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PRINCIPAL-CURVATURE-PRESERVING ISOMETRIES OF SURFACES IN ORDINARY SPACE

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1. Introduction and Results. In this paper we determine all surfaces of revolution and all flat surfaces in $\mathbb{R}^3$ with nonconstant mean curvature that admit a one-parameter family of geometrically distinct nontrivial isometries which preserve both principal curvatures. — An isometry of a surface is nontrivial if it does not extend to an isometry of the whole space. Also two isometries are geometrically distinct if one is not the composition of the other followed by a space-isometry.

The surfaces which admit principal-curvature-preserving isometries have been studied since the time of O. Bonnet [1]. He first showed that all surfaces with constant mean curvature (other than planes and spheres) can be isometrically deformed under preservation of the mean curvature (equivalent to preservation of both principal curvatures). The surfaces of nonconstant mean curvature have been studied by many mathematicians, especially by É. Cartan [2]. Lately S. -S. Chern [3] has given an interesting criterion for their existence. In these papers, several far-reaching general results have been proven, but the differential equations describing these surfaces were not integrated and no direct geometric description of them was available. This is why my interest was motivated in determining explicit examples of surfaces of this type.

We consider a surface of revolution in $\mathbb{R}^3$

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$$X(z,\theta) = \left[r(z)\cos\theta,\ r(z)\sin\theta,\ z\right] \tag{1}$$

where $x = r(z)$ is a function in the x-z plane and $0 \leq \theta \leq 2\pi$. The principal curvatures are known to be:

$$a = k_{\text{meridian}} = \frac{-r''}{[(r')^2 + 1]^{3/2}}, \tag{2}$$

$$c = k_{\text{parallel}} = \frac{1}{r[(r')^2 + 1]^{1/2}}. \tag{3}$$

Away from umbilic points (i.e., $a = c$) we have:

Theorem 1. The surfaces of revolution with nonconstant mean curvature that admit a one-parameter family of geometrically distinct nontrivial isometries which preserve both principal curvatures are exactly those for which the function $x = r(z)$ satisfies the following fourth order ordinary differential equation:

$$\left[\frac{(a+c)'r}{(a-c)\sqrt{(r')^2+1}}\right]' = \left[\frac{(a+c)'}{a-c}\right]^2 \frac{r}{\sqrt{(r')^2+1}} \tag{4}$$

where a, c are given by (2) and (3).

The flat surfaces (i.e., of zero Gaussian curvature) in $\mathbb{R}^3$ with no umbilic points are determined to be pieces of generalized cylinders or cones and pieces of tangential developables. To state and prove the next theorem we need to express these surfaces in their principal parametrization.

A generalized cylinder can be expressed as

$$f(s,z) = C(s) + z\vec{k} \tag{5}$$

where $C(s)$ is a plane curve parametrized by its arclength and lying in a plane with unit normal $\vec{k}$. One principal curvature is zero and the other one is $\pm\kappa(s)$ where $\kappa(s)$ is the plane curvature of $C(s)$ and + or - depends on the orientation.

A generalized cone with vertex at the origin may be given by

$$g(\ell,\lambda) = \lambda e(\ell) \tag{6}$$

where $e(\ell)$ is the curve formed as the intersection of the cone and parametrized by its arclength ℓ, and $\lambda > 0$. One principal curvature is zero and the other one is $\pm\frac{k_g(\ell)}{\lambda}$ where $k_g(\ell)$ is the geodesic curvature of $e(\ell)$ with respect to the unit sphere and again + or - depends on the orientation.

Finally, a tangential developable can be expressed as

$$h(s,t) = C(s) + te_1(s), \quad t > 0 \quad (\text{or } t < 0) \tag{7}$$

where $C(s)$ is a curve in $\mathbb{R}^3$, parametrized by its arclength s and $e_1(s) = \dot{C}(s)$. One principal curvature is zero and the other one is $\pm\frac{\tau(s)}{t\kappa(s)}$ where $\kappa(s)$, $\tau(s)$ are the curvature and torsion of $C(s)$ respectively. Again + or - depends on the orientation. Now we have:

Theorem 2. The flat surfaces in $\mathbb{R}^3$ with nonconstant mean curvature that admit a one-parameter family of geometrically distinct nontrivial isometries which preserve both principal curvatures are exactly the following:

1) Generalized cylinders whose plane basis curves are the logarithmic spirals in polar coordinates (r, θ) expressed by

$$r(\theta) = \frac{1}{\sqrt{\rho^2+1}} e^{\rho\theta}$$

$$\rho \neq 0 \quad \text{constant}, \quad -\infty < \theta < \infty .$$

2) Generalized cones whose intersection with the unit sphere centered at their vertex, as a curve on the unit sphere has geodesic curvature

$$k_g(\ell) = \bar{\rho} \csc \ell$$

$$\bar{\rho} \neq 0 \quad \text{constant}, \quad 0 < \theta < \pi .$$

When $\rho\bar{\rho} = 1$, there is a one-parameter family of nontrivial isometries between the corresponding cylinder and cone, which preserve the principal curvatures at the corresponding points. This family is given by:

$$\ell = \ell(s,z) = \arc\sin \frac{2}{\sqrt{s^2+(z+\varepsilon)^2}} \tag{8}$$

$$\lambda = \lambda(s,z) = \sqrt{s^2+(z+\varepsilon)^2}$$

where ε is the parameter.

After a first version of this paper was written, Prof. K. Kenmotsu kindly informed me about his joint work with Prof. A.G. Colares on this same topic [4]. They have given some equivalence conditions for the existence of the surfaces and they have generalized Theorem 2 in that a surface of the above type with constant Gaussian curvature must be flat. I wish to express my thanks to Prof. Kenmotsu for sending me his preprint, in which my work has been quoted.

2. The proofs of the Theorems

For the proofs of these theorems we use the existence criterion given in [3]. This is described as follows: For a surface in $\mathbb{R}$ without umbilic points and of onconstant mean curvature we define the 1-forms:

$$\alpha_1 = u\omega_1 - v\omega_2, \quad \alpha_2 = v\omega_1 + u\omega_2 \tag{9}$$

where $\{\omega_1, \omega_2\}$ is the principal coframe of the surface and u, v are defined by

$$d(a+c) = (a-c)\ (u\omega_1 + v\omega_2) \tag{10}$$

with a, c the principal curvatures. Then, necessary and sufficient conditions for a surface as above to admit a nontrivial family of geometrically distinct isometries which preserve both principal curvatures are:

$$d\alpha_1 = 0, \quad d\alpha_2 = \alpha_1 \wedge \alpha_2 .$$

Proof of Theorem 1. For a surface of revolution given by (1) we have that the principal coframe is:

$$\omega_1 = \sqrt{(r')^2+1}\ dz, \quad \omega_2 = r\, d\theta .$$

Then, by (2), (3) and (10) we get

$$u = \frac{(a+c)'}{(a-c)\sqrt{(r')^2+1}}, \quad v = 0 .$$

So, by (5) we have

$$\alpha_1 = \frac{(a+c)'}{(a-c)}\ dz, \quad \alpha_2 = \frac{(a+c)'}{(a-c)\sqrt{(r')^2+1}}\ d\theta .$$

We observe that $d\alpha_1 = 0$ always, and the condition $d\alpha_2 = \alpha_1 \wedge \alpha_2$ is equivalent to ordinary differential equation (4).

Proof of Theorem 2. For a cylinder given by (5) we have principal coframe

$$\omega_2 = ds, \quad \omega_2 = dz.$$

With $a = \kappa(s)$ (the plane curvature of $C(s)$) and $c = 0$ we easily get by (10)

$$\alpha_1 = \frac{\kappa'(s)}{\kappa(s)}\, ds, \quad \alpha_2 = \frac{\kappa'(s)}{\kappa(s)}\, dz.$$

We observe that $d\alpha_1 = 0$ and $d\alpha_2 = \alpha_1 \wedge \alpha_2$ give the ordinary differential equation $\left(\frac{\kappa'}{\kappa}\right)' = \left(\frac{\kappa'}{\kappa}\right)^2$. The solutions of this equation are

$$\kappa(s) = \frac{1}{\rho s + \sigma}$$

where ρ, σ are arbitrary constants. For cylinders of nonconstant mean curvature $\rho \neq 0$.

