ASYMPTOTICS FOR SOME NONLINEAR HYPERBOLIC EQUATIONS WITH A ONE-DIMENSIONAL SET OF REST POINTS

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0. Introduction

Let Ω be a bounded, connected, open subset of $\mathbb{R}^N$ and f a non-decreasing function which grows linearly at infinity. For any $h \in L^2(\Omega)$ and any non-decreasing function $g: \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$ we consider the damped wave equation

$$(0.1) \quad \begin{cases} u_{tt} - \Delta u + f(u) + g(u_t) = h(x) & t \geq 0,\ x \in \Omega \\ \dfrac{\partial u}{\partial n}(t,x) = 0 & t \geq 0,\ x \in \partial\Omega \end{cases}$$

The set of rest points (or equilibria) of (0.1) is given by the solutions of the elliptic problem

$$(0.2) \quad \begin{cases} z \in H^2(\Omega), \quad -\Delta z + f(z) = h(x) & \text{in } \Omega \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

It is not difficult to check that the set of solutions of (0.2) is of the form

$$(0.3) \quad E = z_0 + J\mathbb{I}$$

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where z_0 is a solution of (0.2), J a compact interval and the constant function equal to 1 throughout Ω. In [2], necessary and sufficient conditions are given in order for (0.2) to have a non-trivial segment of solutions. In such a case the asymptotic behavior of solutions to (0.1) is not completely obvious: as we established in [5], even solutions which do not depend on x may oscillate between several equilibria as $t \to +\infty$, if the damping term $g(u_t)$ is sufficiently small [namely $o(|u_t|^2)$] for small values of $|u_t|$. If on the other hand $g(v)v \geq c|v|^{3-\varepsilon}$ for $|v|$ small and for some $c > 0$, $0 < \varepsilon \leq 1$, then any solution of the O.D.E.

$$(0.4) \qquad u'' + f(u) + g(u') = 0$$

tends to an equilibrium as $t \to +\infty$. (cf. [5]).

This means that when $h \equiv 0$, all solutions of (0.1) corresponding to initial data independent of x tend to some equilibria as $t \to +\infty$.

The main purpose of this paper is to extend this result to *any* solution of (0.1), and by a method which can be used for different sorts of equations having a similar character. The special case where g is linear is treated in section 1 by a quite simple Liapunov function argument. Weakly nonlinear dampings are treated in section 2. Finally, in section 3, we give a list of examples to clarify more completely the kind of phenomena under investigation. This paper has been motivated in part by the work of H. Matano [6] concerning nonlinear parabolic equations in one dimension. One should note, however, that the conditions we have here are quite restrictive and for instance if in (0.1) we drop u_{tt} and let $g(v) \equiv v$ the problem of asymptotic behavior becomes almost trivial in any space dimension (cf. e.g. [4]).

Therefore the present work must be considered only as a first step to a field of research where almost everything remains to be done.

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1. An abstract result for linearly damped equations

Let H and V be two real Hilbert spaces with norms respectively denoted by $|\ |$ and $\|\ \|$. We assume that $V \subset H$, that the imbedding $V \to H$ is continuous and V is dense in H. The duality pairing on $V' \times V$ is represented by the symbol $<\ ,\ >$ and the inner product of two vectors $u,v \in H$ is written (u,v). We consider

1) A linear operator $A \in \mathcal{L}(V,V')$ such that

$$(1.1) \qquad \forall u \in V, \quad <Au,u> \geq 0.$$

$$(1.2) \qquad \forall u \in V, \quad \forall v \in V, \quad <Au,v> = <Av,u>.$$

2) A linear operator $B \in \mathcal{L}(H)$ such that $B^* = B$ and for some $\alpha > 0$ we have

$$(1.3) \qquad \forall v \in H, \quad (Bv,v) \geq \alpha|v|^2.$$

3) A function $\Phi \in C^1(V)$ such that $\phi := \Phi' \in C(V,V')$ satisfies

$$(1.4) \qquad \phi \in C(V,H)$$

$$(1.5) \qquad \forall u \in V, \quad (\phi(u),u) \geq 0.$$

We assume that A satisfies the condition

$$(1.6) \qquad N := \{v \in V,\ Av = 0\} \text{ is one-dimensional.}$$

Our main result is the following

Theorem 1.1. Let $u: \mathbb{R}^+ \to V$ be such that

(1.7) $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H) \cap C^2(\mathbb{R}^+, V')$

(1.8) $\forall\, t \geq 0, \quad u''(t) + Au(t) + \phi(u(t)) + Bu'(t) = 0$

(1.9) $\bigcup\limits_{t\geq 0} \{[u(t), u'(t)]\}$ is precompact in $V\times H$.

