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MANIFOLDS WITH PURE NON-NEGATIVE CURVATURE OPERATOR

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Abstract. We prove that if a simply connected compact Riemannian manifold has pure non negative curvature operator then its irreducible components (in the de Rham decomposition) are homeomorphic to spheres.

1. Introduction

One of the main problems in Riemannian geometry is the study of the topology of manifolds with non negative-curvatures. Among the varius curvatures the most interesting one is perhaps the sectional curvature and, in this case, the problem is far from being well understood. A stronger condition is the positivity (resp. non-negativity) of the curvature operator and some classification is now available through the work of Gallot, Meyer, Hamilton, Moore, Chow and Yang (see §2).

In this paper we consider manifolds with pure curvature operator, i.e. such that for each point there exists an orthonormal base $\{X_1, \ldots, X_n\}$ of the tangent space satisfying $\langle R(X_i, X_j)X_k, X_e\rangle = 0$ if the set $\{i, j, k, \ell\}$ contains more than two elements (See Definition 3.4). For such manifolds positivity of the curvature operator is equivalent to positivity of the sectional curvature.

Manifolds with pure curvature operator are of interest since they include at least two important classes: the conformally

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flat manifolds and the manifolds which can be immersed isometrically into space forms with flat normal connection.

Then main result we will prove in this paper is the following:

Theorem 1. If a compact simply connected manifolds has pure non-negative curvature operator then its irreducible components (in the de Rham decomposition) are homeomorphic to spheres.

The non-simply connected case will follow by Theorem 3 of [3] combined with Theorem 1 above, i.e..

Theorem 2. Let M be a compact manifold with pure non-negative curvature operator. If its universal covering $\tilde{M}$ is not compact, then

$$\tilde{M} = \mathbb{R}^p \times M_1 \times \ldots \times M_k$$

where each M_i is homeomorphic to a sphere.

2. Known facts and an idea of the proof

For a Riemannian manifold M the curvature operator at $x \in M$ is the linear symmetric map

$$\rho_x : \Lambda^2(T_x M) \to \Lambda^2(T_x M)$$

characterized by

$$<\rho_x(X \wedge Y), W \wedge Z> = <R(X,Y)Z, W>$$

where the scalar product at the left hand side is the one induced at the level of two-forms and R is the Riemann curvature tensor.

Since ρ_x is symmetric it makes sense to talk about the positivity of ρ_x. The results of Gallot Meyer ([4]), Micallef Moore ([6]), Cao Chow ([2]) and Chow Yang ([3]) may be summarized as follows:

2.1 Theorem. Let M be a compact simply connected Riemannian manifold with non-negative curvature operator. Then M is the Riemannian product of manifolds of the following types:

1. Compact symmetric spaces
2. Kähler manifolds biholomorphic to complex projective spaces
3. Manifolds homeomorphic to spheres.

Moreover, if ρ_x is positive for some $x \in M$, the manifold is homeomorphic to a sphere. In order to prove the announced theorem we start by proving that if $M = M_1 \times M_2$ (Riemannian product) and M has pure curvature operator, then M_i has pure curvature operator, $i = 1,2$ (see (3.5)). This allow us to work separately on each irreducible component of M (in the de Rham decomposition).

Second we observe that the purity of the curvature operator implies the vanishing of the Pontryagin form (see (4.1)) and hence the manifold can not be biholomorphic to CP^n, $n > 1$.

Finally, we prove that a symmetric space with pure non-negative curvature operator has constant curvature (see (4.2)).

The above facts, together with Theorem 2.1 will give the desired conclusion.

3. Some linear algebra

For the curvature operator ρ we will consider the following conditions:

P_1: there exist a base of decomposable eigenvectors of ρ, i.e. there exist vectors X_{ij}, Y_{ij} such that the bi-vectors $\omega_{ij} = X_{ij} \wedge Y_{ij}$ are a base of eigenvectors of ρ.

P_2: there exist an orhtonormal base X_i in T_xM such that the bi-vectors $W_{ij} = X_i \wedge X_j$ are eigenvectors of ρ.

Clearly $P_2 \Longrightarrow P_1$ and P_1 implies equivalence between positivity of sectional curvatures and positivity of ρ.

The following examples are of some interest:

3.1 Example. $P_1 \not\Rightarrow P_2$. It is sufficient to describe a counter-example at a single point. Let V be an inner product space with $\dim V \geq 4$ and $X_1,\ldots,X_n$ an orthonormal base of V. For $i < j \leq n$ we set

$$\omega_{ij} = \begin{cases} X_i \wedge X_j & \text{if } (i,j) \neq (1,3),\ (i,j) \neq (2,3) \\ (\cos\Theta X_1 + \sin\Theta X_2) \wedge X_3, & \text{if } (i,j) = (1,3) \\ (-\sin\Theta X_1 + \cos\Theta X_2) \wedge X_3, & \text{if } (i,j) = (2,3), \end{cases}$$

where $\Theta \neq k\frac{\pi}{2}$. Then $\{\omega_{ij}\}$ is a orthonormal base in $\Lambda^2(V)$ consisting of decomposable bi-vectors. Define $\rho\colon \Lambda^2(V) \to \Lambda^2(V)$ by

$$\rho(\omega_{ij}) = \lambda_{ij}\omega_{ij} \quad \text{with} \quad \lambda_{ij} \neq \lambda_{ks} \quad \text{if} \quad \{i,j\} \neq \{k,s\}.$$

Then ρ is a curvature-like operator which satisfies P_1.

