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ÍNDICE

J. L. ARRAUT and N. M. DOS SANTOS – Differentiable conjugation of actions of R^p 1

D. G. COSTA and A. S. OLIVEIRA – Existence of solutions for a class of semilinear elliptic problems at double resonance 21

HERMANO FRID NETO – A stability theorem for entropy solutions of initial value problems for first order quasilinear hyperbolic systems in several space variables 39

JEAN-MARC GAMBAUDO and CHARLES TRESSER – On the dynamics of quasi-contractions 61

F. LEDRAPPIER – Ergodic properties of Brownian motion on covers of compact negatively-curve manifolds 115

ORLANDO LOPES – Asymptotic behavior of solutions of a nonlinear wave equations 141

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DIFFERENTIABLE CONJUGATION OF ACTIONS OF R^p

J. L. ARRAUT AND N. M. DOS SANTOS

Introduction. First we consider non-singular C^r-actions, $r \geq 2$, of R^p on T^p. We prove that, up to linear reparametrization and conjugation by C^r-diffeomorphisms, there is only one action, see 1.6. Next, consider a 1-form $\omega = a_1 dx_1 + a_2 dx_2 + \ldots + a_p dx_p + dx_{p+1}$ on T^{p+1}, $p \geq 1$, with irrational coefficients and denote by F the p-dimensional foliation defined by ω. If the set $\{a_1, \ldots, a_p, 1\}$ is linearly independent over the rationals, then all leaves of F are dense hyperplanes, otherwise they are dense cylinders. We study non-singular C^r-actions, $r \geq 2$, of R^p on T^{p+1} with underlying foliation F. Here we prove that the number of these actions depends on the arithmetic nature of the a_j's. A vector $a = (a_1, \ldots, a_p)$ is *Liouville type* if there is an infinite sequence of integers k_i such that

$$\| k_i a \| < \frac{1}{|k_i|^i}$$

where

$$\| k_i a \| = \inf\{ |k_i a - \ell| ;\ \ell \in Z^p \}$$

If a is not Liouville type there exists a minimum integer $m \geq 2$ and a constant c such that

$$\| ka \| \geq \frac{c}{|k|^{m-1}}$$

for all $k \in Z$. In this case we say that a is *diophantine type* of order m. The 1-form ω is a *Liouville type* or a

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diophantine type form accordingly as a is Liouville or diophantine type, respectively.

2.4 If ω is a diophantine type form, then, up to linear reparametrization and differentiable conjugation, there is only one non-singular action of R^p on T^{p+1}, $p \geq 1$, with underlying foliation F. This generalizes to actions of R^p, a known result on linearization of vector fields [8, Ch. III].

The foliated cohomology, introduced by Reinhart in [6], appears naturally in the study of actions. See 2.5 for the definition of this cohomology and its relation with actions. Using the foliated cohomology we prove

2.9 If $\omega = a_1 dx_1 + \ldots + a_p dx_p + dx_{p+1}$ is a Liouville form then there are infinitely many C^∞ non-singular actions of R^p on T^{p+1}, $p \geq 1$, with underlying foliation F such that none of them is C^1 conjugated to a linear reparametrization of the other.

Denote by $H^1(T^{p+1}, F)$ the 1-cohomology group of the foliated manifold (T^{p+1}, F). Here we computed this group, namely:

2.10 $H^1(T^{p+1}, F) = R^p$ if ω is a diophantine type form. It has infinite dimension if ω is a Liouville type form. For $p = 1$, Heitsch in [3] computed this group in case ω is diophantine and Roger in [7] when ω is liouville. Conversations with G. Hector stimulated our interest in this questions, and we take this oportunity to thank him.

