaré-Bendixon Theorem in such cilinders all the orbits of Y_b outside the discs bounde by the graphs are closed. The phase space of Y_b, $b > 1$, is the one described in Fig. 3.

Similar analysis works for $0 < b < 1$.



$0 < b < 1$ $\qquad$ $b > 1$

Fig. 3

Notice that all the orbits of $\tilde{Y}_b$ except the singularities and the saddle separatrices are closed.

We are going to modify a vector field Y_b as above in order to obtain a vector field with singularities and limit cycles.

Example 1.3: Let $Z_{\varepsilon,b}$ be the vector field given by
$Z_{\varepsilon,b}(x,y) = Y_b(x,y) + \varepsilon(\cos x, 0) = (-\operatorname{sen} y + \varepsilon\cos x, b\operatorname{sen} x)$, where $b >> 1$ and $\varepsilon > 0$ small.

To see that $\tilde{Z}_{\varepsilon,b}$ has limit cycles, it is enough to use the Poincaré-Bendixon Theorem in a cylinder made by closed orbits of $\tilde{Y}_b$ whose boundary are saddle connections, because $\tilde{Z}_{\varepsilon,b}$ is transversal to this boundary pointing inward the cylinder. The closed orbits which appears in the cylinder are isolated because $\tilde{Z}_{\varepsilon,b}$ is analytic.

Example 1.4. Let X be the vector field given by
$X(x,y) = (a\sin(y-\delta_1)-b,\ c\sin x)$, $|b| > |a|$.

The slope $\dfrac{c\sin x}{a\sin(y-\delta_1)-b}$ of X, is odd with respect to the vertical line $x = 0$ for each y fixed. Then the flow is even with respect that line. This implies that the vector field X has all its orbits closed and of type $(1,0)$.

Now we are going to perturb X in order to obtain a vector field on T^2 without singularities and with a finite number of limit cycles.

Example 1.5. Let Y be the vector field given by

$$Y(x,y) = X(x,y) + (0, \lambda\sin y) =$$
$$= (a\sin(y-\delta_1)-b,\ c\sin x+\lambda\sin y), \quad \text{where } |b|>|a|, \lambda\neq 0$$

and $|c|$ sufficiently small so that the difference between the maxima and minima of the graphs of the orbits of X are less than π.

The vector field Y is transversal and points inward the boundary of the cylinder bounded in T^2 by the orbits of $\tilde{X}$ through the points $(0,0)$ and $(0,\pi)$. Then $\tilde{Y}$ has a finite number of limit cycles and has no singularities.

Example 1.6. Consider the family of vector fields

$$X(x,y) = (a_1\sin(x-\delta_1) + a_2\sin(y-\delta_2) + a_3, b_1\sin(x-\gamma_1) +$$
$$+ b_2\sin(y-\gamma_2) + b_3), |a_1| + |a_2| < |a_3| \quad \text{and } |b_1| + |b_3| < |b_2|.$$

Then any vector field $\tilde{X}$ on T^2 has a finite number of limit cycles and has no singularities. This follows from the same argument above considering the cylinder bounded by

$$y = \gamma_2 + \frac{\pi}{2} \quad \text{and} \quad y = \gamma_2 + \frac{3\pi}{2}$$

Example 1.7. *The Cherry Flows*. Here we are going to construct one parameter families of polynomial vector fields on T^2 which exhibit Cherry flows for infinitely many values of the parameter. This shows the richness of the dynamic of such vector fields.

To do this we need the concept of rotation number for degree 1 monotonic continuous endomorphisms of the circle and some of its properties. Let $\Pi: \mathbb{R} \to S^1$ be the covering map $\Pi(t) = e^{it}$. Notice that given a continuous endomorphism $\alpha: S^1 \to S^1$ there is a continuous lifting $\tilde{\alpha}: \mathbb{R} \to \mathbb{R}$, that is $\Pi\circ\tilde{\alpha} = \alpha\circ\Pi$, unique up to translations. The endomorphism α is monotonic an has degree 1 if and only if $\tilde{\alpha}$ is monotonic and $\tilde{\alpha}(t+2\pi) = \tilde{\alpha}(t) + 2\pi$. The rotation number of α is defined by $\rho(\alpha) = \rho(\tilde{\alpha}) = \lim\limits_{n\to\infty} \dfrac{\tilde{\alpha}^n(t)-t}{2\pi n}$.

This number has the following properties [3].

i) $\lim\limits_{n\to\infty} \dfrac{\tilde{\alpha}^n(t)-t}{2\pi n}$ exists and does not depends on t.

ii) $\rho(\alpha) = \dfrac{m}{n}$, $m, n \in \mathbb{Z}$, $n > 0$, if and only if α has a periodic point x of period $n(\alpha^n(x) = x)$, or equivalently, $\tilde{\alpha}^n(t) = t + 2\pi m$, where $\Pi(t) = x$.

iii) Let $\beta: S^1 \to S^1$ be a degree 1 monotonic continuous endomorphism. Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\tilde{\beta}-\tilde{\alpha}\| = \sup\limits_{t\in\mathbb{R}} |\tilde{\beta}(t)-\tilde{\alpha}(t)| < \delta$ then $|\rho(\beta) - \rho(\alpha)| < \varepsilon$.

iv) $\rho(\tilde{\alpha} + 2\pi n) = \rho(\tilde{\alpha}) + n$ for any integer n.