Now we can determine $C(s)$ by $\kappa(s)$. Up to the isometries of the plane, $C(s)$ is the logarithmic spiral given in polar coordinates by

$$r(\theta) = \frac{1}{\sqrt{\rho^2+1}}\, e^{\rho\theta}, \quad \rho \neq 0 \text{ constant}, \ -\infty < \theta < \infty.$$

Next, for a cone given by (6) we have principal coframe

$$\omega_1 = \lambda\, d\ell, \quad \omega_2 = d\lambda.$$

With $a = \frac{k_g(\ell)}{\lambda}$ and $c = 0$ we get

$$\alpha_1 = \frac{k_g'(\ell)}{k_g(\ell)}\, d\ell + \frac{1}{\lambda}\, d\lambda, \quad \alpha_2 = -d\ell + \frac{k_g'(\ell)}{\lambda k_g(\ell)}\, d\lambda.$$

Again $d\alpha_1 = 0$ and $d\alpha_2 = \alpha_1 \wedge \alpha_2$ gives

$$\left[\frac{k_g'}{k_g}\right]' = \left[\frac{k_g'}{k_g}\right]^2 + 1.$$

The solutions of this ordinary differential equation are

$$k_g(\ell) = \bar{\rho} \csc(\ell + \bar{\sigma})$$

where $\bar{\rho} \neq 0$, $\bar{\sigma}$ are arbitrary constants and $0 < \ell < \pi$. We may assume $\bar{\sigma} = 0$ since, by an isometry of the unit sphere we can eliminate it.

When $\rho\bar{\rho} = 1$ the formulae given by (8) determine isometries as claimed in the Theorem. The proof of this part is easy and left to the reader.

The proof of the Theorem will be complete by showing that a piece of tangential developable does not admit any nontrivial family of geometrically distinct isometries which preserve both principal curvatures. For a tangential developable given by (7) we have principal coframe

$$\omega_1 = ds + dt, \quad \omega_2 = t\kappa(s)ds.$$

With $a = 0$ and $c = \frac{\tau(s)}{t\kappa(s)}$ (assume $\frac{\tau(s)}{t} < 0$) we get

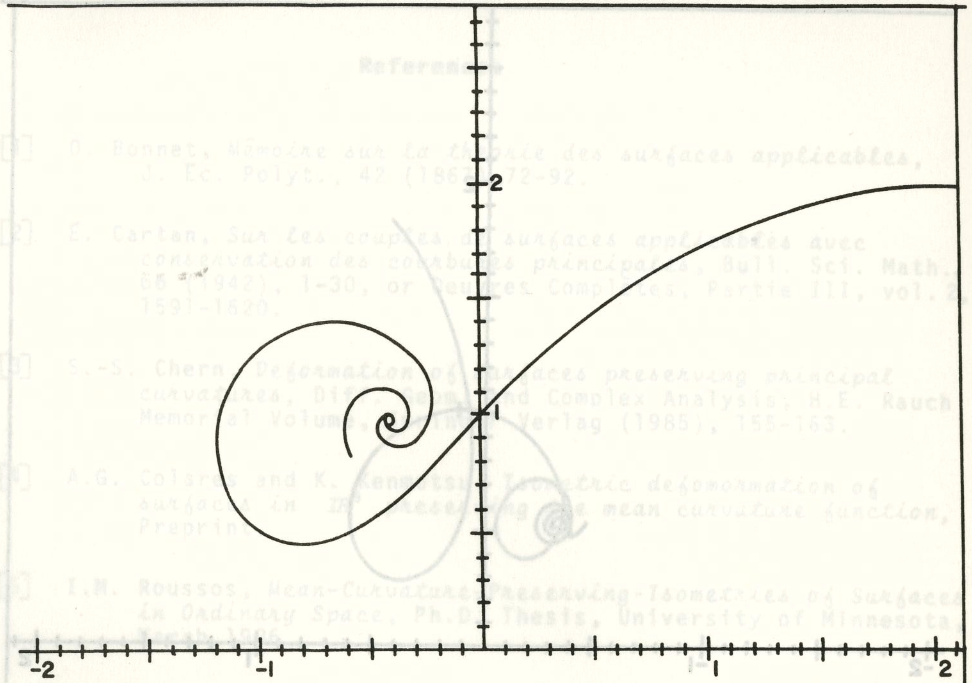
$$\alpha_1 = \left(\frac{2}{t} + \frac{\tau'}{\tau} - \frac{\kappa'}{\kappa}\right) ds + \frac{1}{t}\, dt,$$