Then there exists $a \in N = A^{-1}(\{0\})$ such that $\phi(a) = 0$ and

(1.10) $\lim\limits_{t\to+\infty} \|u(t)-a\| = 0.$

Proof. As a consequence of (1.4) and (1.7) we have $\Phi(u(t)) \in C^1(\mathbb{R}^+)$ with $\forall\, t \geq 0$, $\frac{d}{dt}(\Phi(u(t))) = (\phi(u(t)), u'(t))$. This is indeed obvious when $u \in C^1(\mathbb{R}^+, V)$ and follows then by density when u only satisfies (1.7).

Therefore we have the identity

(1.11) $\frac{d}{dt}\{\frac{1}{2} \langle Au(t), u(t)\rangle + \frac{1}{2}|u'(t)|^2 + \Phi(u(t))\}$

$$= -(Bu'(t), u'(t)), \quad \forall\, t \geq 0.$$

Since $\|u(t)\|$ and $|u'(t)|$ are bounded we deduce

(1.12) $\int_0^{+\infty} |u'(t)|^2 dt \leq \frac{1}{\alpha}\int_0^{+\infty} (Bu'(t), u'(t))dt < +\infty.$

By writing (1.8) as a system in $[u, u']$ we can apply the invariance principle in the closure in $V\times H$ of $\bigcup\limits_{t\geq 0}\{[u(t), u'(t)]\}$. We obtain in particular (cf. e.g. [1], [3])

(1.13) $\lim\limits_{t\to+\infty} |u'(t)| = 0$

(1.14) $\lim\limits_{t\to+\infty} \mathrm{dist}_V(u(t), S) = 0$

where S, the set of equilibria for (1.8), is defined by

(1.15) $S = \{a \in V, \quad Aa + \phi(a) = 0\}.$

First we claim that in fact

(1.16) $S = N \cap \phi^{-1}(\{0\}).$

It is clear that $N \cap \phi^{-1}(\{0\}) \subset S$. Conversely if $a \in S$ we have by (1.1) and (1.5)

$$\langle Aa, a\rangle = (\phi(a), a) = 0.$$

But $\langle Aa, a\rangle = 0$ implies $a \in N$. This proves the claim. We now introduce the function

(1.17) $\psi(t) := (Bu(t), u(t)) + 2(u(t), u'(t)) - 2\int_0^t |u'(s)|^2 ds.$

Obviously ψ is bounded for $t \geq 0$. On the other hand an immediate calculation shows that $\psi \in C^1(\mathbb{R}^+)$ with

$$\frac{d\psi}{dt}(t) = 2(Bu'(t), u(t)) + 2\langle u''(t), u(t)\rangle$$

$$= -2\langle Au(t) + \phi(u(t)), u(t)\rangle \leq 0, \quad \forall\, t \geq 0.$$

Therefore $\psi(t)$ tends to a limit as $t \to +\infty$. By taking account of (1.12), (1.13) and (1.17) we deduce for some $\ell \geq 0$

(1.18) $\lim\limits_{t\to+\infty} (Bu(t), u(t)) = \ell.$

Finally, let

(1.19) $\Sigma = \{a \in S, \ \exists\, t_n \to +\infty$ such that $\lim\limits_{n\to+\infty} \|u(t_n) - a\| = 0\}.$

As a consequence of the general theory of topological dynamics, Σ is *connected* for the topology of V. (cf. [1], [3]). On the other hand, as a consequence of (1.18) we have

(1.20) $\Sigma \subset \{z \in N, \quad (Bz, z) = \ell\}$

and since N is one-dimensional this implies $\Sigma \subset \{z_0, -z_0\}$

for some $z_0 \in N$. By connectedness it follows that $\Sigma = \pm\{z_0\}$ and Theorem 1.1 is completely proved.

2. Some results with a non linear damping

In this section, Ω denotes a positively measured space and the measure on Ω is denoted by dx for simplicity. We choose $H = L^2(\Omega,dx)$ and V is as in section 1. We consider $A \in \mathcal{L}(V,V')$ satisfying (1.1), (1.2) and (1.6) as well as the additional condition

(2.1) $\forall\, u \in V, \ \langle Au,u\rangle \geq \eta|u-Pu|^2$

where $\eta > 0$ and $P: H \to H$ is the orthogonal projection on the line N in the sense of H. We also consider two functions f and $g: \Omega\times\mathbb{R} \to \mathbb{R}$ which satisfy the following properties

(2.2) f and g are measurable in the first variable and continuous in the second variable.