Now we are going to describe what we mean by a Cherry flow on T^2.

The flow of a vector field X on T^2 is a Cherry flow if its orbit structure is as follows:

(1) X has exactly two singularities, a source F and a saddle S, both hyperbolic;

(2) X has a non-trivial recurrence in $\Lambda = T^2 - W^u(F)$, that is there is $x \in \Lambda$ such that $x \in \omega(x)$.

It follows from the above facts about rotation number and the Denjoy Schwartz Theorem that if X is a vector field in T^2 satisfying (1), for which we have a transversal circle Σ and an associated degree 1 monotone Poincaré Transformation $f : \Sigma \to \Sigma$ such that $\rho(f)$ is irrational, X induces a Cherry flow.

Now we are going to prove that

$X_\alpha(x,y) = (1 + \cos x + \sin y,\ \alpha(1 + \sin x) + \cos y), \quad \alpha \in [0,1]$

induces a Cherry flow for infinitely many values of α.

Let $X_\alpha(x,y) = (1 + \cos x + \sin y, \alpha(1 + \sin x) + \cos y)$, $\alpha \in [0,1]$.

We have $DX_\alpha(x,y) = \begin{pmatrix} -\sin x & \cos y \\ \alpha\cos x & -\sin y \end{pmatrix}$. Thus $(\frac{3\pi}{2}, \frac{3\pi}{2}) = F$ is a source of X_α for all α.

The horizontal component of X_α, in the square $[0,2\pi]\times[0,2\pi]$, is zero along the closed convex curve Γ given implicitly by the equation $1 + \cos x + \sin y = 0$, and is contained in the square $[\frac{\pi}{2}, \frac{3\pi}{2}] \times [\pi, \frac{3\pi}{2}] = Q$.

The vertical component of X_α in that square is zero along the curve H_α given implicitly by the equation $\alpha(1+\sin x)+\cos y = 0$.

In Q H_α is graph of a C^0 increasing monotonic function defined in an interval $[a(\alpha), \frac{3\pi}{2}]$, where $a(\alpha)$ is a C^0 nondecreasing monotonic function, $a(\alpha) = \frac{\pi}{2}$ if $\alpha \le \frac{1}{2}$ and $(a(\alpha),\pi) \in H_\alpha$ if $\alpha \ge \frac{1}{2}$. Moreover, $H_{\alpha_1}(x) > H_{\alpha_2}(x)$ whenever $\alpha_1 < \alpha_2$ and $x \in [a(\alpha), \frac{3\pi}{2}]$, $F \in H_\alpha$ for all α, and H_0 is horizontal. Since Γ in the square $Q_1 = [\frac{\pi}{2},\pi] \times [\pi, \frac{3\pi}{2}]$ is graph of a C^0 decreasing monotonic function connecting the vertices $(\frac{\pi}{2}, \frac{3\pi}{2})$ and (π,π), we have that X_α has exactly two singularities: the source F and a singularity $S_\alpha \in \Gamma \cap Q_1$.

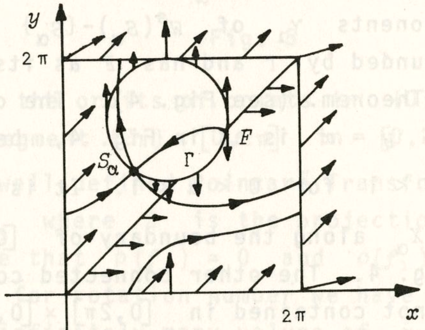


Fig. 4

We claim that S_α is hyperbolic saddle. In fact, the eigenvalues of $DX_\alpha(S_\alpha)$ are given by

$$\lambda = \frac{-(\sin x_\alpha + \sin y_\alpha) \pm \sqrt{(\sin x_\alpha - \sin y_\alpha)^2 + 4\alpha\cos x_\alpha \cos y_\alpha}}{2}$$

where $S_\alpha = (x_\alpha, y_\alpha) \in Q_1$. We have $\pi - x_\alpha > y_\alpha - \pi$ and thus $\sin x_\alpha + \sin y_\alpha > 0$, because Γ is a convex curve. Here $0 < \alpha < 1$. Since $(x_\alpha, y_\alpha) \in Q_1$, $(\sin x_\alpha - \sin y_\alpha)^2 + 4\alpha\cos x_\alpha \cos y_\alpha > 0$ and $\alpha\cos x_\alpha \cos y_\alpha > \sin x_\alpha \sin y_\alpha$. Then

$$\lambda_1 = \frac{-(\sin x_\alpha + \sin y_\alpha) - \sqrt{(\sin x_\alpha - \sin y_\alpha)^2 + 4\alpha\cos x_\alpha \cos y_\alpha}}{2} < 0$$

and

$$\lambda_2 = \frac{-(\sin x_\alpha + \sin y_\alpha) + \sqrt{(\sin x_\alpha - \sin y_\alpha)^2 + 4\alpha\cos x_\alpha \cos y_\alpha}}{2} > 0,$$

and the claim is proved for $0 < \alpha < 1$. For $\alpha = 0$ and $\alpha = 1$ the claim also holds and the proof is easy. The eigenspace associated to λ_1 is given by the equations

$$\begin{cases} -(\sin x_\alpha + \lambda_1)x + (\cos y_\alpha)y = 0 \\ (\alpha \cos x_\alpha)x - (\sin y_\alpha + \lambda_1)y = 0 \end{cases}$$