1. Generalities and Actions of R^p on T^p

Let M be a closed orientable connected m-dimensional manifold and F: $R^p \times M \to M$ a differentiable non-singular action of R^p on M. The orbit of $x \in M$ under F is the map $F_x: R^p \to M$ defined by $F_x(v) = F(v,x)$. The tangent fields to M given by

$$X_j(x) = DF_x(0)(e_j), \qquad 1 \leq j \leq p$$

are called the *infinitesimal generators* of F. F is said to be of class C^r if its infinitesimal generators are of class C^r. $X = \{X_1, \ldots, X_p\}$ is a commutative p-frame of the underlying foliation F of F. Any set of 1-forms $\xi = \{\xi_1, \ldots, \xi_p\}$ such that $\xi_i(X_j) = \delta_{ij}$ is called a *p-coframe adapted* to F. Let

1.1 $$I(F) = \{\omega \in \Lambda(M) : \omega \text{ annihilates } TF\}$$

$I(F)$ is a differential ideal i.e., $dI(F) \subset I(F)$, and $I(F)^{q+1} = 0$ where $q = m-p$. A form ω is called *F-closed* if $d\omega \in I(F)$. If ξ is a p-coframe adapted to F then every ξ_j is F-closed. It is interesting to observe that an action F determines uniquely its frame of infinitesimal generators X, but there are many coframes adapted to F. In fact, if ξ is one such coframe and $\omega_1, \ldots, \omega_p$ are 1-forms in $I(F)$, then $\xi' = \{\xi_1 + \omega_1, \ldots, \xi_p + \omega_p\}$ is another. Any p-coframe ξ adapted to F determines uniquely the frame X of infinitesimal generators. It is precisely this versatility of coframes adapted to a given action F that makes it more adequate to regard actions as forms rather than as a vector fields.

Let $A = [A_{ij}]$ be a element of $GL(p,R)$ and $v \in R^p$ thought as a column vector. If F is an action, then F_A given by $F_A(v,x) = F(Av,x)$ is an action too. The infinitesimal generators of F_A, in terms of those of F are

$$X'_j = A_{1j} X_1 + \ldots + A_{pj} X_p, \qquad 1 \leq j \leq p$$

or in matrix notation $X' = {}^tAX$. If ξ is a p-coframe adapted to F then $\xi' = A^{-1}\xi$ is a p-coframe adapted to F_A. Denote by $A^r(R^p, M)$ the set of non-singular C^r-actions of R^p on M.

Two actions $F, G \in A^r(R^p,M)$ are said to be *linearly equivalent* if $G = F_A$ for some $A \in GL(p,R)$. The class of F under this relation will be written $\underline{F}$ and the set of all such classes by $\underline{A}^r(R^p,M)$. Let $\xi = \{\xi_1,\dots,\xi_p\}$ be a coframe adapted to an action F and V_ξ the real vector subspace of $\Lambda^1(M)$ generated by the elements of ξ. To every $F_A \in \underline{F}$ corresponds a base $\xi' = A^{-1}\xi$ of V_ξ and thus $\underline{F}$ is determined by any one of these vector spaces V_ξ.

To every $F \in A^r(R^p,M)$ is associated a symmetric $(q+1)$-linear mapping

1.2 $\alpha_F : R^p \times \dots \times R^p \to H^{2q+1}_{DR}(M)$, $q = m-p$

called the *characteristic mapping* of F. Take a coframe ξ adapted to F and consider the map from R^p into $\Lambda^1(M)$ given by $v = (v^1,\dots,v^p) \to v\cdot\xi = v^1\xi_1 + \dots + v^p\xi_p$ then

1.2' $\alpha_F(v_1,\dots,v_{q+1}) = [\eta_1 \wedge d\eta_2 \wedge \dots \wedge d\eta_{q+1}]$ where $\eta_j = v_j\cdot\xi$, $1 \le j \le q+1$. It is shown in [1, 1.4] that α_F does not depend on the adapted coframe taken.

