THE GENERALIZED GAUSS MAP OF MINIMAL SURFACES IN H^3 AND H^4

CÉLIA C. GÓES AND PLINIO A.Q. SIMÕES

1. Introduction

The object of this paper is to establish conditions for a C^∞ map of a Riemann surface M into Q_{n-2}, the hyperquadric $z^1z^1 + \dots + z^nz^n = 0$ of $\mathbb{C}P^{n-1}$, to be the generalized Gauss map of a minimal conformal immersion of M into H^3 and H^4, the hyperbolic space of dimensions three and four respectively. Using the upper half-hyperplane as model for the hyperbolic space and exploiting the conformality between the metrics induced on M, by the euclidean metric and the hyperbolic metric through the immersion we can adapt the theory developed by Hoffman and Osserman [H-O,2] to obtain the conditions.

2. Basic facts

Let $<\ ,\ >$ be the usual euclidean metric on $\mathbb{R}^n$ and let H^n and $\mathbb{R}^n_+$, $n=3,4$, the set $\{(x,t) \mid x \in \mathbb{R}^{n-1},\ t > 0\}$ endowed with the metrics $(,)_{(x,t)} = \frac{1}{t^2} <\ ,\ >$ and $<\ ,\ >$ respectively. Given $(x,t) \in H^n$, let $\hat{x} = (x^1,\dots,x^{n-1},0) \in \mathbb{R}^{n-1}$. Thus $a,b \in H^n$ implies that $L_{a,b}((x,t)) = \frac{b^n}{a^n}[(x,t)-\hat{a}] + \hat{b}$ is an isometry of H^n such that $L_{a,b}(a) = b$, $(L_{a,b})_*(v) = \frac{b^n}{a^n} v$ for all v in the tangent space $T_a(H^n)$. Let M be a Riemann surface and $\tilde{\theta}(p) = (x(p),t(p))$ be a conformal immersion of M into H^n. If

Recebido em 10/01/87.

$q = (0,\dots,0,1) \in H^n$, $(L_{\tilde{\theta}(p),q})_*$ sends $T_{\tilde{\theta}(p)}(H^n)$ isometrically onto $T_q(H^n)$, which is $\mathbb{R}^n$ endowed with its usual inner product. Let $G_2(\mathbb{R}^n)$ be the grassmannian of the oriented 2-vector subspaces of $\mathbb{R}^n$. The map $\tilde{G}: M \to G_2(\mathbb{R}^n)$ defined by $\tilde{G}(p) = (L_{\tilde{\theta}(p),q})_*(\tilde{\theta}_{*p}(T_p(M)))$ is the generalized Gauss map of $\tilde{\theta}$. It is well known that $G_2(\mathbb{R}^n)$ can be identified with the hyperquadric $Q_{n-2} = \{[z] \in \mathbb{C}P^{n-1} / z = (z^1,\dots,z^n)$ and $\sum_{k=1}^{n} z^k z^k = 0\}$ of the $(n-1)$-dimensional complex projective space. Such identification will be assumed throughout this paper.

Now let $z = u+iv$ be local isothermal parameters for M and let θ be a conformal immersion of M into $\mathbb{R}^n_+$ given by $\theta(p) = \tilde{\theta}(p)$ $(\forall\, p \in M)$. Then

$$(2.1) \qquad \tilde{G}(z) = \left[\frac{\partial\theta}{\partial u} - i\frac{\partial\theta}{\partial v}\right] = \left[\frac{\partial\theta}{\partial z}\right],$$

where $\frac{\partial\theta}{\partial z} = 1/2\left(\frac{\partial\theta^1}{\partial u} - i\frac{\partial\theta^1}{\partial v},\dots,\frac{\partial\theta^n}{\partial u} - i\frac{\partial\theta^n}{\partial v}\right) \in \mathbb{C}^n$.