Then its slope is $\dfrac{y}{x} = \dfrac{\lambda_1 + \sin x_\alpha}{\cos y_\alpha} > 0$. This fact, the convexity of Γ and the orientation of X_α along Γ implies that one of the connected components γ_1 of $W^s(S_\alpha)-\{S_\alpha\}$ lies entirely inside the disc bounded by Γ and has F as its α-limit, by the Poincaré-Bendixon Theorem. (see Fig. 4) The direction of X_α along the diagonal $y = x$ is as in Fig. 4, because $\dfrac{\alpha(1+\sin x)+\cos x}{1+\sin x+\cos x} < 1$ for $0 < \alpha < 1$. It is easy to see that the direction of X_α along the boundary of $[0,2\pi] \times [0,2\pi]$ is as indicated in Fig. 4. The other connected component γ_2 of $W^s(S_\alpha) - \{S_\alpha\}$ is not contained in $[0,2\pi] \times [0,2\pi]$. In fact, since X_α has no closed orbit inside $[0,2\pi] \times [0,2\pi]$, if $\gamma_2 \subset [0,2\pi] \times [0,2\pi]$ then by the Poincaré-Bendixon Theorem again $\alpha(\gamma_2) = F$ or $\alpha(\gamma_2) = S_\alpha$. Thus X_α has at least three singularities which is a contradiction.

Then by the configuration of X_α in $[0,2\pi]\times[0,2\pi]$ γ_2 crosses the segment $\{0\} \times [0,2\pi]$.

With the same kind of arguments one can check that the connected components β_1, β_2 of $W^u(S_\alpha) - \{S_\alpha\}$ are such that β_1 crosses the segment $[0,2\pi] \times \{2\pi\}$ and β_2 crosses the segment $\{2\pi\}\times[0,2\pi]$.

This gives a description of the phase space of X_α for $0 < \alpha < 1$. The phase space of X_0 and X_1 is given in Fig. 5.

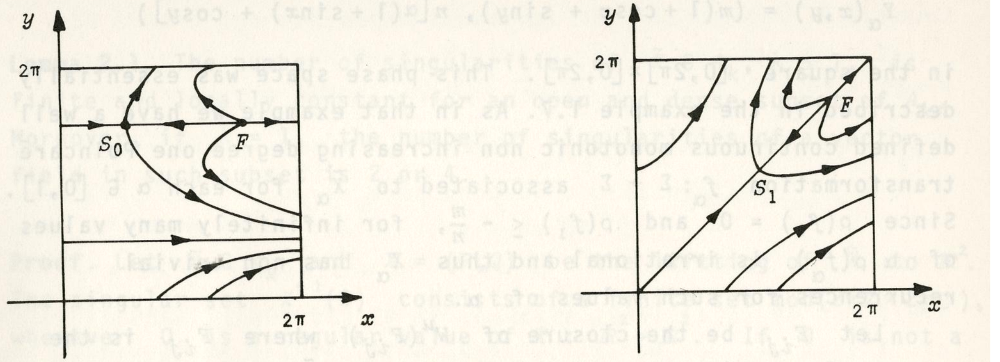


Fig. 5

We have that all the orbits of points in $\{0\} \times [0,2\pi] - \gamma_2$ cross positively the segment $\{2\pi\} \times [0,2\pi]$ in $[0,2\pi]\times[0,2\pi] \equiv T^2$. Then there is a well defined Poincaré Transformation $f_\alpha : \Sigma \to \Sigma$ associated to $\tilde{X}_\alpha$ where Σ is the projection of $\{0\}\times[0,2\pi]$ on T^2 [3]. We have that $\rho(f_0) = 0$ and $\rho(f_1) = -1$. Then from the facts stated for rotation number we have that X_α induces a Cherry flow for infinitely many values of $\alpha \in [0,1]$.

In the Cherry flow above the set $W^s(S_\alpha) \cup W^u(S_\alpha)$ is called a "Cherry fork".

Our last example is a one parameter family X_α of polynomial vector fields of degree $k = \max\{m,n\}$, which for infinitely many values of α presents $m.n$ "Cherry forks" such that the unstable manifolds of the sources does not contains saddles in its closures, and also non trivial recurrences.

Example 1.8. Cherry flow with finitely many forks

Let $X_\alpha(x,y) = (1+\cos(mx)+\sin(ny),\ \alpha(1+\sin(mx))+\cos(ny))$, $0 \leq \alpha \leq 1$, m,n positive integers.

We divide the square $[0,2\pi]\times[0,2\pi]$ into $m \cdot n$ retangles R_{ij} whose sides has lengths $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$. U_p to a translation and a change of coordinates of the form $(u,v) \to (mu,nv)$, we can see that the phase space of X_α in R_{ij} is the same as the phase space of the vector field Y_α given by

$$Y_\alpha(x,y) = (m(1+\cos x + \sin y),\ n[\alpha(1+\sin x) + \cos y])$$

in the square $[0,2\pi]\times[0,2\pi]$. This phase space was essentially described in the example 1.7. As in that example we have a well defined continuous monotonic non increasing degree one Poincaré transformation $f_\alpha : \Sigma \to \Sigma$ associated to $\tilde{X}_\alpha$ for each $\alpha \in [0,1]$. Since $\rho(f_0) = 0$ and $\rho(f_1) \leq -\frac{m}{n}$, for infinitely many values of α $\rho(f_\alpha)$ is irrational and thus X_α has non trivial recurrences for such values of α.