1.3 As an example consider actions of R^p on T^p. Let $\partial/\partial x = \{\partial/\partial x_1,\dots,\partial/\partial x_p\}$ be the canonical frame of the tangent bundle $T(T^p)$ and $dx = \{dx_1,\dots,dx_p\}$ its dual coframe. The *canonical action* E of R^p on T^p is the one generated by $\partial/\partial x$. Identify $H_{DR}(T^p)$ with R^p by the isomorphism $[dx_j] \to e_j$, $1 \le j \le p$. Now, let $F \in A^r(R^p,T^p)$ and ξ its adapted coframe, the characteristic mapping $\alpha_F: R^p \to H_{DR}(T^p)$ is given by $\alpha_F(e_j) = [\xi_j]$, $1 \le j \le p$. In the standard base of R^p, α_F is represented by the matrix $[\alpha_{ij}]$, where $\alpha_{ij} = \frac{1}{2\pi}\int_{\sigma_i}\xi_j$, and ξ_i are generators of $\pi_1(T^p)$. In particular $\alpha_E = I$.

1.4 **Lemma**. Let F be a non-singular C^r-action, $r \ge 2$, of R^p on T^p, $p > 1$. Then the rows of the matrix α_F are a set of generators of the isotrophy group of F.

Proof. Denote the matrix α_F by A and let $G(v,x) = F({}^tAv,x)$. Straight computation gives $\alpha_G = \alpha_F \circ A^{-1} = A \circ A^{-1} = I$. First we show that the rows of the identity matrix are generators of the isotropy group of G. Let ξ be the coframe adapted to G. Since $\alpha_G = I$, one has $[\xi_j] = [dx_j]$, $1 \le j \le p$, and so the group of periods of each ξ_j is Z. This implies that the foliations F_j defined by the closed forms ξ_j are compact. As the one-dimensional foliations G^i given by the infinitesimal generators X_i of G, $1 \le i \le p$, are just $\bigcap_{j \ne i} F_j$, they are also compact. Fix a point $x \in T^p$ and let τ_i be the period of the orbit γ_i of X_i through x. γ_i is homotopic to $\sum_j n_{ij}\sigma_j$, where $\sigma_1,\dots,\sigma_p$ are the canonical generators of $\pi_1(T^p,x)$ and the n_{ij}'s are integers. Since $[\xi_k] = [dx_k]$, one gets

$$\frac{1}{2\pi}\int_{\gamma_i}\xi_k = \frac{1}{2\pi}\int_{\gamma_i} dx_k = \sum_j n_{ij}\frac{1}{2\pi}\int_{\sigma_j} dx_k = n_{ik}$$

and by direct computation one also gets $\int_{\gamma_i}\xi_k = \tau_i\delta_{ik}$, therefore γ_i is homotopic to $\tau_i\sigma_i$ and τ_i is an integer. Finally, since γ_i is the orbit of a vector field, it follows that $\tau_i = 1$, and so the vectors e_i, i.e. the rows of I, generate the isotropy group of x under G. Now, since $G(e_i,x) = F({}^tAe_i,x) = x$, the lemma follows.

1.5 Two actions F and G in $A^r(R^p,M)$, $r \ge 2$, are said to be C^r-*conjugated* if there exists $\phi \in \mathrm{Diff}^s(M)$, $s \ge 1$, such that $G(v,\phi(x)) = \phi \circ F(v,x)$ for all $v \in R^p$ and all $x \in M$. In this situation $\alpha_F = \phi^* \circ \alpha_G$ [1, 2.7]. Two actions $\underset{\sim}{F}$ and $\underset{\sim}{G}$ in $A^r(R^p,M)$ are said to be C^s-conjugated if there exists $F \in \underset{\sim}{F}$ and $G \in \underset{\sim}{G}$ which are C^s-conjugated. Let F be conjugated by ϕ to G, X and Y the p-frames of infinitesimal generators of F and G respectively, then $\phi_* X_j = Y_j$, $1 \le j \le p$. For every coframe $\eta = \{\eta_1,\ldots,\eta_p\}$ adadpted to G, $\phi^*\eta = \{\phi^*\eta_1,\ldots,\phi^*\eta_p\}$ is a coframe adapted to F, thus ϕ induces an isomorphism $\phi^*: V_\eta \to V_{\phi^*\eta}$.