If $\Phi(z) = (\phi^1(z),\dots,\phi^n(z)) \in \mathbb{C}^n$ is a homogeneous local expression of $\tilde{G}(z)$, there is $\psi: M \to \mathbb{C}-\{0\}$ such that

$$(2.2) \qquad \frac{\partial\theta}{\partial z} = \psi\Phi.$$

Let ds^2 be the riemannian metric induced on M by θ, $\frac{\partial}{\partial\bar{z}} = 1/2\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$, Δ the Laplace-Beltrami operator of M with respect to ds^2, $\lambda^2 = \left|\frac{\partial\theta}{\partial u}\right|^2 = \left|\frac{\partial\theta}{\partial v}\right|^2$, H the mean curvature vector of θ. It is well known that

$$(2.3) \qquad \Delta\theta = \frac{4}{\lambda^2}\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}\theta = 2H.$$

Indicating by $\langle\langle,\rangle\rangle$ the usual hermitian inner product on $\mathbb{C}^n$, let

$$(2.4) \qquad V = \Phi_{\bar{z}} - \eta\Phi,$$

where $\Phi_{\bar{z}} = \frac{\partial\Phi}{\partial\bar{z}}$ and $\eta = \frac{\langle\langle\Phi_{\bar{z}},\Phi\rangle\rangle}{|\Phi|^2}$.

Then by (2.2), (2.3) and (2.4) we have

$$(2.5) \qquad V = |\Phi|^2\bar{\psi}H,$$

$$(2.6) \qquad (\log\psi)_{\bar{z}} = -\eta.$$

3. The case n=3

Let $\nu = (\nu^1,\nu^2,\nu^3)$ be the normal unitary vector field to $\theta(M)$ in $\mathbb{R}^3_+$, $\hat{\mathbb{C}}$ the extended complex plane and $f: M \to \hat{\mathbb{C}}$ the composition of the classical Gauss map of θ with the stereographic projection with respect to the north pole of the euclidean unitary sphere. Then for $z \in U \subset \mathbb{C}$, we have

$$(3.1) \qquad \nu(z) = \frac{1}{1+|f(z)|^2}\left(2\,\mathrm{Re}\, f(z),\ 2\,\mathrm{Im}\, f(z), |f(z)|^2-1\right).$$

Identifying $\hat{\mathbb{C}}$ biholomorphically with Q_1, through the correspondence $\xi(\omega) = [1-\omega^2, i(1+\omega^2), 2\omega]$, $\xi(\infty) = [-1,i,0]$, $\xi^{-1}([z^1,z^2,z^3]) = -\frac{z^1+iz^2}{z^3}$, where $\omega \in \hat{\mathbb{C}}$ and z^1,z^2,z^3 are homogeneous coordinates on $Q_1 \subset \mathbb{C}P^2$, we obtain

$$(3.2) \qquad [\Phi] = [1-f^2,\ i(1+f^2),\ 2f].$$

Then from (2.4) we obtain

$$(3.3) \qquad \eta = \frac{2\bar{f}f_{\bar{z}}}{1+|f|^2},$$

$$(3.4) \qquad V = -2f_{\bar{z}}\,\nu.$$

If h is defined by $H = h\nu$, (2.5) and (3.4) imply

$$h = \frac{-f_{\bar z}}{\bar\psi(1+|f|^2)^2}. \tag{3.5}$$

Therefore

$$f_{\bar z} = 0 \Longleftrightarrow h = 0. \tag{3.6}$$

Restricting to those $z \in U$, where h is non-zero, we obtain

$$(\log h)_z = \frac{f_{z\bar z}}{f_{\bar z}} - 2\frac{\bar f f_z}{1+|f|^2} - \bar n - (\log \bar\psi)_z . \tag{3.7}$$

Then (2.6) implies

$$(\log h)_z = \frac{f_{z\bar z}}{f_{\bar z}} - 2\frac{\bar f f_z}{1+|f|^2}. \tag{3.8}$$

Let $\tilde H$ be the mean curvature vector of $\tilde\theta = (x^1, x^2, t)$. Then we have $\tilde H = t^2 H + t\nu^3\nu$ [G-S]. Hence the minimality of $\tilde\theta$ implies

$$h = -\frac{\nu^3}{t}. \tag{3.9}$$

Then from (3.1) we obtain

$$h = -\frac{|f|^2 - 1}{t(|f|^2+1)}. \tag{3.10}$$

Therefore

$$f_{\bar z} = 0 \Longleftrightarrow h = 0 \Longleftrightarrow |f|^2 = 1 \Longleftrightarrow \nu^3 = 0. \tag{3.11}$$

