# A LOWER BOUND FOR THE FIRST EIGENVALUE OF A FINITE-VOLUME NEGATIVELY CURVED MANIFOLD

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## 0. Introduction

In this paper $(M,g)$ will be a complete Riemannian manifold with a metric $g$ and finite volume $V$, of pinched negative sectional curvature $K$, $-1 \le K \le k^2 < 0$, and of dimension greater than two. We prove the following geneneralization of the main results of [D-R] and [S].

**Theorem.** Let $M$ be as above and let $\lambda = \lambda_1(M) = \inf\{\mu > 0 \mid \mu \in \mathrm{Spec}(\Delta)\}$ be the greatest lower bound of the positive part of the spectrum of the Laplace operator (considered as an unbounded operator on $L^2(M)$). Then

$$\lambda \geq \frac{c(n)}{V^2} k^2,$$

where the constant $c(n) > 0$ depends only on $n = \dim M$.

The method of proof is the same as in [D-R] but there are certain technical complcations. The main difficulty turns out to be avoiding the appearance of derivatives of the curvature tensor in the estimate of $L^\infty$ norm of the gradient of an eigenfunction belonging to $\lambda_1(M)$. This is accomplished by smoothing the metric using the result of [B-M-R]. The paper is organized as follows. In Section 1 we introduce the notation, discuss the "thick and thin" decomposition of $M$, and show how to smooth out the metric.

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Section 2 deals with proving that "thin pieces have large Dirichlet eigenvalues." This was done by an explicit calculation in [D-R] and has to be replaced by a comparison argument. Section 3 contains the proof of the theorem.

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## 1. Smoothing the metric

We begin by describing the "thick and thin" decomposition of $M$ (cf. §10 of [B-G-S]). There exists a positive number $\mu = \mu(n)$ depending only on $n = \dim M$ such that the set $\{x \in M \mid \operatorname{inj}(x) \leq \mu\}$ is either empty or consists of components, each of which is either

a) a closed embedded tubular neighborhood of a simple closed geodesic of length smaller than $2\mu$, or

b) a cusp, i.e. a quotient of a horoball in the universal covering $\tilde{M}$ by a discrete group of parabolic isometries acting with compact quotient $H$ on the boundary horosphere $\tilde{H}$.

Every cusp can be parametrized as $H \times [0,M)$ by mapping curves $t \to (x,t)$ into unit speed geodesics $\gamma_x$ emanating from $x$ and orthogonal to $H$. The hypersurfaces $H_c$ defined by $t = c$ are $C^\infty$, are perpendicular to geodesics $\gamma_x$, and have sectional curvatures $K$ satisfying $|K| \leq 2$. This, except for the smoothness of $H_c$, is either proved in or follows easily from [H-I]. The smoothness of horospheres follows from the fact that they are images of strongly stable submanifolds for the geodesic flow in the unit tangent bundle of $M$. The geodesic flow is Anosov and it is a general fact that strongly stable manifolds of an Anosow flow are as smooth as the flow itself (cf. [H-P]).



The "size" of components of either type can be estimated as follows. If $\gamma$ is a short (of length $< 2\mu$) simple closed geodesic, then the normal exponential mapping is injective on a tube of radius $r(n)$ in the normal bundle. Denote the image of this tube by $T$. By choosing $\mu$ sufficiently small we can assume that $r(n) > 1$ and that the shell $S = \{x \in M \mid r(n) \geq d(x,\gamma) \geq r(n) - 1\}$ of $T$ consists of points with injectivity radius greater than or equal to $\mu$. Similarly, if $T$ is a cusp, we can assume that all points $x$ between $H = H_0$ and $H_1$ satisfy $\operatorname{inj}(x) \geq \mu$. As above this set will be called the shell of $T$. The union of all tubes and cusps will be denoted $M_{\text{thin}}$. The pieces of this union are disjoint and, for each component $T$ of $M_{\text{thin}}$, the volume satisfies $\operatorname{vol}(T) \geq c(n) > 0$. Here and in the sequel the constants depend only on the quantities indicated. The same symbol appearing in different inequalities may denote different constants.

Let $S$ be the union of all shells of components of $M_{\text{thin}}$. By definition $M_{\text{thick}} = (M - M_{\text{thin}}) \cup S$. We will denote by $M_t$, $t \geq 0$, $M$ with all cusps cut off at distance $t$ from the bounding horocycle.