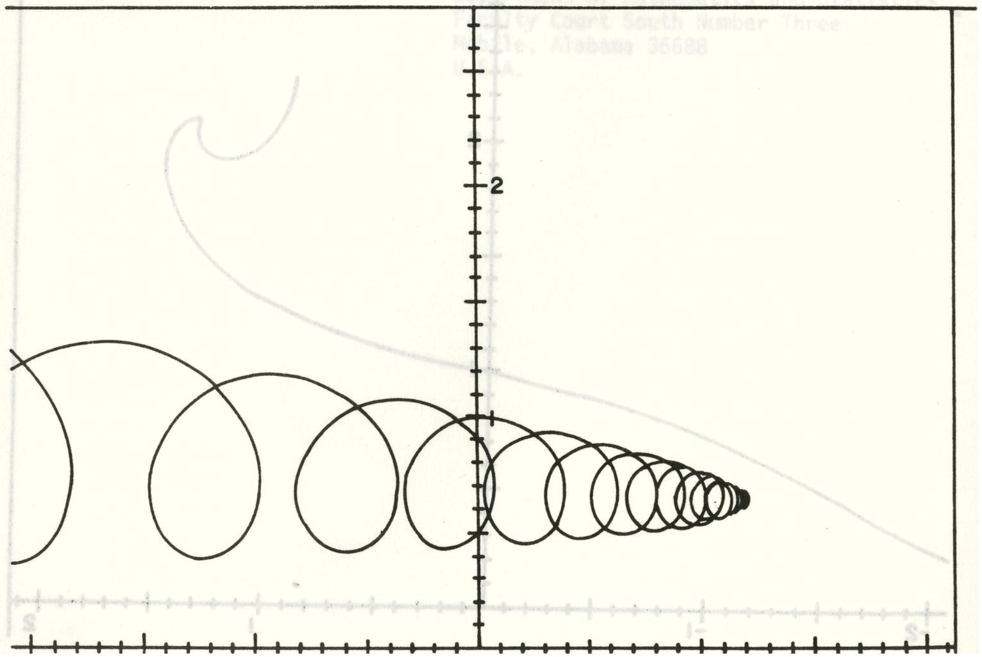
$$\alpha_2 = \left(\frac{\kappa'}{t\kappa^2} - \frac{\tau'}{t\kappa\tau} - \frac{1}{t^2\kappa} + \kappa\right) ds + \left(\frac{\kappa'}{t\kappa^2} - \frac{\tau'}{t\kappa\tau} - \frac{1}{t^2\kappa}\right) dt.$$

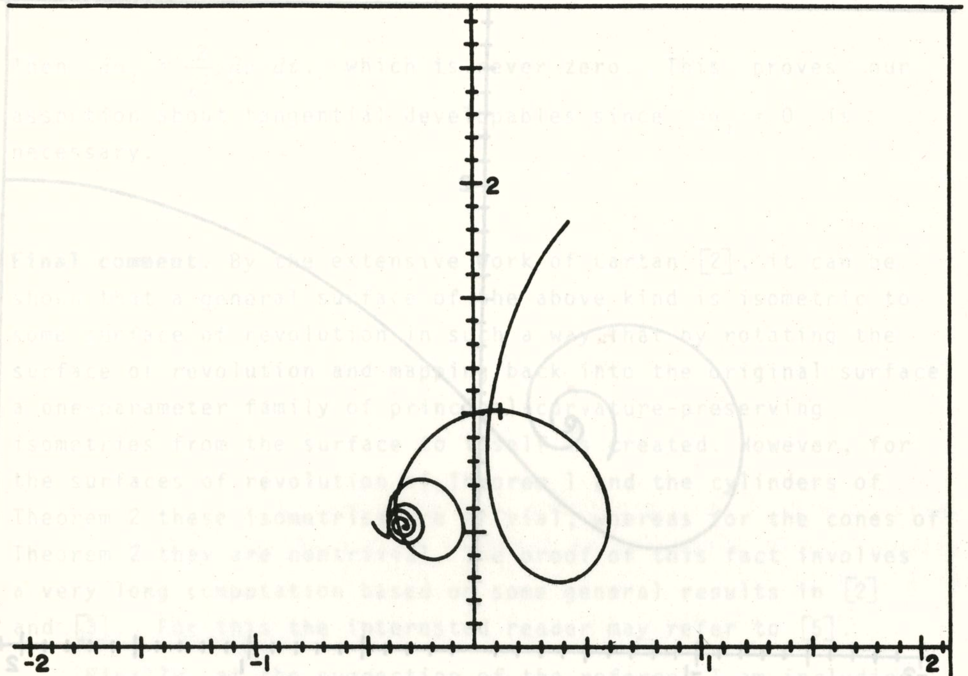
Then $d\alpha_1 = \frac{2}{t^2}\, ds \wedge dt$, which is never zero. This proves our assertion about tangential developables since $d\alpha_1 = 0$ is necessary.