From the expressions of h given by (3.10) and (3.5) we obtain

$$\bar\psi = \frac{t f_{\bar z}}{|f|^4 - 1}. \tag{3.12}$$

Since $\bar\psi$ is well defined on U, we can extend the ratio $\frac{f_{\bar z}}{|f|^4-1}$ to U. Then it is straightforward to verify that

$$\alpha = \frac{\bar f_z}{|f|^4-1} dz \tag{3.13}$$

is a differential 1-form globally defined on M. Also (3.12) and (2.2) imply

$$\theta_z = \frac{t\bar f_z}{|f|^4-1}(1-f^2, i(1+f^2), 2f). \tag{3.14}$$

From (3.10) and (3.13) we obtain

$$(|f|^2-1)f_{z\bar z} - \frac{2\bar f|f|^2}{|f|^2+1} f_z f_{\bar z} = 0 \tag{3.15}$$

Now it is straightforward to see that the differential 2-form

$$\omega = \left\{(|f|^2 - 1)f_{z\bar z} - \frac{2\bar f|f|^2}{|f|^2+1} f_z f_{\bar z}\right\}|dz|^2 \tag{3.16}$$

is globally defined on M and so its vanishing is a necessary condition for $\tilde G$ to be the generalized Gauss map of $\tilde\theta$.

Remark: Through the identification of Q_1 with $\hat{\mathbb{C}}$, the existence of ω and α is independent from $\tilde\theta$. ω is defined on all M and α at all points of M where $|f|^2 \neq 1$.

Then assuming the existence of $\tilde\theta$ we have that α is globally defined and so that $\beta_1 = (1-f^2)\alpha$, $\beta_2 = i(1+f^2)\alpha$ and $\beta_3 = 2f\alpha$ are differential 1-forms globally defined on M.

Let us now state the results of this section.

Theorem 1. Let M be a connected Riemann surface and let $\tilde G: M \to Q_1$ be smooth. Then the following conditions are necessary for $\tilde G$ to be the generalized Gauss map of a minimal conformal immersion of M into H^3.

1) $\omega \equiv 0$,
2) α is globally defined.

We then have

$$(3.17)\qquad \begin{cases} t = \exp\left(2\int_M \operatorname{Re}\beta_3\right) \\ x^2 = 2\int_M t \operatorname{Re}\beta_2 \\ x^1 = 2\int_M t \operatorname{Re}\beta_1 \end{cases}$$

Proof. The conditions 1) and 2) were already verified and the formulas in (3.17) are consequence of (3.14).

Theorem 2. Let M be a connected Riemann surface. If $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are two minimal conformal immersions of M into H^3 having the same generalized Gauss map, then there is an isometry $T: H^3 \to H^3$ such that $T\circ\tilde{\theta}_1 = \tilde{\theta}_2$.

Proof. From Hoffman and Osserman [H-O,2] there is $c \in \mathbb{R}$, $c \neq 0$ and a vector d such that $\theta_2 = c\theta_1 + d$. If $\theta_1 = (x^1, x^2, t)$, $\theta_2 = (y^1, y^2, s)$ and $d = (d^1, d^2, d^3)$ we have $s = ct + d^3$. Then $s_z = ct_z$. But from (3.14) we obtain $\frac{t_z}{t} = \frac{2f}{|f|^4 - 1}\bar{f}_z = \frac{s_z}{s}$. Then $\frac{s_z}{s} = \frac{ct_z}{ct+d^3} = \frac{t_z}{t}$ and so $d^3 = 0$. Now it is straightforward to see that $T(x^1, x^2, t) = c(x^1, x^2, t) + (d^1, d^2, 0)$ is an isometry of H^3.