(2.3) $f(x,u)$ is non-decreasing as a function of u for all $x \in \Omega$ fixed.

(2.4) $\forall\, x \in \Omega, \ f(x,0) = 0$

(2.5) The (nonlinear) operator defined on V by $[\phi(u)](x) = f(x,u(x))$ a.e. in Ω carries V into H and $\phi \in C(V,H)$.

In order to make the method more transparent we shall only prove a result involving rather strong conditions on the term $g(x,v)$. A more general result will we treated elsewhere, c.f. Remark 2.2 below.

The main result of this section is as follows

Theorem 2.1. In addition to the above conditions on A, f and g, assume that we have

(2.6) $\forall\ (x,v) \in \Omega\times\mathbb{R}, \ g(x,v)v \geq 0$

(2.7) $\exists\, c > 0,\ C \geq c$ such that for all $(x,v) \in \Omega\times\mathbb{R}$, $c|v| \leq |g(x,v)| \leq C|v|$.

Let $u: \mathbb{R}^+ \to V$ be a solution of

(2.8) $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H) \cap C^2(\mathbb{R}^+, V')$

(2.9) $\forall\, t \in \mathbb{R}, \ u''(t) + Au(t) + f(x,u(t)) + g(x,u'(t)) = 0.$

Then if u also satisfies (1.9), there exists $a \in N = A^{-1}(\{0\})$ such that $f(x,a(x)) = 0$ a.e. in Ω and $\lim_{t\to+\infty} \|u(t)-a\| = 0$.

Proof. We introduce $F: \Omega\times\mathbb{R} \to \mathbb{R}$ given by the formula

(2.10) $\forall\, x \in \Omega, \ \forall\, u \in \mathbb{R}, \ F(x,u) = \int_0^u f(x,s)ds.$

It is clear that F is *convex* in u for any $x \in \Omega$ fixed and $F_u = f$. Similarly to (1.11) we have the energy identity

(2.11)
$$\frac{d}{dt}\left\{\frac{1}{2}\langle Au(t),u(t)\rangle + \frac{1}{2}|u'(t)|^2 + \int_\Omega F(x,u(t,x))dx\right\} = -\int_\Omega g(x,u'(t,x))u'(t,x)dx, \quad \forall\, t \geq 0.$$

As a consequence of (2.6)-(2.7) we prove (1.13)-(1.14) in the same way as in [5].

Moreover the hypotheses on A and f imply that (1.16) is satisfied with $\mathcal{F}$ defined by (1.15) and ϕ as in (2.5). We also note that the set Σ defined in (1.19) has the form

(2.12) $\Sigma = J z_0$

where $\{z_0\}$ is a fixed basis of N and J a compact interval of $\mathbb{R}$ depending on u. We have to show that J is reduced to a single point.

Assuming that it is not the case, let $\lambda_0 \in \mathrm{Int}(J)$ and $t_n \to +\infty$ such that

(2.13) $$\lim_{n\to+\infty} \|u(t_n) - \lambda_0 z_0\| = 0.$$

We introduce $\lambda(t)$ and $\bar{u}(t)$ defined by

(2.14) $$\forall\, t \geq 0, \quad Pu(t) = \bar{u}(t) = \lambda(t) z$$

and we define the "energy"

(2.15) $$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}<Au(t),u(t)> + \int_\Omega F(x, u(t,x))dx.$$

Since for any $a \in S$ we have $F(x,a(x)) \equiv 0$ it follows from (2.15) and (1.13)-(1.14) that

(2.16) $$\lim_{t\to+\infty} E(t) = 0.$$

We will now derive a sharper estimate which will imply $J = \{\lambda_0\}$ and therefore contradict our hypothesis that J is not a singleton. In order to do that, we use the inequality

(2.17) $$\frac{dE}{dt} \leq -c|u'(t)|^2$$

in conjunction with the estimate

(2.18) $$\frac{d}{dt}(u-Pu,u') = |u'|^2 - (u',Pu') + <u'',u-Pu>$$

$$\leq |u'|^2 - <Au,u-Pu> - (f(x,u),u-Pu) + C|u'|\,|u-Pu|$$

which follows easily from (2.7) and (2.9).