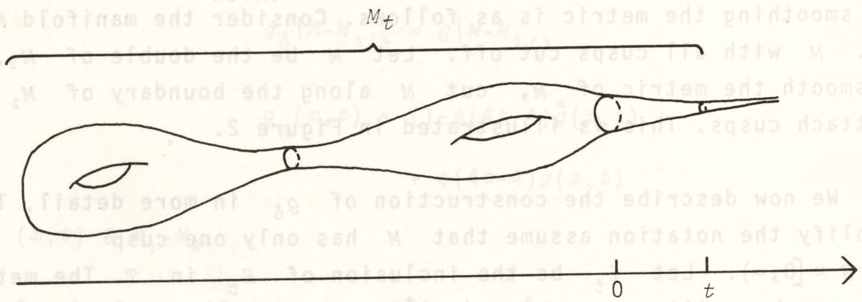


Figure 1

We now state our smoothing result.

**Proposition.** For every $\delta \in (0,1)$ and an integer $k \geq 0$ there exists a metric $g_\delta$ on $M$ such that

(i) $(1-\delta)^2 g \leq g_\delta \leq (1+\delta) g$

(ii) $\|Rm_\delta\|_{C^k(M_{\text{thick}})} \leq c_1(n,k,\delta)$

(iii) $\text{inj}(x) \geq c(n,k,\delta) > 0$ for every $x \in M_{\text{thick}}$.

(iv) $g_\delta = g$ on $M-M_2$.

Here $Rm_\delta$, $\text{inj}_\delta$ denote the Riemann curvature tensor and the injectivity radius, respectively, for the metric $g_\delta$. Moreover

$$\|Rm_\delta\|_{C^k(M_{\text{thick}})} = \sum_{\ell=0}^{k} \sup_{x \in M_{\text{thick}}} |\nabla^\ell Rm_\delta(x)|,$$

where the pointwise norm is computed with respect to the metric $g_\delta$.

**Proof.** We would like to use the result of [B-M-R] which applies to compact manifolds without boundary. Note howerver, that we need $C^k$ bound only on the compact part $M_{\text{thick}}$ of $M$. The idea for smoothing the metric is as follows. Consider the manifold $M_2$, i.e. $M$ with all cusps cut off. Let $M$ be the double of $M_2$. We smooth the metric of $M$, cut $M$ along the boundary of $M_2$ and reattach cusps. This is illustrated in Figure 2.

We now describe the construction of $g_\delta$ in more detail. To simplify the notation assume that $M$ has only one cusp $T \cong H \times [0,\infty)$. Let $i_t$ be the inclusion of $H_t$ in $T$. The metric $g|T$ can be written as $dt^2 + (i_t)^* g$. Choose a smooth function $\phi\colon [0,1] \to [0,1]$ so that $\phi|[0,1/4] \equiv 0$, $\phi|[3/4,1] \equiv 1$ and define a new metric $\tilde{g}$ on $M$ as follows.

$$\tilde{g} \mid M_1 = g$$

$$\tilde{g} \mid M_2 - M_1 = dt^2 + (1-\phi(t-1))(i_t^* g)(x) + \phi(t-1)(i_2^* g)(x).$$



Thus, near $\partial M_2$, $\tilde{g}$ is a product metric so that we can smoothly double $(M_2, \tilde{g})$. Denote the resulting Riemannian manifold $(\bar{M}, \bar{g})$. The metric $\bar{g}$ agrees with $g$ on $M_{5/4}$ and satisfies

$$|\overline{Rm}(x)| \leq c(n)$$

for all $x \in M$. Here, $\overline{Rm}$ stands for the Riemann curvature tensor of the metric $\bar{g}$. This inequality is a consequence of the uniform boundedness of sectional curvatures of the hypersurfaces $H_t$.

We now apply the result of [B-M-R] to the metric $\bar{g}$ on the double $M$ and obtain a metric $\bar{\bar{g}}$ such that

$$(1-\delta)^2 \bar{g} \leq \bar{\bar{g}} \leq (1+\delta)^2 \bar{g}$$

$$\|\bar{\bar{Rm}}\|_{C^k(M)} \leq c(n,k,\delta).$$

Since $M_{\text{thick}} \subset M_1 \subset M_{5/4}$ and $g$ agrees with $\bar{g}$ on $M_{5/4}$, (i) and (ii) will be satisfied on $M_{\text{thick}}$ if $g_\delta|M_{\text{thick}} = \bar{\bar{g}}|M_{\text{thick}}$. We now re-attach the cusps, i.e. define the metric $g_\delta$ on $M$ as follows.

$$g_\delta|M_1 = \bar{\bar{g}}|M_1$$

$$g_\delta|M-M_{5/4} = g|M-M_{5/4}$$

$$g_\delta(x,t) = (1-\phi(4t-4))\bar{\bar{g}}(x,t) + \phi(4t-4)g(x,t)$$

for $(x,t) \in M_1 - M_2$.