On the other hand ρ does not satisfy P_2. In fact if $\{Y_1,\ldots,Y_n\}$ is a base of V the bi-vectors $\pi_{ij} = Y_i \wedge Y_j$, viewed as 2-planes, satisfy the followin property: There exist two lines L_1, L_2 in π_{ij} such that $\pi_{ij} \cap \pi_{ks}$ is either zero, or L_1 or L_2. But ω_{12} is intersected by ω_{13}, ω_{14}, ω_{23} and ω_{24} along four distinct lines.

3.2 Example. Let $\dim M \leq 3$ or M be conformally flat. Then ρ satisfies P_2 (see [5]).

3.3. Example. Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion with flat normal connection. Then ρ satisfies P_2.

The above examples justify our interest in property P_2.

3.4 Definition. We will say that M has pure curvature operator (p.c.o. for short) if ρ satisfies P_2. In this case an orthonormal base $\{X_1,\ldots,X_n\}$ in T_xM such that $\rho(X_i \wedge X_j) = \lambda_{ij} X_i \wedge X_j$ will be called a ρ-base.

It is clear that if M_1 and M_2 are Riemannian manifolds with p.c.o. ρ_1 and ρ_2 respectively, then the product $M_1 \times M_2$ has p.c.o. since the union of a ρ_1-base and a ρ_2-base gives a ρ-base. The converse was proved under additional conditions in [5]; we will now prove it in full generality.

3.5 Proposition. Let $M = M_1 \times M_2$ be a decomposable Riemannian manifold with p.c.o. Then M_1 and M_2 have p.c.o.

Proof. Let $\{X_1,\ldots,X_n\}$ be a ρ-base for M. We will denote by ρ_i, $i = 1,2$ the curvature operator of M_i and, for $X \in T_xM$, by X' and X'' the projection of X onto TM_1 and TM_2 respectively. With these notations we have:

3.5.1

$$\begin{cases} \rho(X_i \wedge X_j) = \lambda_{ij}(X_i' \wedge X_j' + X_i' \wedge X_j'' + X_i'' \wedge X_j' + X_i'' \wedge X_j'') \\ \rho(X_i \wedge X_j) = \rho_1(X_i' \wedge X_j') + \rho_2(X_i'' \wedge X_j'') \end{cases}$$

therefore

3.5.2.

$$\begin{cases} \rho_1(X_i' \wedge X_j') = \lambda_{ij} X_i' \wedge X_j' \\ \rho_2(X_i'' \wedge X_j)'' = \lambda_{ij} X_i'' \wedge X_j'' \\ \lambda_{ij}(X_i' \wedge X_j'' + X_i'' \wedge X_j'') = 0 \end{cases}$$

3.5.3. Claim. If $\lambda_{ij} \neq 0$ then either $X_j' = 0 = X_j'$ or $X_i'' = 0 = X_j''$.

Proof of the claim: Let us suppose $X_j'' \neq 0$ and set $\omega = X_i' \wedge X_j'' + X_i'' \wedge X_j'$. Taking interior product with X_i' we get

$$0 = i(X_i')\omega = \|X_i'\|^2 X_j'' - \langle X_j', X_i' \rangle X_i''$$

and therefore $X_j'' = \langle X_j', X_i' \rangle \|X_i'\|^{-2} X_i''$.

Taking interior product with X_j' we get

$$0 = i(X_j')\omega = \langle X_i', X_j' \rangle X_j'' - \|X_j'\|^2 X_i'' =$$

$$= \|X'\|^{-2}(\langle X_i', X_j' \rangle^2 - \|X_i'\|^2 \|X_j'\|^2) X_i''$$

If $X_i \neq 0$ the above relation gives $X_j' = \lambda X_i'$ and by 3.5.1 and 3.5.2 we get $X_i \wedge X_j = X_i'' \wedge X_j''$ and this implies (taking interior product with X_i') that $X_i' = 0$, which contradicts our assumption. So $X_i' = 0$.

Reversing the roles of X_i and X_j we get again, either $X_j' = 0$ or $X_j'' = 0$ and therefore X_i and X_j are tangent to M_1 or M_2. Again, since $\lambda_{ij} \neq 0$ they must be tangent to the same M_i, which proves our claim.