Theorem 3. Let M be a simply connected, non compact and non parabolic Riemann surface and let $\tilde{G}: M \to Q_1$ be smooth. If $f: M \to \hat{\mathbb{C}}$ is the expression of $\tilde{G}$, obtained through the natural identification of Q_1 with $\hat{\mathbb{C}}$ and $|f|^2 \neq 1$ on M, the differential forms α and ω are globally defined and the condition $\omega \equiv 0$ implies the existence of a minimal conformal immersion $\tilde{\theta}$ of M into H^3 that has $\tilde{G}$ as its generalized Gauss map.

Remarks. 1) There is no minimal conformal immersion of Riemann surfaces that are either compact or have parabolic type in H^n. [G-S].

2) The condition $|f|^2 \neq 1$ implies the following

(a) the image of the classical Gauss map of θ is contained in one of the hemisphere of the unitary euclidean sphere;

(b) the mean curvature vector of θ never vanishes;

(c) $f_{\bar{z}}$ never vanishes and so f is nowhere conformal.

Proof. $|f|^2 \neq 1$ implies the global existence of α and $\omega \equiv 0$ is the integrability condition of the system (3.14). This gives the local existence of $\tilde{\theta}$. The global existence follows from theorem 2 and the simply-connectedness of M, through a standard monodromy argument.

Theorem 4. Let M be a Riemann surface and $\tilde{\theta}: M \to H^3$ be a minimal conformal immersion. If $f: M \to \hat{\mathbb{C}}$ is the generalized Gauss map (through the identification of Q_1 with $\hat{\mathbb{C}}$) then the quadratic differential form $\gamma = \frac{f_z\bar{f}_z}{|f|^4 - 1}dz^2$ is globally defined and holomorphic on M.

Proof. (3.12) implies that γ is globally defined. The condition $\omega \equiv 0$ implies that $\left(\frac{f_z\bar{f}_z}{|f|^4-1}\right)_{\bar{z}} = 0$.

Corollary. If $\tilde{\theta}$ is non totally geodesic its umbilic points are isolated.

Proof. $p \in M$ is umbilic for $\tilde{\theta}$ if, and only if, it is umbilic for θ, that is if, and only if, $f_z(p) = 0$ [K]. Then p is umbilic for $\tilde{\theta}$ if p is a zero of γ; but the zeroes of γ are isolated points.

4. The case $n = 4$

In this case, the hyperquadric Q_2 is biholomorphically identified with $\hat{\mathbb{C}}\times\hat{\mathbb{C}}$ through the correspondence given by

$$\xi(\omega^1,\omega^1) = [1+\omega^1\omega^2, i(1-\omega^1\omega^2), \omega^1-\omega^2, -i(\omega^1+\omega^2)],$$

$$\xi(\infty,\omega^2) = [\omega^2, i\omega^2, 1, -i], \quad \xi(\omega^1,\infty) = [\omega^1, -i\omega^1, -1, -i],$$

$$\xi(\infty,\infty) = [1,-i,0,0], \quad \xi^{-1}([z^1,z^2,z^3,z^4]) = \left(\frac{z^3-iz^4}{z^1-iz^2}, \frac{-z^3+iz^4}{z^1-iz^2}\right)$$

where $(\omega^1,\omega^2)\in\hat{\mathbb{C}}\times\hat{\mathbb{C}}$ and z^1,z^2,z^3,z^4 are homogeneous coordinates on $Q_2\subset \mathbb{C}P^3$. Taking on $\hat{\mathbb{C}}\times\hat{\mathbb{C}}$ the metric $ds^2 = \dfrac{2|d\omega^1|^2}{(1+|\omega^1|^2)^2} + \dfrac{2|d\omega^2|^2}{(1+|\omega^2|^2)^2}$ and on Q_2 the metric induced by the Fubini-Study's metric on $\mathbb{C}P^3$, ξ becomes an isometry. Then the generalized Gauss map decomposes naturally into a pair of functions $(f_1,f_2)\colon M\to\hat{\mathbb{C}}\times\hat{\mathbb{C}}$. Moreover, if $z = u+iv \in U\subset\mathbb{C}$ are isothermal parameters on M, we have $\tilde{G}(z) = [\Phi(z)]$ where