Let E_{ij} be the closure of $W^u(F_{ij})$ where F_{ij} is the source of X_α in R_{ij}. We claim that if $\tilde{X}_\alpha$ has non trivial recurrences, then $E_{ij} \supset E_{i_1 j_1}$ does not occurs.

This follows from the fact that

$$X_\alpha(x + \frac{2\pi}{m},\ y + \frac{2\pi}{n}) = X_\alpha(x,y),$$

and then $E_{ij} \supset E_{i_1 j_1}$ implies that $E_{i_1 j_1} \supset E_{ij}$. Then $\tilde{X}$ exhibits a closed curve Γ, which does not bounds a disc, made up by singularities and regular orbits, which is a contradiction.

2. Proof of the theorem

The proof of our Theorem follows essentially the same steps of the Peixoto's proof of the density of Morse-Smale vector fields on two-dimensional manifolds [4], except that we need to deal only with global perturbations instead of local ones. Let $X = (P,Q)$ be a polynomial vector field on $\mathbb{R}^2$. The perturbations to be used along the proof are mostly orthogonal perturbations, i.e., we consider the vector field $X^\perp = (-Q,P)$ and perturbations of the form $X_\varepsilon = X + \varepsilon X^\perp$. $\tilde{X}_\varepsilon$ is called an orthogonal perturbation of $\tilde{X}$, on T^2.

Our first step is to prove that the set B_k of the degree k polynomial vector fields on T^2 with all singularities hyperbolic, is dense in the set A_k, of the degree k polynomial vector fields on T^2 with singularities.

We start with the following

Lemma 2.1. The number of singularities of $\tilde{X} \in A_k$, $k \geq 1$, is finite and locally constant for an open and dense subset of A_k. Moreover, if $k = 1$, the number of singularities of a vector field in such subset is 2 or 4.

Proof. Let $\tilde{X} \in A_k$ and $X = (P,Q)$ be the lifting of $\tilde{X}$ to $\mathbb{R}^2$. The singular set $X^{-1}(0)$ consists of a finite set mod$(2\pi\mathbb{Z}\times 2\pi\mathbb{Z})$, whenever 0 is a regular value of $X : \mathbb{R}^2 \to \mathbb{R}^2$. If 0 is not a regular value of X, by Sard's Theorem there is an arbitrarily small vector $v \in \mathbb{R}^2$ which is a regular value of X. Thus 0 is a regular value of $X - v$. By the Inverse Function Theorem it follows that if $\tilde{X}, \tilde{Y} \in A_k$ are sufficiently close and 0 is a regular value of $X : \mathbb{R}^2 \to \mathbb{R}^2$, and thus of $Y : \mathbb{R}^2 \to \mathbb{R}^2$, then $\# X^{-1}(0)$ (mod$(2\pi\mathbb{Z}\times 2\pi\mathbb{Z})$) $= \# Y^{-1}(0)$ (mod$(2\pi\mathbb{Z}\times 2\pi\mathbb{Z})$). This finishes the proof of the first part of (2.1).

For the second part of (2.1) it is enough to show that if $\tilde{X} \in A_1$, is in the open and dense set satisfying the first part of (2.1) then there is $\tilde{Y} \in A_1$ arbitrarily close to $\tilde{X}$ such that $\tilde{Y}$ has exactly 2 or exactly 4 singularities. We have

$$X(x,y) = (a_1\cos x + a_2\sin x + a_3\cos y + a_4\sin y + a_5,\ b_1\cos x + b_2\sin x + b_3\cos y + b_4\sin y + b_5) = (0,0)$$

if, and only if, $A.\begin{pmatrix}\cos x\\ \sin x\end{pmatrix} = B\begin{pmatrix}\cos y\\ \sin y\end{pmatrix} - \begin{pmatrix}a_5\\ b_5\end{pmatrix}$, where

$A = \begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}$ and $B = \begin{pmatrix}-a_3 & -a_4\\ -b_3 & -b_4\end{pmatrix}$. By a small perturbation of $\tilde{X}$ we may assume that A and B are invertible matrices. It is enough to consider $(x,y) \in [0,2\pi] \times [0,2\pi]$. Then

$G : x \in [0,2\pi] \to A.\begin{pmatrix}\cos x\\ \sin x\end{pmatrix} \in \mathbb{R}^2$ and $H : y \in [0,2\pi] \to B\begin{pmatrix}\cos y\\ \sin y\end{pmatrix} - \begin{pmatrix}a_5\\ b_5\end{pmatrix}$

are parametrizations of two ellipses Γ_1 and Γ_2 in $\mathbb{R}^2$. Thus, after a small translation of $\tilde{X}$, we have that $\#(\Gamma_1 \cap \Gamma_2) = 2$ or $\#(\Gamma_1 \cap \Gamma_2) = 4$.