Now we are going to show that in $A^r(R^p,T^p)$ there is only one class of C^r-conjugation, namely, the class of the canonical action E.

1.6 Theorem. If F is a C^r, $r \ge 2$, non-singular action of R^p on T^p, $p \ge 1$, then there exists $G \in \underset{\sim}{F}$ and $\phi \in \mathrm{Diff}^r(T^p)$ homotopic to the identity which conjugates E with G.

Proof. Let G be as in the proof of 1.4, so that $\alpha_G = I$. By 1.4 the vectors $e_1,\ldots,e_p$ of R^p are generators of the isotropy group of G and so the map $G_x : R^p \to T^p$ induces a C^r-diffeomorphism $\phi: T^p \to T^p$ such that $\phi_* \partial/\partial x_j = X_j$, $1 \le j \le p$, where X is the frame of infinitesimal gererators of G. This implies that ϕ conjugates E with G. Let ξ be the closed coframe adapted to G. Since $\phi^*\xi_j = dx_j$ and $[\xi_j] = [dx_j]$, $1 \le j \le p$, it follows that ϕ induces the identity in $H^1_{DR}(T^p)$ and therefore ϕ is homotopic to the identity.

2. Actions of R^p on T^{p+1}, $p \ge 1$, with Linear and Dense underlying Foliations

Let $\omega = a_1 dx_1 + \ldots + a_p dx_p + dx_{p+1}$ be a 1-form on T^{p+1} with $a_j \in R-Q$ for $1 \le j \le p$. Denote by $\mathcal{F} = \mathcal{F}(\omega)$ the codimension 1 foliation defined by ω. If the set $\{a_1,\ldots,a_p, 1\}$ is linearly independent over Q, then all leaves of $\mathcal{F}$ are dense hyperplanes. They are dense cylinders of the type $T^{p-k} \times R^k$ in case $k+1$ is the cardinality of the maximum linearly independent subset. The action $E = E(\omega)$ of R^p on T^{p+1} defined by the vector fields

2.1 $E_j = \partial/\partial x_j - a_j \partial/\partial x_{p+1}$, $1 \le j \le p$,

is called the *canonical action* with underlying foliation $\mathcal{F}$. Note that $\{dx_1,\ldots,dx_p\}$ is a coframe of $\mathcal{F}$ adapted to E for any such ω. Denote by $A^r(R^p,M,\mathcal{F})$ the set of all non-singular C^r-actions of R^p on M with the same underlying foliation $\mathcal{F}$. An element $F \in A^r(R^p,T^{p+1},\mathcal{F})$ is the same as a p-coframe ξ of $\mathcal{F}$ which is $\mathcal{F}$-closed i.e., such that $d\xi_j \wedge \omega = 0$, $1 \le j \le p$, [1, 1.3]. We will use for computations the frame $\{E_1,\ldots,E_p, \partial/\partial x_{p+1}\}$ and the coframe $\{dx_1,\ldots,dx_p,\omega\}$ of T^{p+1}. Let $\mu = f_1 dx_1 + \ldots + f_p dx_p + h\omega$ be any 1-form on T^{p+1}. Straight computation gives:

2.2 μ is $\mathcal{F}$-closed if and only if $E_i f_j - E_j f_i = 0$ for all $1 \le i < j \le p$.

It was proven in [1, 4.7] that for any F in $A^r(R^2,T^3,\mathcal{F})$, $r \ge 2$, α_F is the zero map; we remark as an aside that the proof can be generalized for $A^r(R^p,T^{p+1},\mathcal{F})$. Thus, the characteristic mapping does not distinguish among these actions.

It seems natural to ask how many conjugation classes exists in $A^r(R^p, T^{p+1}, F)$. It appears that the answer to this question depends on the arithmetic nature of the coefficients of ω.