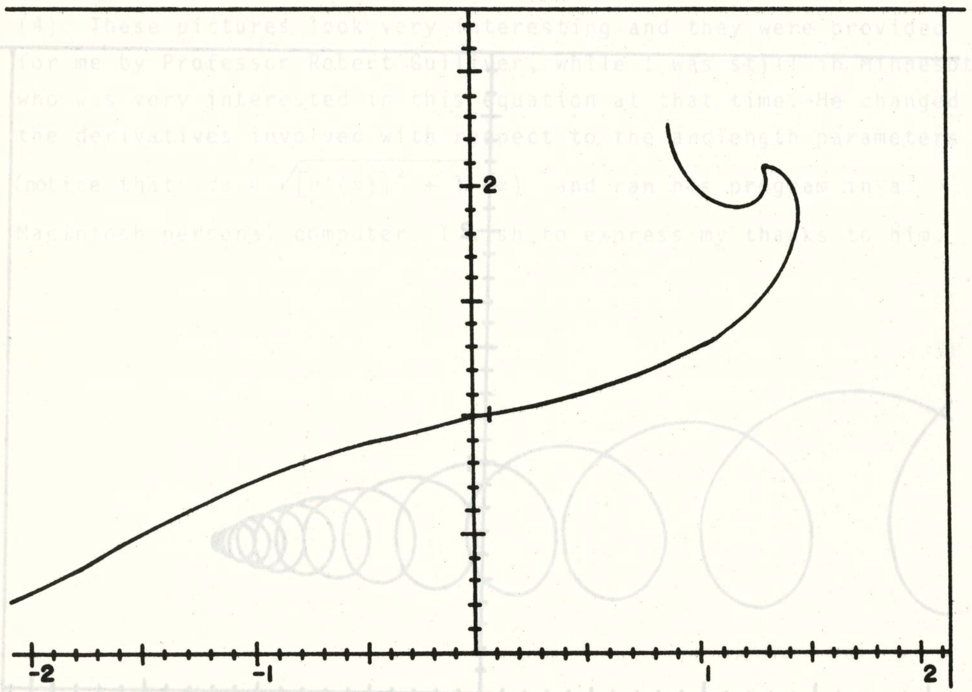
Final comment. By the extensive work of Cartan [2], it can be shown that a general surface of the above kind is isometric to some surface of revolution in such a way that by rotating the surface of revolution and mapping back into the original surface, a one-parameter family of principal-curvature-preserving isometries from the surface to itself is created. However, for the surfaces of revolution of Theorem 1 and the cylinders of Theorem 2 these isometries are trivial, whereas for the cones of Theorem 2 they are nontrivial. The proof of this fact involves a very long computation based on some general results in [2] and [3]. For this the interested reader may refer to [5].

Finally, at the suggestion of the referee, I am including some pictures of curves which satisfy the differential equation (4). These pictures look very interesting and they were provided for me by Professor Robert Gulliver, while I was still in Minnesota, who was very interested in this equation at that time. He changed the derivatives involved with respect to the arclength parameters s (notice that $ds = \sqrt{[r'(z)]^2 + 1}\; dz$) and ran his program in a Macintosh personal computer. I wish to express my thanks to him.









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