Now we can prove

Lemma 2.2. The set B_k is open and dense in A_k.

Proof. Given $(x_0, y_0) \in [0,2\pi] \times [0,2\pi]$, the vector field $X_{(x_0,y_0)}$, given by $Z_{(x_0,y_0)}(x,y) = (\sin(x-x_0), \sin(y-y_0))$ in $\mathbb{R}^2$, is such that $Z_{(x_0,y_0)}(x_0,y_0) = 0$ and $DZ_{(x_0,y_0)}(x_0,y_0) = I =$ = identity. Therefore, if (x_0,y_0) is a singularity of a vector field X on $\mathbb{R}^2$, then (x_0,y_0) is an hyperbolic singularity of $X + \varepsilon Z_{(x_0,y_0)}$ for all $\varepsilon > 0$ sufficiently small. Now, the Lemma follows from this fact and (2.1).

A closed orbit γ *of* $\tilde{X} \in P_k$ *is isolated* if there is a neighbourhood U of γ in T^2 such that γ is the unique closed orbit of $\tilde{X}$ contained in U.

Now we are going to prove that there exists an open and dense subset C_k of B_k such that if $\tilde{X} \in C_k$, then $\tilde{X}$ has no closed orbits or has only isolated closed orbits.

In order to do this we first state a simple and useful remark about closed orbits of vector fields in P_k. It is a consequence of the fact that $\tilde{X} \in P_k$ is analytic and then any first return map (Poincaré Transformation) induced by $\tilde{X}$ is analytic.

Remark 2.3. Any closed orbit γ of $\tilde{X} \in P_k$ is isolated or belongs to an *open band* of *closed orbits* (i.e., $\gamma \subset U$, where $U \subset T^2$ is an open set homeomorphic to $(0,1) \times S^1$, and U is a union of closed orbits of $\tilde{X}$).

A *graph* of $\tilde{X} \in B_k$ is a collection $\{p_1, \ldots, p_n, \gamma_1, \ldots, \gamma_n\}$ of saddles $p_1, \ldots, p_n$ and regular orbits $\gamma_1, \ldots, \gamma_n$ of $\tilde{X}$ such that $\alpha(\gamma_i) = p_i$, $\omega(\gamma_i) = p_{i+1}$, $i = 1, \ldots, n-1$, $\alpha(\gamma_n) = p_n$ and $\omega(\gamma_n) = p_1$. Let $\Gamma = \bigcup_{i=1}^{n} \{p_i\} \cup \gamma_i$. We also refer to Γ as a ghaph of $\tilde{X}$.

To obtain the above mentioned set C_k we need to analyse the boundary of an open band of closed orbits of $\tilde{X} \in B_k$. This is done in the next lemma.

Lemma 2.4. Let U be an open band of closed orbits of $\tilde{X} \in B_k$. Then ∂U is a finite union of graphs of $\tilde{X}$.

Proof. By the hyperbolicity of the singularities of $\tilde{X} \in B_k$, the set of singularities in ∂U, which is non empty, are made up necessarily by saddles. Let $x \in \partial U$ be a regular point of $\tilde{X}$ and γ the $\tilde{X}$ - orbit of x. We claim that $\omega(\gamma)$ and $\alpha(\gamma)$ are saddles. In fact, if this is not the case, $\omega(\gamma)(\alpha(\gamma))$ contains a regular point and thus, by the Jordan Curve Theorem, γ is a closed orbit. This implies, by the analyticity of $\tilde{X}$, that $x \in U$, which is a contradiction.

Now, using that if $p \in \partial U$ is a saddle then at least one stable and one unstable separatrix of p is contained in ∂U, and that the number of saddles of $\tilde{X}$ is finite, we get that ∂U is a finite union of graphs of $\tilde{X}$ as we want. A consequence of the proof of (2.4) is the following

Corollary 2.5. Let $\tilde{X} \in B_k$ and $\gamma_1, \gamma_2, \gamma_3, \ldots$ be a sequence of closed orbits of $\tilde{X}$. If there exists a sequence $\{x_n\}$, $x_n \in \gamma_n$ such that $x_n \to x$ then $\overline{(\bigcup_{n=1}^{\infty} \gamma_n)} - \bigcup_{n=1}^{\infty} \gamma_n$ is a finite union of graphs of $\tilde{X}$.

Let $p \in T^2$ be a saddle of $\tilde{X} \in B_k$. Denote by $W_+^u(p)$ and $W_-^u(p)$ the connected components of $W^u(p) - \{p\}$ (called saddle separatrices). We say that $W_+^u(p)$ ($W_-^u(p)$) is a *stabilized unstable separatrix* if $\omega(W_+^u(p))$ ($\omega(W_-^u(p))$) is an hyperbolic sink or an hyperbolic attracting closed orbit. Similar definition for *stabilized stable separatrix*.

Remark. Let γ be a saddle separatrix of $\tilde{X} \in B_k$. It $\omega(\gamma)(\alpha(\gamma))$ is a closed orbit, then by a convenient small orthogonal perturbation of $\tilde{X}$ we stabilize the corresponding saddle separatrix of the perturbed vector field. (See [1], Theorem 71, 72, pag. 399-400).