$$(4.1)\qquad \Phi(z) = (1+f_1f_2, i(1-f_1f_2), f_1-f_2, -i(f_1+f_2)).$$

If we set

$$(4.2)\qquad A(z) = (f_2-\bar f_1, -i(f_2+\bar f_1), 1+\bar f_1 f_2, -i(1-\bar f_1 f_2))$$

then

$$(4.3)\qquad \langle\langle\Phi(z), A(z)\rangle\rangle = 0;$$

$$(4.4)\qquad |\Phi|^2 = |A|^2 = 2(1+|f_1|^2)(1+|f_2|^2).$$

Moreover

$$(4.5)\qquad e_1 = \frac{\sqrt{2}\,\mathrm{Re}\,\Phi}{|\Phi|},\ e_2 = \frac{\sqrt{2}\,\mathrm{Im}\,\Phi}{|\Phi|},\ e_3 = \frac{\sqrt{2}\,\mathrm{Re}\,\bar A}{|A|},\ e_4 = \frac{\sqrt{2}\,\mathrm{Im}\,\bar A}{|A|},$$

is an orthonormal frame of $\mathbb{R}^4$ adapted to the immersion θ. Then, from (2.4), (4.1) and (4.5), we obtain

$$(4.6)\qquad \eta = \bar f_1 F_1 + \bar f_2 F_2,$$

$$(4.7)\qquad V = (F_1-F_2)\frac{|A|}{\sqrt{2}}\, e - i(F_1+F_2)\frac{|A|}{\sqrt{2}}\, e_4,$$

where $F_1 = \dfrac{(f_1)_{\bar z}}{1+|f_1|^2}$ and $F_2\ \dfrac{(f_2)_{\bar z}}{1+|f_2|^2}$.

From (2.5), (4.7) and $H = h_1e_3 + h_2e_4$ we obtain

$$(4.8)\qquad h_1 = \frac{F_1-F_2}{2\bar\psi\sqrt{(1+|f_1|^2)(1+|f_2|^2)}},\quad h_2 = \frac{-i(F_1+F_2)}{2\bar\psi\sqrt{(1+|f_1|^2)(1+|f_2|^2)}}.$$

Then

$$(4.9)\qquad H = 0 \iff (f_1)_{\bar z} = (f_2)_{\bar z} = 0$$

But $\tilde H = t^2H + t(U_4)^N$, where $U_4 = (0,0,0,1)$ and $(\ )^N$ stands for the orthogonal projection over the normal fibre bundle of $\tilde\theta$. Then assuming that $\tilde\theta$ is minimal we have

$$(4.10)\qquad H = -\frac{(U_4)^N}{t},\quad \text{[G-S]}.$$

This together with (4.5) imply

$$(4.11)\qquad h_1 = \frac{\mathrm{Im}(1-f_1\bar f_2)}{t\sqrt{(1+|f_1|^2)(1+|f_2|^2)}},\quad h_2 = \frac{-\mathrm{Re}(1-f_1\bar f_2)}{t\sqrt{(1+|f_1|^2)(1+|f_2|^2)}}$$

Therefore

$$(4.12)\qquad (f_1)_{\bar z} = (f_2)_{\bar z} = 0 \iff H = 0 \iff 1 - f_1\bar f_2 = 0.$$

From the expressions of h_1 and h_2 given by (4.11) and (4.8) we obtain

$$(4.13)\qquad \bar\psi = \frac{itF_1}{1-f_1\bar f_2} = \frac{it(f_1)_{\bar z}}{(1+|f_1|^2)(1-f_1\bar f_2)};$$

$$F_1(1-\bar f_1 f_2) = F_2(1-f_1\bar f_2). \tag{4.14}$$

Since ψ is defined at all points of U, the ratios $\dfrac{(f_1)_{\bar z}}{1-f_1\bar f_2}$ and $\dfrac{(f_2)_{\bar z}}{1-\bar f_1 f_2}$ may be extended to all U.

Then, the differential 1-forms

$$\alpha_1 = \frac{(\bar f_1)_z}{(1-\bar f_1 f_2)(1+|f_1|^2)}\,dz, \quad \alpha_2 = \frac{(\bar f_2)_z}{(1-f_1\bar f_2)(1+|f_2|^2)}\,dz, \tag{4.15}$$

are globally defined on M.