Now we reorder the vectors $X_1, \ldots, X_n$ in such a way that $X_1, \ldots, X_k$ are tangent to M_1 and there exists, for any $i = 1, \ldots, k$, an index j such that $\lambda_{ij} \neq 0$ (in particular X_j is tangent to M_1) $X_{k+1}, \ldots, X_m$ are such that $\lambda_{ij} = 0$ for $i = k+1, \ldots, m$ and $\{X_1, \ldots, X_k, X_{k+1}', \ldots, X_m'\}$ span the tangent space to M_1.

We observe that $\{X_1, \ldots, X_k\}$ are orthonormal and orthogonal to span $\{X_{k+1}', \ldots, X_m'\}$. Let $\{Y_{k+1}, \ldots, Y_m\}$ be an orthonormal base for span $\{X_{k+1}', \ldots, X_m'\}$. From 3.5.1 it follows easily that $\{X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_m\}$ is a ρ_1-base and therefore M has p.c.o. In the same say we see that M has p.c.o. and therefore the proof is completed.

3.6. Remark. We do not know whether a totally geodesic submanifold of a manifold with p.c.o. has p.c.o. However if M is a complete open manifold with pure non negative curvature operator and $S \subseteq M$ is a soul, then S has pure curvature operator (see [1], Lemma (4.1)).

4. Proof of the theorem

By the previous section we can reduce Theorem 1 to the case where M is irreducible. By 2.1, M is therefore a symmetric space or a manifold biholomorphic to CP^n or homeomorphic to a sphere. In the last case we do not have anything to prove. The second possibility is ruled out by the following:

4.1 Proposition. If a Riemannian manifold M has pure curvature operator, then the Pontrjagin forms of M vanish.

Proof. See [5]

4.2 Proposition. Let M be a irreducible symmetric space. Then M has pure curvature operator if and only if M has constant sectional curvature.

Proof. In a small neighborhood A of a given point, we may find an orthonormal frame field $\{X_1,\dots,X_n\}$ such that the only essential components of R are $\langle R(X_i,X_j)X_j,X_i\rangle = \lambda_{ij}$. Since M is symmetric λ_{ij} are constant on A which implies that R has eigenvalues with constant multiplicity in A. Then the frame field $\{X_1,\dots,X_n\}$ can be chosen differentiably on A. We need the following:

4.2.1 Claim. Let j,ℓ be fixed and suppose $\lambda_{ij}\neq\lambda_{i\ell}$ for some $i=1,\dots,dimM$. Then, for all $X\in TM$

$$\langle\nabla_X X_j, X_\ell\rangle = 0.$$

Proof of the claim. Since M is symmetric, $\nabla R\equiv 0$. Moreover, ρ being pure implies that if the set $\{i,j,k,\ell\}$ contains more then two elements, $\langle R(X_i,X_j)X_k,X_\ell\rangle = 0$. Therefore we have:

$$0 = (\nabla_X R)(X_i,X_j,X_i,X_\ell) = -R(X_i,\nabla_X X_j,X_i,X_\ell)$$

$$-R(X_i,X_j,X_i,\nabla_X X_\ell) = -\langle\nabla_X X_\ell,X_j\rangle(\lambda_{ij}-\lambda_{i\ell})$$

and the claim is proved.

Let us suppose that the curvature of M is not constant. Then there exists $i_0\in\{1,\dots,n\}$ such that $\{\lambda_{i,j}\}$ contains at least two elements. Up to reordering the base we can suppose

$$\lambda_{12} = \dots = \lambda_{1p}, \lambda_{1r}\neq\lambda_{12} \quad \text{if} \quad r>p$$

4.2.2 Claim. There exists $r_0>p$, such that if $i\notin\{1,r_0\}$, $\lambda_{1i}=\lambda_{r_0 i}$

Proof of the claim. Let us suppose that for all $r>p$ there exists $i\notin\{1,r\}$ with $\lambda_{1i}\neq\lambda_{ri}$. From 3.1.1 we get

$$\langle\nabla_X X_1, X_r\rangle = 0 \quad \forall\, X\in TM,\ r>p.$$

Again 3.1.1. gives:

$$\langle\nabla_X X_j, X_r\rangle = 0 \quad j=2,\dots,p;\quad r>p$$

and therefore the distributions $span\{X_1,\dots,X_p\}$ and $span\{X_{p+1},\dots,X_n\}$ would be parallel which contradicts irreducibility of M and proves our claim.

Using now 3.1.1. for $i=r_0$, $j=2,\dots,p$ we get:

$$\langle\nabla_X X_j, X_1\rangle = 0 = \langle\nabla_X X_j, X_s\rangle,\quad s>p$$

Therefore the distributions $span\{X_2,\dots,X_p\}$ and $span\{X_1,X_{p+1},\dots,X_n\}$ are parallel, which again contradicts irreducibility of M.

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