Lemma 2.6. There exists an open and dense subset of B_k, made up by vector fields with no graphs.

Proof. Let $\tilde{X} \in B_k$. We are always using that in B_k the number of stabilized separatrices is locally maximal. First assume that $\tilde{X}$ has a graph bounding a disc on T^2. The graphs bounding discs are partially ordered by the inclusion of such discs. We may assume by a small perturbation of $\tilde{X}$, if necessary, that a minimal graph Γ of $\tilde{X}$ satisfies the following condition: $\tilde{X}_\lambda = \tilde{X} + \lambda\tilde{X}^\perp$ has no graphs inside the disc bounded by Γ, for small values of λ. By the Poincaré-Bendixon Theorem and by the kind of our perturbation, for small non zero values of λ, $\tilde{X}_\lambda$ has for each of its saddles in Γ at least on more stabilized separatrix. Because $\tilde{X}$ has a locally finite number of hyperbolic saddles, this process shows that there is an open and dense subset of B_k where all the vector fields has no graphs bounding a disc on T^2.

Now we consider a vector field $\tilde{X}$ in the set above, holding at least a graph Γ does not bounding a disc. We may assume that there are graphs Γ_1, Γ_2 of $\tilde{X}$ bounding a topological cylinder $C \subset T^2$ such that $\tilde{X}$ has no graphs in C. It may happens that $\Gamma_1 = \Gamma_2$. Let $p \in \Gamma_1$ be a saddle of $\tilde{X}$. Let $p = p_0 < p_1 < \ldots < p_r$ be a maximal chain of saddles of $\tilde{X}$ contained in C. We have that $p_r \notin \Gamma_1 \cup \Gamma_2$ because $\tilde{X}$ has no graph bounding a disc, unless $p_r = p_0$. Then the ω-limit set of one of the unstable separatrix of p_r is either a hyperbolic attractor or the graph Γ_2. If the first situation occurs for one of the unstable separatrices of p_r, by a small orthogonal perturbation of $\tilde{X}$ we stabilize one more saddle separatrix of the perturbed vector field.

Otherwise, we consider a maximal chain of saddles of $\tilde{X}$, $p'_s < \ldots < p'_0 = p$ contained in C. Then, the α-limit set of one of the stable separatrices of p'_s are hyperbolic reppelors. Thus, again by a small orthogonal perturbation of $\tilde{X}$ we stabilize one more saddle separatrix of the perturbed vector field, and the proof of (2.6) is complete.

Corollary 2.6. There exists an open and dense subset C_k of B_k such that if $\tilde{X} \in C_k$, then $\tilde{X}$ has no closed orbits or has only isolated closed orbits.

Lemma 2.7. There exists an open and dense subset of C_k, made up by vector fields such that they have no closed orbits or their closed orbits are hyperbolic, finite and locally constant.

Proof. Let $\tilde{Y} \in C_k$, $N(\tilde{Y})$ an arbitrary neighbourhood of $\tilde{Y}$ and $\tilde{Y}_1 \in N(\tilde{Y})$. We can assume that $\tilde{Y}_1$ has at least an isolated closed orbit. By [1], Theorem 71, 72, pag. 339 - 400 we may assume that this closed orbit is hyperbolic. If there exists a neighbourhood of $\tilde{Y}_1$, made up by vector fields that has only this closed orbit, we finish the proof. Otherwise for every neighbourhood of $\tilde{Y}_1$, $N_1(\tilde{Y}_1)$, contained in $N(\tilde{Y})$, there exists $\tilde{Y}_2 \in N_1(\tilde{Y}_1)$ with at least two closed orbits which we can suppose hyperbolics. In this way, or the lemma is proved, or we can construct a sequence of vector fields $\{\tilde{Y}_n\}$, and a sequence of compact nested neighbourhoods $\{N_n(\tilde{Y}_n)\}$, such that if $\tilde{Z} \in N_n(\tilde{Y}_n)$ then the number of hyperbolic closed orbits of $\tilde{Z}$ is greater or equal to n, $n = 1,2,3,\ldots$, and $\operatorname{diam} N_n(\tilde{Y}_n) \to 0$ when $n \to \infty$. Then $\prod_{n=1}^{\infty} (N_n(\tilde{Y}_n)) = \{\tilde{Z}_0\}$. Here we use that P_k is a finite dimensional vector space. Therefore, $\tilde{Z}_0$ has infinitely many closed orbits which is a contradiction with (2.5) because $\tilde{Z}_0$ has no graphs. This finishes the proof of our Lemma.

Remark. Here there is a difference between the polynomial and the analytic case. The convergence of a subsequence of $\tilde{Y}_n$ in the analytic case follows from Montel's Lemma [2].

End of the proof of the theorem

Let $\tilde{X}$ be a vector field such that $\tilde{X}$ is in an open and dense set consisting of vector fields with no graphs, all singularities hyperbolic and with finitely many hyperbolic closed orbits, whose number is locally constant. To finish the proof of our Theorem it is enough to show that if $\tilde{X}$ has a nontrivial recurrent orbit then by an arbitrarily small orthogonal perturbation of $\tilde{X}$ we get a vector field $\tilde{Y}$ satisfying the above conditions and holding at least one more stabilized saddle separatrix. Since the number of saddle separatrices is locally constant, it follows that there exists an open and dense subset of C_k such that all $\tilde{X}$ in this set does not have non-trivial recurrences. Now, using the Denjoy-Schwartz Theorem [5] and a finite number of small ortogonal perturbations, we can break the remaining saddle connections stabilizing all the saddle separatrices, and the Theorem is proved.