From (2.2), (4.13) and (4.14) we obtain

$$\theta_z = \frac{-it}{1-\bar f_1 f_2}\,\frac{(\bar f_1)_z}{1-|f_1|^2}\,(1+f_1f_2,\, i(1-f_1f_2),\, f_1-f_2,\, -i(f_1+f_2)), \tag{4.16}$$

$$\begin{cases} (f_1\bar f_2-1)(f_1)_{z\bar z} - \dfrac{2|f_1|^2\bar f_2}{1+|f_1|^2}(f_1)_z(f_1)_{\bar z} + \dfrac{\bar f_1-\bar f_2}{1+|f_1|^2}(f_1)_z(f_1)_{\bar z} = 0 \\[2ex] (\bar f_1 f_2-1)(f_2)_{z\bar z} - \dfrac{2|f_2|^2\bar f_1}{1+|f_2|^2}(f_2)_z(f_2)_{\bar z} + \dfrac{\bar f_2-\bar f_1}{1+|f_2|^2}(f_2)_z(f_2)_{\bar z} = 0. \end{cases} \tag{4.17}$$

If we set

$$\begin{cases} \omega_1 = \left[(f_1\bar f_2-1)(f_1)_{z\bar z} - \dfrac{2|f_1|^2\bar f_2}{1+|f_1|^2}(f_1)_z(f_1)_{\bar z} + \dfrac{\bar f_1-\bar f_2}{1+|f_1|^2}(f_1)_z(f_1)_{\bar z}\right]|dz|^2 \\[2ex] \omega_2 = \left[(\bar f_1 f_2-1)(f_2)_{z\bar z} - \dfrac{2|f_2|^2\bar f_1}{1+|f_2|^2}(f_2)_z(f_2)_{\bar z} + \dfrac{\bar f_2-\bar f_1}{1+|f_2|^2}(f_2)_z(f_2)_{\bar z}\right]|dz|^2 \end{cases} \tag{4.18}$$

we have two differential 2-forms globally defined on M.

The existence of the minimal conformal immersion $\tilde\theta$ having $\tilde G$ as its generalized Gauss map implies that $\alpha_1, \alpha_2, \omega_1$ and ω_2 are globally defined and $\alpha_1 \equiv \alpha_2$ and $\omega_1 \equiv \omega_2 \equiv 0$. Then the differential 1-forms $\beta_1 = -i(1+f_1f_2)\alpha_1$, $\beta_2 = (1-f_1f_2)\alpha_1$, $\beta_3 = -i(f_1-f_2)\alpha_1$ and $\beta_4 = -(f_1+f_2)\alpha_1$, are globally defined on M.

Let us now state the results of this section.

Theorem 5. Let M be a connected Riemann surface and let $\tilde G: M \to Q_2$ be smooth. Then the following conditions are necessary for the existence of a minimal conformal immersion $\tilde\theta = (x^1, x^2, x^3, t)$ of M into H^4:

1) α_1 and α_2 are globally defined and $\alpha_1 \equiv \alpha_2$,
2) $\omega_1 \equiv \omega_2 \equiv 0$.

Moreover, if these conditions are satisfied, we have

$$\begin{cases} t = \exp\left[2\displaystyle\int_M \operatorname{Re}\beta_4\right] \\ x^3 = 2\displaystyle\int_M t\operatorname{Re}\beta_3 \\ x^2 = 2\displaystyle\int_M t\operatorname{Re}\beta_2 \\ x^1 = 2\displaystyle\int_M t\operatorname{Re}\beta_1 \end{cases} \tag{4.19}$$

Proof. It remains only to prove the formulas in (4.19), but they follow from (4.16).

Theorem 6. Let M be a Riemann surface and let $\tilde\theta_1, \tilde\theta_2$ be minimal conformal immersions of M into H^4. It $\tilde\theta_1$ and $\tilde\theta_2$ have the same generalized Gauss map, then there is an isometry $T: H^4 \to H^4$ such that $T\circ\tilde\theta_1 = \tilde\theta_2$.

Proof. It is similar to that one of theorem 2.