We note that $<Au,Pu> = <A(Pu),u> = 0$ and by using (2.1) the inequality (2.18) becomes $\frac{d}{dt}(u-Pu,u') \leq C_1|u'|^2 - \frac{1}{2}<Au,u> - (f(x,u),u-Pu)$.

By the convexity of F in u we have $F(x,Pu) - F(x,u) \geq f(x,u)(Pu-u)$ a.e in Ω for all $t \geq 0$ and by integrating we obtain

$$-(f(x,u),u-Pu) \leq \int_\Omega F(x,Pu)\,dx - \int_\Omega F(x,u)dx$$

Finally, for any $t \geq 0$ we find

(2.19) $$\frac{d}{dt}(u(t)-Pu(t),u'(t)) \leq C_1|u'(t)|^2 - \frac{1}{2}<Au(t),u(t)>$$

$$+ \int_\Omega F(x,Pu(t,x))dx - \int_\Omega F(x,u(t,x))dx.$$

As a consequence of (2.13) we have in particular

(2.20) $$\lim_{n\to+\infty} |Pu(t_n) - \lambda_0 z_0| = 0.$$

In particular, for $n \geq n_0$ we have $Pu(t_n) \in (\mathrm{Int}\, J)z_0$ and therefore $F(x,Pu(t_n,x)) = 0$ a.e. in Ω.

In the sequel we set, with $\lambda(t)$ given by formula (2.14):

(2.21) $$\forall\, n \geq n_0, \quad \tau_n = \mathrm{Inf}\{t > t_n,\ \lambda(t) \notin J\}.$$

For all $n \geq n_0$ and $t \in [t_n,\tau_n]$ we have

(2.22) $$\frac{d}{dt}(u(t)-Pu(t),u'(t)) \leq C_1|u'(t)|^2$$

$$- \frac{1}{2}<Au(t),u(t)> - \int_\Omega F(x,u(t,x))dx.$$

By computing $\frac{d}{dt}[E(t) + \varepsilon(u(t)-Pu(t),u'(t))]$ for $\varepsilon > 0$ small enough, it now follows classically that for some $C \geq 0$, $\delta > 0$ we have

(2.23) $$E(t) \leq C \exp[-2\delta(t-t_n)]\ (t_n)$$

for all $t \in [t_n,\tau_n]$, $n \geq n_0$.

In particular we find

(2.24) $$|u'(t)| \leq \{2CE(t_n)\}^{\frac{1}{2}} \exp(-\delta(t-t_n))$$

for all $t \in [t_n,\tau_n]$, $n \geq n_0$.

By integrating on $[t_n,t]$ this implies

(2.25) $$|Pu(t)-Pu(t_n)| \leq |u(t)-u(t_n)| \leq \frac{\{2CE(t_n)\}^{\frac{1}{2}}}{\delta}.$$

Since by (2.16), $E(t_n)$ tends to 0 as $n \to +\infty$, (2.25) implies obviously that for $n \geq n_1$, we have

$$\tau_n = +\infty \tag{2.26}$$

$$\forall\, t \geq t_n,\quad |u(t)-u(t_n)| \leq \frac{\{2CE(t_n)\}^{\frac{1}{2}}}{\delta}. \tag{2.27}$$

By combining (2.13) and (2.27) we deduce

$$\lim_{t\to+\infty} |u(t)-\lambda_0 z_0| = 0. \tag{2.28}$$

This clearly implies $J = \{\lambda_0\}$ and this contradiction completes the proof of Theorem 2.1. ■

Remark 2.2. The results of Theorem 2.1 is still valid if the conditions (2.7) on g are replaced by the much weaker assumption

$$c \inf\{|v|, |v|^{a}\} \leq |g(x,v)| \leq C(|v|+|v|^{s}) \tag{2.29}$$

for all $(x,v) \in \Omega\times\mathbb{R}$

where $0 < c \leq C$, $a \in [1,2[$ and $s \geq 1$ is such that $V \subset L^{s+1}(\Omega)$ with continuous and dense imbedding. Then the convergence to an equilibrium can be deduced for solutions in $C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H) \cap W^{2,1}_{loc}(\mathbb{R}^+, V')$ for which $E(t)$ is absolutely continuous with $\frac{dE}{dt} \leq -\int_\Omega g(x, u'(t,x))u'(t,x)dx$ a.e. on $\mathbb{R}^+$. If $a > 1$ we also need to assume that meas$(\Omega) < +\infty$. The detailed proof of such a result, quite technical and combining ideas from the proofs of [5], Theorem 1 and Theorem 2.1 above, will be given elsewhere.