Let γ be a non-trivial ω-recurrent $\tilde{X}$-orbit. Ordering the critical elements (singularities and closed orbits) of $\tilde{X}$ in the natural way (i.e. $\sigma_1 < \sigma_2 \Longleftrightarrow W^u(\sigma_1) \cap W^s(\sigma_2) \neq \emptyset$, where, σ_i, $=1,2$, are critical elements of $\tilde{X}$), we may assume that there is a saddle $p \in \bar{\gamma}$ with at least one unstable separatrix which is a non-trivial ω-recurrent orbit, and we can choose a chain of saddles $q < p_1 < \dots < p_r < p$ whose minimal element is q. Let Γ be a piecewise analytic curve made up by saddle separatrices joining q to p. Assume first that Γ has at least two saddles. Let $\tilde{C}$ be a C^∞ transversal circle to $\tilde{X}$ crossing γ at a point $x \in \gamma$. The lifting of $\tilde{C}$ is the set $C^* = \{C + (2m\pi,\ 2n\pi) = C_{mn},\ m,n \in \mathbb{Z}\}$ where C is the image of a C^∞ embedding $c: \mathbb{R} \to \mathbb{R}^2$ wich separates the plane in two unbounded connected open sets.

Then, it is clear that X is transversal to C_{mn} for all $m,n \in \mathbb{Z}$, point to the right, as we may assume, and the same occurs for small orthogonal perturbations of X. Suppose firstly that one of the stable separatrices of q crosses $\tilde{C}$ finitely many times. Thus, cutting the torus along $\tilde{C}$ and using the Poincaré-Bendixon Theorem, we get that such separatrix borns in a source (i.e., repellor critical element). Therefore by a small orthogonal perturbations of $\tilde{X}$ we stabilize a stable separatrix of a saddle of Γ.

Suppose now that both stable separatrices of q cross $\tilde{C}$ infinitely many times. There exists a disc in $\mathbb{R}^2$ bounded by arcs of the stable separatrices of q_1 and of a transversal curve C_{mn} to X containing a unstable separatrix of q_1, where q_1 is projected on q. This follows from the fact that one of the unstable separatrices of p is a non-trivial ω-recurrent orbit, and from our hypotesis on the stable separatrices of q. Thus by the Poincaré-Bendixon Theorem we may assume that the unstable separatrix of q not in Γ is stabilized. Otherwise we already got a new stabilized saddle separatrix for a vector field arbitrarily near $\tilde{X}$. By Denjoy--Schwartz Theorem [5] we may assume that there exists a saddle q_2 of X with an unstable separatrix crossing C_{mn} in a point b arbitrarily near the unique intersection point a of a stable separatrix of q_1 with C_{mn}. Our vector field X has no graphs and all its critical elements are hyperbolic. Moreover, all small orthogonal perturbations of X remains transversal to C_{mn} for all $m,n \in \mathbb{Z}$.

We can choose an order in C_{mn} such that $a < b$. Given $\varepsilon > 0$, there exists a flow box B at $a \in C_{mn}$ and a saddle q_2 of X with an unstable separatrix crossing C_{mn} in a point $b \in B$, $a < b$, such that, the orbit of $X_\lambda = X + \lambda X^\perp$ through the intersection of the X-orbit of b with the left side of B, intersects C_{mn} in a point less than the intersection point of the X_λ-orbit by the intersection of the X-orbit of a with the right side of B, for some $0 < \lambda < \varepsilon$. Let α_λ be the unstable separatrix of X_λ by q_2 nearest α_0 in compact parts, and β_λ be the stable separatrix of q_1 nearest β_0 in compact parts.

Consider the images $[b_{\lambda_0}, b]$, $[a, a_{\lambda_0}] \subset C_{mn}$ of the continuous functions

$$\lambda \in [0,\lambda_0] \to \alpha_\lambda \cap C_{mn} \in C_{mn},$$

$$\lambda \in [0,\lambda_0] \to \beta_\lambda \cap C_{mn} \in C_{mn}.$$

We notice that these functions are well defined for $\lambda_0 \in [0,\varepsilon)$ sufficiently small.

If there is $\lambda_0 \in [0,\varepsilon)$ such that $b_{\lambda_0} < a_{\lambda_0}$ we get a vector field X_λ holding a saddle connection joining q_2 with q_1, for some value of $\lambda \in (0,\lambda_0]$. Now, by a convenient small orthogonal perturbation of $\tilde{X}_\lambda$ (for this λ) we get a vector field on T^2 with one more stabilized saddle separatrix.