A vector $a = (a_1,\dots,a_p)$ is said to be *Liouville type* if there is an infinite sequence of integers k_i such that

$$\|k_i a\| < \frac{1}{|k_i|^i}$$

where

$$\|k_i a\| = \inf\{|k_i a - \ell|;\ \ell \in Z^p\}.$$

If a is not Liouville type then there exists a minimum integer $m \geq 2$ and a constant $c > 0$ such that

$$\|k_1 a\| \geq \frac{c}{|k|^{m-1}}$$

for all integers k. In this case we say that a is *diophantine type of order* m. A 1-form $\omega = a_1 dx_1 + \dots + a_p dx_p + dx_{p+1}$ is said to be *diophantine type* or *Liouville type* accordingly as $a = (a_1,\dots,a_p)$ is diophantine type or Liouville type, respectively. The order of a diophantine type 1-form ω is the order of its diophantine type vector a. If at least one of the coefficients of ω is a diophantine number, then ω is a diophantine type form. Let α be a Liouville number [5] and $a_j = p_j(\alpha)$ where $p_j(x)$, $1 \leq j \leq p$ are polynomials with rational coefficients. Then ω is a Liouville type form.

2.3 Lemma. Let $\omega = a_1 dx_1 + \dots + a_p dx_p + dx_{p+1}$ be a diophantine form of order m on T^{p+1}, $p \geq 1$. For any C^{m+s}, $s > \left[\frac{p+1}{2}\right] + 1$, F-closed 1-form μ there exists a unique, up to additive constants, C^ℓ-function, $\ell = s - \left[\frac{p+1}{2}\right] - 1$, $h: T^{p+1} \to R$ such that $\mu + h\omega$ is closed.

Proof. To simplify, we give the proof in T^3 for $\omega = a\,dx_1 + b\,dx_1 + dx_2$. One can assume, without loss of generality, that $\mu = f\,dx_1 + g\,dx_2$. Straight calculation gives: $\mu + h\omega$ is closed if and only if $E_1 h = \partial f/\partial x_3$, $E_2 h = \partial g/\partial x_3$ and $E_1 g = E_2 f$. Since the last equation is equivalent to μ being F-closed, see 2.2, to prove the lemma it is enough to find an h such that:

2.3.1 $$E_1 h = \partial f/\partial x_3$$

2.3.2 $$E_2 h = \partial g/\partial x_3$$

Denote the lifting of objects from T^3 to R^3 with the same letters. Let $f = \sum_k f_k e^{2\pi i(k.x)}$ and $g = \sum_k g_k e^{2\pi i(k.x)}$ with $k = (k_1,k_2,k_3) \in Z^3$ and $(x_1,x_2,x_3) \in R^3$ be the Fourier series of f and g, respectively. Equations 2.3.1 are equivalent to the following collection of relations

2.3.1' $$(k_1 - ak_3)h_k = k_3 f_k$$

2.3.2' $$(k_2 - bk_3)h_k = k_3 g_k$$

Assume ω is a diophantine form of order m. Define $h_k = 0$ if $k_3 = 0$. As ω is diophantine, then for each k, $k_3 \neq 0$ one has either

2.3.4 $$|k_3 a - k_1| \geq \frac{1}{\sqrt{2}} \cdot \frac{c}{|k_3|^{m-1}}$$

or

2.3.5 $$|k_3 b - k_2| \geq \frac{1}{\sqrt{2}} \cdot \frac{c}{|k_3|^{m-1}}.$$

Equation $E_1 g = E_2 f$ is equivalent to the relations $(k_1 - ak_3)g_k = (k_2 - bk_3)f_k$ and from them it follows that if h_k satisfies one of the relations 2.3.1 or 2.3.2', then it also satisfies the other. Thus by 2.3.4. and 2.3.5., for each k, $k \neq 0$, we can find a solution h_k of 2.3.1' and 2.3.2' satisfying

$$|h_k| = \frac{|k_3|}{|k_1 - ak_3|}\,|f_k| \leq |k_3|^m |f_k|\sqrt{2}$$

Since f is a C^{m+s}-function, the h_k's are the Fourier coefficients of a function h of the Sobolev space H^s. By Sobolev lemma h is a C^ℓ-function, $\ell = s - \left[\frac{p+1}{2}\right] - 1$ and therefore the form $\mu + h\omega$ is closed.