3. Examples

In this section, we show how to apply the abstract results of sections 1 and 2 to some concrete examples of semilinear hyperbolic P.D.E.

The examples are chosen in a way to point out some typical situations.

Example 1. Let Ω be a bounded, open, connected domain with smooth boundary and consider the problem

$$\begin{cases} u_{tt} - \Delta u + c \operatorname{sgn} u(\sin|u|)^+ + \alpha u_t = 0 & \text{on } \mathbb{R}^+\times\Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \mathbb{R}^+\times\partial\Omega \end{cases} \tag{3.1}$$

where $c > 0$, $\alpha > 0$.

We can apply Theorem 1.1 with $H = L^2(\Omega)$, $V = H^1(\Omega)$ and $[\phi(u)](x) = c \operatorname{sgn} u(x)[\sin|u|(x)]^+$. It is easily verified that for all $u \in V$,

$$\Phi(u) = c\int_0 \left[\int_0^{|u(x)|} (\sin s)^+ ds\right] dx \geq \eta\int_\Omega |u(x)|dx - C$$

sor some $\eta > 0$, $C \geq 0$.

Therefore all solutions of (3.1) on $\mathbb{R}^+$ are in $C_B(\mathbb{R}^+, V) \cap C^1_B(\mathbb{R}^+, H)$. Since $[\phi(u)](t) \in W^{1,\infty}(\mathbb{R}^+, G)$ it is easy to check that any solution of (3.1) satisfies (1.9). Therefore for each solution u there exists a *constant* $\bar{u} \in \pm \bigcup_{k\in\mathbb{N}\setminus\{0\}} [(2k-1)\pi,\ 2k\pi]$ such that $u(t,x) \to \bar{u}$ in $H^1(\Omega)$ as $t \to +\infty$.

Example 2. Let f_1, f_2, f_3, f_4 be 4 C^1 and Lipschitz continuous functions such that

$$\forall\, i \in \{1,2,3,4\},\qquad \forall\, u \in \mathbb{R},\qquad f_i(u)u \geq 0.$$

Let a, b, α, β be 4 positive constants and consider the system

$$(3.2)\quad \begin{cases} u_{tt} - a\Delta u + F_2(v)f_1(u) + f_3(u) + \alpha u_t = 0 & \text{on } \mathbb{R}^+\times\Omega \\ v_{tt} - b\Delta v + F_1(u)f_2(v) + f_4(v) + \beta v_t = 0 & \text{on } \mathbb{R}^+\times\Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \mathbb{R}^+\times\partial\Omega \end{cases}$$

where Ω is a smooth bounded, connected domain in $\mathbb{R}^n$ and

$$F_i(s) = \int_0^s f_i(\sigma)d\sigma, \quad \forall\, i \in \{1,2,3,4\}, \quad \forall\, s \in \mathbb{R}.$$

Assume that $|f_k(u)| \geq \eta_k|u| - C_k$ for $k \in \{3,4\}$ and $f_i \equiv 0$ on $[-R,R]$ for $i \in \{1,2,3,4\}$ with $R > 0$. Then

i) The system (3.2) has a two-dimensional set of equilibria, namely a subset of $\mathbb{R}^2$ including $[-R,R]^2$.

ii) The Cauchy problem for (3.2) is well set in $[H^1(\Omega)]^2$ and for any solution (u,v) we have the energy identity

$$(3.3)\quad \frac{d}{dt}\Big(\int_\Omega\{\tfrac{1}{2}u_t^2 + \tfrac{1}{2}v_t^2 + \tfrac{a}{2}|\nabla u|^2 + \tfrac{b}{2}|\nabla v|^2 + F_1(u)F_2(v) + F_3(u) + F_4(v)\}dx\Big) = -\alpha\int_\Omega u_t^2\,dx - \beta\int_\Omega v_t^2\,dx.$$