On checks easily that conditions (i), (ii) and (iv) of the Proposition are satisfied. The third condition follows from the first two by an argument of [K] as follows. First note that for every $x \in M_{\text{thick}}$ the radius of the largest ball around the origin in the tangent space at $x$ on which the exponential mapping is

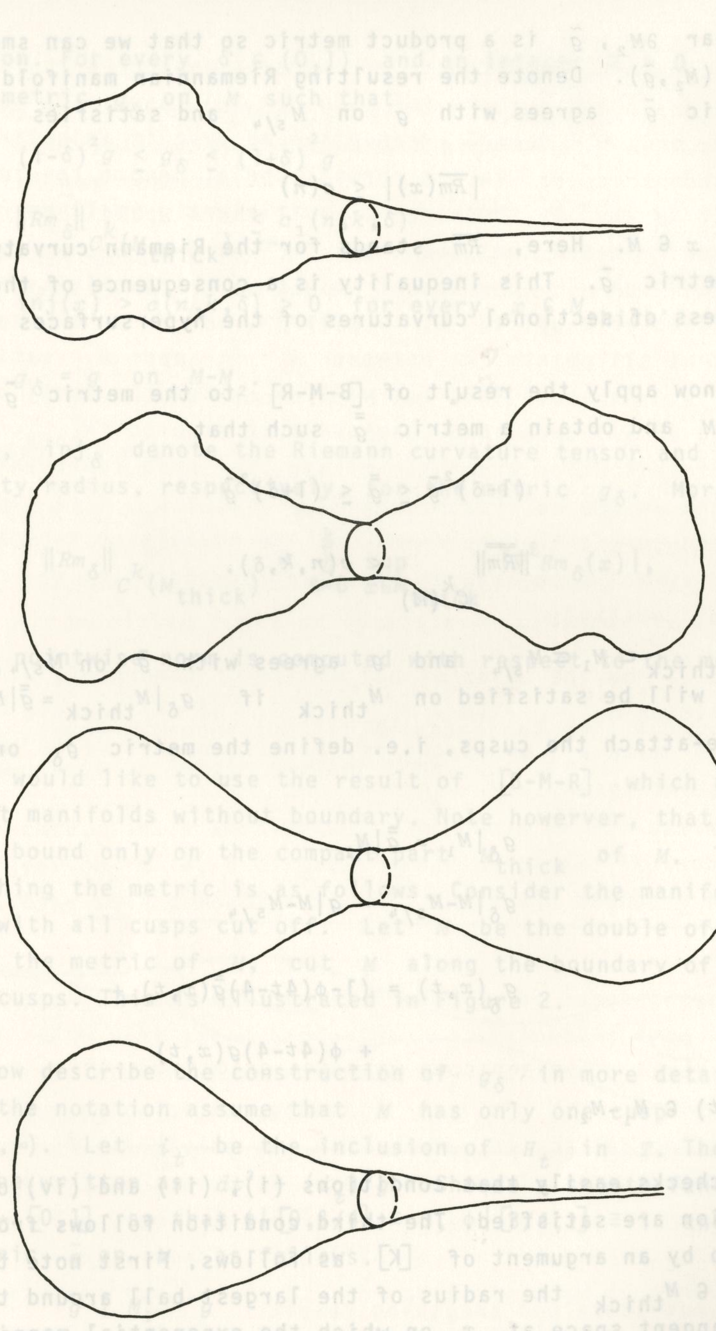

Figure 2



nonsingular (for the metric $g_\delta$) can be bounded from below by a constant depending only on the upper bound of the norm of the curvature tensor. Thus very short curves in $M$ can be lifted to the tangent space at their initial point. If the injectivity radius at $x \in M_{\text{thick}}$ were very small there would exist a short closed loop consisting of two geodesic segments emanating from $x$. By (i) such loop is short with respect to the metric $g$ and can be assumed to be contained in a ball or radius $r < \text{inj}(x)$. Therefore this loop is contractible by a "short" homotopy. The homotopy in turn is "short" with respect to $g_\delta$ and can be lifted by the inverse of the exponential mapping for $g_\delta$ to the tangent space at $x$. It follows that the original closed loop lifts to a closed curve. This is a contradiction, since the two geodesic segments lift to segments of different rays in the tangent space.

## 2. First Dirichlet eigenvalue of a thin component

Let $T$ be a component of $M_{\text{thin}}$. In this section we outline the proof of the following estimate.