Theorem 7. Let M be simply connected, non compact and non parabolic Riemann surface and let $\tilde G: M \to Q_2$ be smooth.

If $f_1, f_2: M \to \hat{\mathbb{C}}$ are the components of $\tilde G$ obtained through the natural identification of Q_2 with $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, and $1-f_1\bar f_2 \neq 0$, the differential forms $\alpha_1, \alpha_2, \omega_1$ and ω_2 are globally defined and the conditions $\alpha_1 \equiv \alpha_2$ and $\omega_1 \equiv \omega_2 \equiv 0$ imply the existence of a minimal conformal immersion $\tilde{\theta}$ of M into H^4 that has $\tilde{G}$ as its generalized Gauss map.

Remark. The hypothesis imply that the mean curvature vector of θ never vanishes and that $(f_1)_{\bar{z}}$ and $(f_2)_{\bar{z}}$ never vanish.

Proof. $1-f_1\bar{f}_2 \neq 0$ implies the global existence of α_1 and α_2. Besides, the conditions $\alpha_1 \equiv \alpha_2$, $\omega_1 \equiv \omega_2 \equiv 0$ are the integrability conditions of (4.16). Therefore $\tilde{\theta}$ exists locally. The global existence of $\tilde{\theta}$ follows from theorem 6 by a straightforward monodromy argument.

Theorem 8. Let M be a connected Riemann surface, $\tilde{\theta}$ be a minimal conformal immersion of M into H^4 and $f_1, f_2 : M \to \hat{\mathbb{C}}$ the components of its generalized Gauss map, obtained through the natural identification of Q_2 with $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Then the quadratic differential form

$$\gamma = \frac{(f_1)_z (f_2)_z (\bar{f}_1)_z (\bar{f}_2)_z}{(1-\bar{f}_1 f_2)(1-f_1\bar{f}_2)(1+|f_1|^2)(1+|f_2|^2)}\, dz^2$$

is globally defined and holomorphic on M.

Proof. The global existence follows from (4.13) and the conditions $\alpha_1 \equiv \alpha_2$ and $\omega_1 \equiv \omega_2 \equiv 0$ imply that γ is holomorphic.

Corollary. If $\tilde{\theta}$ is non totally geodesic the umbilic points and the umbilic directions of $\tilde{\theta}$ are isolated.

5. Final remarks

1. If in (3.1) we consider the stereographic projection relative to the south pole of the unitary euclidean sphere, the identification of Q_1 with $\hat{\mathbb{C}}$ is made through the correspondence $\xi(\omega) = [\omega^2-1, i(\omega^2+1), 2\omega]$, $\xi(\infty) = [1, i, 0]$, $\xi^{-1}([z^1, z^2, z^3]) = \frac{z^1 - iz^2}{z^3}$, where $\omega \in \hat{\mathbb{C}}$ and z^1, z^2, z^3 are homogeneous coordinates on $Q_1 \subset \mathbb{C}P^2$. Then instead of (3.14) we have

$$\theta_z = \frac{t\bar{f}_z}{|f|^4 - 1}\,(f^2 - 1,\ i(f^2 + 1),\ 2f)$$

and the conclusions are similar.

2. If we set $f_1 \equiv f_2 \equiv if$, on the case $n = 4$, we recapture the case $n = 3$ on the totally geodesic submanifold

$$x^3 = 0 \quad \text{of} \quad H^4.$$

References

[G-S] C. Góes and P. Simões. *Some remarks on minimal immersions in the hyperbolic space*. Bol. Soc. Bras. Mat., vol. 16 nº 2 (1985), 55-65.

[H-O,1] D.A. Hoffman and R. Osserman. *The Geometry of the generalized Gauss map*. AMS Memoirs 236 (1980).

[H-O,2] ————————. *The Gauss map of surfaces in* $\mathbb{R}^n$. J. of Diff. Geometry 18 (1983) 733-794.

[K] K. Kenmotsu. *Weierstrass formula for surfaces of prescribed mean curvature*. Math. Ann. 245 (1979) 89-99.

Instituto de Matemática e Estatística
Universidade de São Paulo
05.508 - São Paulo-SP