Therefore if $\eta_3 > 0$ and $\eta_4 > 0$ the trajectories of (3.2) are bounded in $(H^1(\Omega))^2$ and (u_t,v_t) is bounded in $(L^2(\Omega))^2$. Then the analogs of (1.9) and (1.13)-(1.14) are easily checked. Although Theorem 1.1 is not directly applicable in this case, it follows from a calculation similar to the proof of (1.18) that $\int_\Omega u^2(t,x)dx$ and $\int_\Omega v^2(t,x)dx$ have limits as $t \to +\infty$. Therefore any solution (u,v) of (3.2) is such that

$$\lim_{t\to+\infty} u(t) = \bar{u}, \qquad \lim_{t\to+\infty} v(t) = \bar{v}$$

in $H^1(\Omega)$, where $(\bar{u},\bar{v})$ are two real constants such that

$$f_3(\bar{u}) = f_4(\bar{v}) = 0 \quad \text{and}$$

either $\quad f_1(\bar{u}) = f_2(\bar{v}) = 0 \quad$ or $\quad F_1(\bar{u})F_2(\bar{v}) = 0$

(note that if $F_j(s) = 0$, then $f_j(s) = 0$).

This example shows that the method of proof of Theorem 1.1 is applicable to systems in some special cases. Of course some of the growth or coerciveness conditions on the f_i can be relaxed.

Example 3. In both examples 1 and 2, we can also take $\Omega = \prod_{j=1}^{n}\,]0,\ell_j[$ and replace the Neumann boundary condition by an Ω-periodicity condition.

Example 4. Let $f \in C^1(\mathbb{R})$ be such that $f(u)u \geq 0$ for all $u \in \mathbb{R}$ and assume

$$(3.4)\qquad |f'(u)| \leq C(1+|u|^s)$$

with $s > 0$ if $n \in \{1,2\}$, $s \in [0, \frac{2}{n-2}]$ if $n \geq 3$.

Let Ω be a bounded, open, connected domain in $\mathbb{R}^n$ and consider the problem

$$(3.5)\quad \begin{cases} u_{tt} - \Delta u - \lambda_1 u + f(u) + a(x)u_t = 0 & \text{on } \mathbb{R}^+\times\Omega \\ u = 0 & \text{on } \mathbb{R}^+\times\partial\Omega \end{cases}$$

where $a \in L^\infty(\Omega)$ and $0 < \alpha \leq a(x)$ a.e. in Ω, and λ_1 is the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. Then any solution of (3.4) in the class $C(\mathbb{R}^+, H_0^1(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega))$ such that $\bigcup_{t\geq 0}[u(t),u_t(t)]$ is precompact in $H_0^1(\Omega)\times L^2(\Omega)$ is such that $\lim_{t\to+\infty}\int_\Omega |u_t|^2dx = 0$ and $\lim_{t\to+\infty} u(t,\cdot) = c\phi_1$ where $c \in \mathbb{R}$ is such that $f(c\phi_1(x)) = 0$, $\forall\, x \in \Omega$.

This result follows as an immediate consequence of Theorem 1.1. Note that all solutions tend to 0 except if $f \equiv 0$ in some interval containing 0.

Also precompactness of trajectories is easily achieved assuming that for some $\eta > 0$, $C \geq 0$ we have

(3.6) $\qquad \forall u \in \mathbb{R}, \quad F(u) \geq \eta |u| - C.$

(3.7) If $n \geq 3$, (3.4) is satisfied with $s < \frac{2}{n-2}$.

Example 5. Consider equation (0.1) with f as in example 4 satisfying (3.6)-(3.7) and g satisfying (2.6)-(2.7). Let $z_0(x)$ be a solution of (0.2) and let us introduce

(3.8) $\qquad \tilde{u}(t,x) = u(t,x) - z_0(x).$

Then $\tilde{u}$ is a solution of

$$(3.9) \quad \begin{cases} \tilde{u}_{tt} - \Delta\tilde{u} + \tilde{f}(x,\tilde{u}) + g(\tilde{u}_t) = 0 & \text{on } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial \tilde{u}}{\partial n} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega \end{cases}$$

with $\tilde{f}(x,w) := f(z_0(x)+w) - f(z_0(x))$.

By applying Theorem 2.1 to (3.9) we obtain that $\tilde{u}(t)$ tends to a limit in $H^1(\Omega)$ as $t \to +\infty$. Therefore in these conditions $u(t,.)$ tends to a solution of (0.2) as $t \to +\infty$.

Remark 3.1. It is possible to imagine many other examples. For instance $(-\Delta)$ with Neumann conditions can also be replaced by $\Delta^2 - \lambda_1 I$ in $H_0^2(\Omega)$ when the first eigenvalue λ_1 of Δ^2 in $H_0^2(\Omega)$ is simple.

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