Otherwise, $a_{\lambda_0} < b_{\lambda_0}$ for all $\lambda_0 \in [0,\varepsilon)$ such that $\lambda \in [0,\lambda_0] \to a_\lambda \in C_{mn}$ and $\lambda \in [0,\lambda_0] \to b_\lambda \in C_{mn}$ are well defined. Let a_1 be the least upper bound of such a_{λ_0} and b_1 be the greatest lower bound of such b_{λ_0}. Let $\lambda_1 = \text{lub}\,\{\lambda_0 \in [0,\varepsilon) : \lambda \in [0,\lambda_0] \to a_\lambda \in C_{mn}$ and $\lambda \in [0,\lambda_0] \to b_\lambda \in C_{mn}$ are both well defined$\}$. It is clear that $\alpha_{\lambda_1} \cap C_{mn} = \emptyset$ or $\beta_{\lambda_1} \cap C_{mn} = \emptyset$.

We claim that if $\beta_{\lambda_1} \cap C_{mn} = \emptyset$ then q_1 belongs to a saddle connection of X_{λ_1}, and so, as above, by a small orthogonal perturbation of $\tilde{X}_{\lambda_1}$ we get a vector field on T^2 with one more stabilized saddle separatrix.

In fact, β_{λ_1} is in the boundary of $\{(X_\lambda)_t(a_\lambda),\ 0 \le \lambda < \lambda_1,\ t \ge 0\}$ by the continuity of the flow $(X_\lambda)_t$ with respect to λ and with the initial conditions (close to q_1). In this case, β_{λ_1} is contained in a band of $\mathbb{R}^2$ bounded by C_{mn} and $C_{m'n'}$ for some $m', n' \in \mathbb{Z}$. Thus, except for a compact part of $\beta_{\lambda_1} \cup \{q_1\}$, the projection of β_{λ_1} on T^2 is contained in a cilinder. Then by the Poincaré-Bendixon Theorem and our hypothesis if follows that $\alpha(\tilde{\beta}_{\lambda_1})$ is a saddle, where $\tilde{\beta}_{\lambda_1}$ is the projection of β_{λ_1} on T^2.

Now we assume that $\beta_{\lambda_1} \cap C_{mn} \neq \emptyset$ and so $\alpha_{\lambda_1} \cap C_{mn} = \emptyset$. Using the same argument as above, we get a finite chaim of saddle connections starting at q_2 and ending at a saddle q_3, with all regular orbits contained in the boundary of $\{(X_\lambda)_t(b_\lambda),\ 0 \le \lambda < \lambda_1,\ t \le 0\}$. Thus we may repeat that procedure changing q_2 by q_3, and $[a,b] \subset C_{mn}$ by $[a_1,b_1] \subset [a,b] \subset C_{mn}$. Continuing in this way and using that the number of saddles of $\tilde{X}$ is finite, we get either a perturbation $\tilde{X}_\lambda$ of $\tilde{X}$, $\lambda_1 < \lambda < \varepsilon$, with a saddle connection containing q_1, and thus by a small orthogonal perturbation we achievea vector field with one more stabilized separatrix, or we get a vector field X_{λ_k}, $\lambda_1 < \lambda_k < \varepsilon$, such that X_λ, $\lambda \in [\lambda_k, \varepsilon)$ satisfies our first hypothesis about the family X_λ, $\lambda \in [0,\varepsilon)$, with respect to X and so we get a saddle connection joining q_k with q_1 for some vector field X_λ with $\lambda_k < \lambda < \varepsilon$. By a last small orthogonal perturbation we obtain a vector field with one more stabilized saddle separatrix.

If Γ has only one saddle p, remainding that in this case one of the unstable separatrices of p is ω-recurrent, then the Theorem can be proved by the same arguments as above. For γ α-recurrent the proof is similar. This ends the proof of the Theorem.

The Analytic Case. Initially we remark that given an analytic vector field $\tilde{Y}$ on T^2 whose orbits are all dense, there exists $\lambda \neq 0$, arbitrarily small, such that the analytic vector field $\tilde{X} = \tilde{Y} + \lambda \tilde{Y}^\perp$ has at least one closed orbit. So in this case it remains to prove that given an analytic vector field $\tilde{X}$ on T^2 whose orbits are all closed it is possible to perturb it in an analytic fashion in such a way that the perturbed vector field has an isolated closed orbit. This can be done as follows. We consider the vector field $\tilde{X}^{\perp}$ and its orbit through a point p belonging to a closed orbit γ of $\tilde{X}$. Let α be the orbit of $\tilde{X}^{\perp}$ through p.

Let $q \in \gamma$ be the first intersection of α with γ after p. Assume that $q \neq p$. When $p = q$ the proof is similar and easier. Consider an analytic flow box $\phi: U \to (-\delta, 1+\delta)\times(-\varepsilon,\varepsilon) \subset \mathbb{R}^2$ of $\tilde{X}$ such that $\phi(p) = (0,0)$, $\phi(q) = (1,0)$, $\phi([p,q]) = \{(t,0),\ t \in [0,1]\}$. Let $\theta: T^2 \to [p,q]$ be defined by $\theta(x) = \gamma_x \cap [p,q)$ where γ_x is the closed orbit of $\tilde{X}$ through x. Consider the function $f: T^2 \to \mathbb{R}$ defined by $f(x)=\sin(2\Pi(\phi\cdot\theta(x)))$. This function is an analytic first integral of $\tilde{X}$. The vector field $\tilde{X} + \varepsilon\cdot f\cdot\tilde{X}^{\perp}$ is the desired perturbation of $\tilde{X}$, and then the Peixoto's Theorem is true in the analytic case when the manifold is the 2-torus.

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