**Lemma 1.** If $f \in C^\infty(T)$ and $f|\partial T \equiv 0$, then

$$\int_T |df|^2 \, dV \geq \frac{(n-1)^2}{4} k^2 \int_T f^2 \, dV$$

**Remark.** Here $T$ is equipped with the original metric $g$.

**Proof:** We sketch the proof for a cusp. In terms of the parametrization of a cusp $T$ as $H \times [0,\infty)$ described in §1, the volume element can be written as $dV|T = h(x,t)dt \wedge dV_0$, where $dV_0$ is the volume element of $H = H_0$ in the induced metric.

A standard comparison argument using stable Jacobi vector fields (cf. [H-I]) yields

(2.1) $$\frac{1}{h}\frac{\partial h}{\partial t} \geq -(n-1)k.$$

We now apply the argument of McKean (cf. [M], [D-R]) as follows.

$$\left[\int_0^\infty \left(\frac{\partial f}{\partial t}\right)^2 hdt\right]^{1/2}\left[\int_0^\infty f^2 hdt\right]^{1/2} \geq$$
$$\geq \int_0^\infty \frac{\partial f}{\partial t} fhdt = \frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}(f^2)hdt \geq$$
$$-\frac{1}{2}\int_0^\infty f^2\frac{\partial h}{\partial t}\,dt \geq \frac{(n-1)}{2}\,k\int_0^\infty f^2 hdt.$$

Therefore

$$\int_0^\infty |df|^2 hdt \geq \frac{(n-1)^2}{4}k^2\int_0^\infty f^2 hdt.$$

integration over $H$ yields the Lemma.

**Remark.** The proof for the case when $T$ is a tubular neighborhood of a geodesic is identical to the proof above and uses an estimate for the volume element in terms of Fermi coordinates analogous to (2.1).

The following is an easy corollary and an approximate version of Le-ma 1.

**Lemma 2.** There exists a constant $\eta = \eta(n) > 0$ such that if $T$ is a component of $M_{thin}$ with shell $S$, and if $f$ is a function on $T$ satisfying

(i) $$\int_T f^2 dV \geq b > 0$$

(ii) $$\int_S |dt|^2 dV \leq b\eta$$

(iii) $$\int_S f^2 dV \leq b\eta,$$

then

$$\int_T |df|^2 dV \geq b\,\frac{(n-1)^2k^2}{8}.$$



The proof of the corresponding fact in [D-R] carries over verbatim.

## 3. The proof

We first remark that by Lemma 1 the spectrum of the Laplacian of $M$-$M_1$ with Dirichlet boundary conditions is contained in $[k^2(n-1)^2/4,\infty)$. Therefore (cf. [D-L])

$\mathrm{Spec}(\Delta) \cap [0,k^2(n-1)^2/4)$ is discrete. It follows that either $\lambda_1(M,g) \geq (n-1)^2k^2/4$ or $\lambda_1(M,g)$ is an eigenvalue. Since $g$ and $g_\delta$ are isometric near infinity the same alternative holds for $g_\delta$ for every $\delta \in (0,1)$. We have to apply the Proposition of §1 with specific values of $\delta$ and $k$. $\delta$ can be chosen arbitrarily, e.g. $\delta = 1/2$; the choice of $k = k(n)$ will be described below. Denote the corresponding metric by $g'$.

Observe that it is enough to prove our theorem for the metric $g'$. Indeed, if

$$I(f,M,g) = \frac{\int_M |df|_g^2 dV_g}{\int_M f^2 dV_g}$$

then, by min-max characterization of eigenvalues,

(3.1) $$\lambda_1(M,g) = \inf_W \sup_{f\in W-\{0\}} I(f,M,g)$$

where $W$ runs over all two dimensional subspaces of $C^\infty(M)$.

It follows from (1.1)(i) that

(3.2) $$I(f,M,g) \geq c_1(n)\ I(f,M,g') \quad \text{for } f\in C_0^\infty \text{ and}$$
$$\mathrm{vol}(M,g) \leq c_2(n)\ \mathrm{vol}(M,g').$$

Hence, if

$$\lambda_1(M,g') \geq \frac{c(n)k^2}{\mathrm{vol}(M,g')^2}$$

then, by (3.1),

$$\lambda_1(M,g) \geq \frac{c(n)c_1(n)}{c_2(n)^2} \frac{k^2}{\mathrm{vol}(M,g)^2}.$$

Observe also that Lemma 1 and Lemma 2 of §2 give lower bounds for $I(f,T,g)$. Therefore both remain true for the metric $g'$ if we replace $(n-1)^2/4$ by $c(n)$.