2.4 Theorem. Let $\omega = a_1dx_1+\dots+a_p dx_p + dx_{p+1}$ be a diophantine form of order m and F the foliation of T^{p+1}, $p \geq 1$, defined by ω. Any action $F \in \underset{\sim}{A}^{m+s}(R^p, T^{p+1}, F)$, $s > \left[\frac{p+1}{2}\right] + 1$, is C^ℓ-conjugated, $\ell = s - \left[\frac{p+1}{2}\right] - 1$, to the canonical action $E(\omega)$ by a diffeomorphism homotopic to the identity.

Proof. Take an $F \in \underset{\sim}{F}$ and let ξ be a p-coframe adapted to F. Since the C^{m+s}-forms ξ_j, are F-closed, by 2.3 there exist C^ℓ-functions h_j such that the forms $\xi_j + h_j\omega$ are closed, $1 \leq j \leq p$. Thus, one can assume that the p-coframe ξ is closed. The set of forms $\{\xi_1,\dots,\xi_p,\omega\}$ is closed C^ℓ-coframe of T^{p+1} and define a non-singular C^ℓ-action $\bar{F}$ of R^{p+1}. By 1.6 there exists $A \in GL(p+1,R)$ and $\phi \in \mathrm{Diff}^\ell(T^{p+1})$, homotopic to the identity, which conjugates $\bar{F}_A$ with the canonical action giving

2.4.1 $\phi^*\eta_j = dx_j$, $1 \leq j \leq p+1$

where η is the coframe of T^{p+1} adapted to $\bar{F}_A$. Since $\bar{F}$ and $\bar{F}_A$ are linearly equivalent $\omega \in V_\eta$ and of course $\omega \in V_{dx}$, too. By 2.4.1, ϕ^* as an operator on forms restricts to an isomorphism of V_η onto V_{dx}, and as a map in $H_{DR}(T^{p+1})$ it is the identity, then necessarily $\phi^*\omega = \omega$, and thus leaves F invariant. Finally, the action G whose infinitesimal generators are ϕ_*E_j, $1 \leq j \leq p$, has $\{\eta_1,\dots,\eta_p\}$ as an adapted coframe. In fact $\eta_i(\phi_*E_j) = \phi^*\eta_i(E_j) = dx_i(\partial/\partial x_j - a_j\partial/\partial x_{p+1}) = \delta_{ij}$. Therefore $G \in \underset{\sim}{F}$ and by 2.4.1, ϕ conjugates $E(\omega)$ with G.

2.5 Next we prove that for Liouville type forms ω there are infinitely many C^1-conjugation classes in $A(R^p, T^{p+1}, F)$. For this purpose we use the cohomology of the foliated manifold (T^{p+1}, F).

From now on we regard the 1-forms coming from actions as sections of T^*F, the dual bundle of the subbundle TF of TM. Let F be a locally free action of R^p on a closed orientable connected m-dimensional manifold M and F its underlying foliation. A C^s *k-form along* F is a C^s section of the k-th exterior power of T^*F; we denote by $\Lambda^k_s(TF)$ the set of all C^s k-forms along F. The *exterior differential along* F,

$$d_F\colon \Lambda^k_s(TF) \longrightarrow \Lambda^{k+1}_{s-1}(TF), \quad s \geq 1$$

is defined by the standard formula

$$d_F\alpha(X_0,\dots,X_k) = \sum_{i=0}^{k} (-1)^i X_i\alpha(X_0,\dots,\hat{X}_i,\dots,X_k)$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)$$

where $X_0,\dots,X_k$ are smooth sections of TF. Thus the C^∞ forms along F, $\Lambda(TF)$, is a differential complex with respect to d_F. The cohomology $H^*(M,F)$ of this complex is called the *foliated cohomology* of (M,F) and is the cohomology of M with value in the sheaf of germs of differentiable functions on M locally constant on the leaves of F [9,I.5].

A *p-coframe* ξ *along* F is an ordered sequence $\xi = \{\xi_1,\dots,\xi_p\}$ of 1-forms along F giving a basis of T_x^*F at each point x of P. A coframe ξ is d_F-closed if $d_F\xi_i = 0$ for $i=1,\dots,p$. There is a one-to-one correspondence between locally free C^r-actions of R^p on M and d_F-closed C^r p-coframes along F and conversely. Two locally free C^r, $r \geq 1$, actions F and G with the same underlying foliation F are conjugated by a diffeomorphism ϕ preserving F if $\eta = \phi^*\xi$, where ξ and η are the coframe along F corresponding to F and G, respectively.

If F is the foliation on T^{p+1} defined by $\omega = a_1dx_1+\dots+a_pdx_p+dx_{p+1}$ then we denote by ε_j the restrictions of dx_j to TF, $1 \leq j \leq p$. Thus $\varepsilon = \{\varepsilon_1,\dots,\varepsilon_p\}$ is a d_F-closed coframe along F and the C^∞ action E corresponding to ε is called the *canonical action* of F.

2.6 Lemma. Let $\omega = a_1dx_1+\dots+a_pdx_p+dx_{p+1}$, with $\{1,a_1,\dots,a_p\}$ linearly independent over the rationals, be a Liouville type form and F the foliation defined by it. There exists an infinite sequence (μ_n) of d_F-closed C^∞ 1-forms along F such that

i) $\{\mu_n,\varepsilon_2,\dots,\varepsilon_p\}$ are p-coframes along F.

ii) given any two positive integers $i > j$, any constants b, $c_2,\dots,c_p$ and any translation τ of T^{p+1}, then the equation

$$\tau^*\mu_j + b\mu_i + c_2\varepsilon_2 + \dots + c_p\varepsilon_p = d_Fh$$

has no C^1 solution $h\colon T^{p+1} \to R$.

Proof. To simplify we give the proof in T^3. Let F be defined by $\omega = a_1dx_1 + a_2dx_2 + dx_3$. Since $a = (a_1,a_2)$ is a Liouville type vector and $\{1,a_1,a_2\}$ is linearly independent over the rationals there exist sequences of integers q_i and $n_i > i$ and $\ell_i = (\ell_{i1},\ell_{i2})$ in Z^2 such that

$$\text{2.6.1} \qquad \frac{1}{|q_i|^{n_i+2}} \leq |q_ia - \ell_i| < \frac{1}{|q_i|^{n_i+1}}$$

Thus for infinitely many i and some j, $1 \leq j \leq 2$, say for simplicity, $j = 1$, we have

$$\text{2.6.2} \qquad \frac{1}{\sqrt{2}}\cdot\frac{1}{|q_i|^{n_i+2}} \leq |q_ia_1 - \ell_{i1}| < \frac{1}{|q_i|^{n_i+1}}$$

and

$$\text{2.6.3} \qquad |q_ia_2 - \ell_{i2}| < \frac{1}{|q_i|^{n_i+1}}$$

If $\mu = f\varepsilon_1 + g\varepsilon_2$ is d_F-closed then the Fourier coefficients of f and g must satisfy the relations

$$\text{2.6.4} \qquad (k_1 - k_3a_1)g_k = (k_2 - k_3a_2)f_k.$$

We construct the sequence μ_n inductively. First consider the C^∞ function f whose Fourier coefficients are $f_0 = c_1$, $f_k = 0$ if $k \neq (\ell_{i1}, \ell_{i2}, q_i)$ and

2.6.5 $$f_{(\ell_i, q_i)} = \frac{1}{\left|\ell_{i1}^{n_i/3}\, \ell_{i2}^{n_i/3}\, q_i^{n_i/3}\right|} = f_{(-\ell_i, -q_i)}.$$