As remarked above we can assume that $\lambda = \lambda_1(M,g')$ is smaller than $k^2(n-1)^2/4$ and corresponds to an eigenfunction $\phi$ satisfying

$$\text{(3.3)} \quad \begin{aligned} &\text{(i)} \quad \Delta\phi + \lambda\phi = 0 \\ &\text{(ii)} \quad \int_M \phi^2 dV = 1 \\ &\text{(iii)} \quad \int_M \phi\, dV = 0 \\ &\text{(iv)} \quad \int_M |d\phi|^2 dV = \lambda. \end{aligned}$$

$\Delta$, $dV$, $|\ |$ above are with respect to the metric $g'$.

We now show that $|d\phi(x)|$ can be estimated in terms of $\lambda$ for $x \in M_{\text{thick}}$. If $0 < r < \mathrm{inj}(x)$ then by standard elliptic theory (cf. [C-G-T])

$$|d\phi(x)| \leq b \sum_0^N \|\Delta^i d\phi\|_{L^2(B_r(x))} \leq$$

$$\text{(3.4)} \qquad \leq b \sum_0^N \lambda^i \|d\phi\|_{L^2(M)} \leq$$

$$\leq b\lambda^{1/2} \sum_0^N \lambda^i$$

where we used (3.3) and the commutativity $\Delta d = d\Delta$. Unlike in the case of functions considered in [C-G-T], the constant $b$ depends on $r$ and on $C^k(B_r(x))$ norm of the curvature tensor for some $k = k(n)$, and $N = [n/4]+1$. From now on we use the metric $g'$ of §1 with $k = k(n)$ and $\delta = 1/2$. Since we are trying to prove a lower bound for $\lambda$, assume $\lambda \leq 1$. By the Proposition of §1, we can choose a lower bound for $\mathrm{inj}(x)$, and upper bounds for $|\nabla^\ell Rm(x)|$, $0 \leq \ell \leq k(n)$ depending only on $n$ provided $x \in M_{\text{thick}}$.



Therefore we can assume that the constant $b$ in (3.4) depends only on $n$. We thus have an estimate

$$\text{(3.5)} \qquad |d\phi(x)| \leq c(n)\lambda^{1/2},$$

holding for all $x \in M_{\text{thick}}$. We are now in a position to repeat the argument of [D-R]. Assume that $\lambda \leq \alpha V^{-2}$. This will lead to a contradiction if $\alpha$ is sufficiently small. Note tha $M_{\text{thick}}$ is nonempty and therefore $V = \mathrm{vol}(M) > v(n)$. As in [D-R], (3.5) and our assumption on $\lambda$ imply that the oscillation of $\phi$ on $M_{\text{thick}}$ is less than or equal to $c_1(n)(\alpha/V)^{1/2}$. Hence, if

$$\sup_{x \in M_{\text{thick}}} |\phi(x)| > c_1(n)(\alpha/V)^{1/2}$$

then $\phi|M_{\text{thick}}$ is of constant sign and we may assume that it is positive. If $M_{\text{thin}} = \phi$ this is an immediate contradiction by (3.3)(ii). If $M_{\text{thin}}$ is nonempty, then it follows that $\phi(x) < 0$ for a point $x \in M_{\text{thin}}$. Let $T$ be the component of $M_{\text{thin}}$ containing $x$. Since $\partial T \subset M_{\text{thick}}$, $\phi|\partial T > 0$ and we conclude that $\phi$ is an eigenfunction with eigenvalue $\lambda$ for a domain $D \subset T$. It then follows from Lemma 1 and domain monotonicity of Dirichlet eigenvalues that $\lambda > c_2(n)k^2$.

If, on the other hand,

$$\sup_{x \in M_{\text{thick}}} |\phi(x)| \leq c_1(n)(\alpha/V)^{1/2},$$

then for small $\alpha$ the integral of $\phi^2$ over $M_{\text{thin}}$ is at least $1/2$. It follows that there exists a component $T$ of $M_{\text{thin}}$ for which

$$\int_T \phi^2 dV \geq c_3(n) \frac{\mathrm{vol}(T)}{V} \geq c_4(n)\frac{1}{V}.$$

Now apply Lemma 2 with $b = c_4(n)/V$, $\eta = c_5(n)\alpha$, to conclude that

$$\lambda \geq \int_T |d\phi|^2 dV \geq \frac{c_6(n)}{V} k^2.$$

For small values of $\alpha$ this contradicts the assumption that $\lambda < \alpha V^{-2}$.

It follows that

$$\lambda \geq c_7(n) \min\left(1, \frac{1}{v^2}, k \ , \frac{k^2}{v}\right) \geq \frac{c_8(n)k^2}{V^2}.$$

This concludes the proof.

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