Define g by $g = c$, $g_k = 0$ if $k \neq (\ell_i, q_i)$ and

2.6.6 $$g_{(\ell_i, q_i)} = \frac{\ell_{i2} - q_i a_2}{\ell_{i1} - q_i a_1} f(\ell_i, q_i) = g_{(-\ell_i, -q_i)}$$

From 2.6.2, 2.6.3 and 2.6.6 we get

2.6.7 $$|g_{(\ell_i, q_i)}| < \sqrt{2}\,|q_i|\,|f_{(\ell_i, q_i)}|$$

showing that g is a C^∞ function. Thus $\mu = \mu_1 = f\varepsilon_1 + g\varepsilon_2$ is d_F-closed. Let $\mu_1 = \mu$. To construct μ_2 take a proper subsequence of $(\ell_{i1}, \ell_{i2}, q_i)$ given in 2.6.1 which contains infinitely many terms and disregards infinitely many of them. Starting with this subsequence, the same construction made for μ gives $\mu_2 = f^2\varepsilon_1 + g^2\varepsilon_2$. Following this procedure we construct inductively the sequence (μ_n). It is clear from the above construction that $\{\mu_i = f^i\varepsilon_1 + g^i\varepsilon_2, \varepsilon_2\}$ is a p-coframe along F. To prove ii), suppose that the equation

2.6.8 $$b\mu_i + \tau^*\mu_j + c\varepsilon_2 = d_F h$$

has a C^1 solution h. Since $j < i$ there are infinitely many $k = (\ell_{i1}, \ell_{i2}, q_i)$ such that $f^i_k = 0$ and $f^j_k \neq 0$. Thus for the above k, 2.6.8 implies that the Fourier coefficients of f^j and h must satisfy the following relations

2.6.9 $$(\ell_{i1} - q_i a_1) h_{(\ell_i, q_i)} = f^j_{(\ell_i, q_i)} e^{2\pi i \tau \cdot (\ell_i, q_i)}, \quad \tau \in R^{p+1}.$$

Since a_1 is irrational from 2.6.2, 2.6.5 and 2.6.9 we get

$$|h_{(\ell_i, q_i)}| = \frac{|f^j(\ell_i, q_i)|}{|\ell_{i1} - q_i a_1|} > \left|\frac{q_i}{\ell_{i1}}\right|^{n_i/3} \left|\frac{q_i}{\ell_{i2}}\right|^{n_i/3} |q_i|$$

Since $\frac{q_i}{\ell_{ij}} \to \frac{1}{a_j}$, then $|h_{(\ell_i, q_i)}| \to \infty$ as $i \to \infty$ and by the Riemann-Lebesgue Lemma h_k are not the Fourier coefficients of a continuous function. Assume $|a_j| < 1$, $1 \leq j \leq p$,

2.7 Let ϕ and ψ be C^s-maps, $s > 1$, from M into M that leave a foliation F invariant. According to [4,II] ϕ and ψ are said to be *integrably homotopic* if there exists a C^s-map $H: (M \times R, F \times R) \to (M, F)$ taking leaves of $F \times R$ into leaves of F and such that $H(, t) = \phi$ for $t < 0$ and $H(, t) = \psi$ for $t > 1$. It is shown there, that ϕ and ψ induce the same homomorphism in the *foliated cohomology*.

2.8 Lemma. Let $\omega = a_1 dx_1 + \ldots + a_p dx_p + dx_{p+1}$, $\{a_1, \ldots, a_p, 1\}$ linearly independent over Q, be a Liouville type form on T^{p+1}. If $\phi: T^{p+1} \to T^{p+1}$ is a C^1-diffeomorphism preserving the foliation F given by ω, then

i) ϕ is C^1-homotopic either to the identity or the map *inv*, induced by $-I$.

ii) ϕ is integrably homotopic either to a translation τ or to $\tau \circ$*inv*.

Proof. To simplify, we give the proof in T^3 for $\omega = a_1 dx_1 + a_2 dx_2 + dx_3$. By hypothesis $\phi^*\omega = g\omega$ for some continuous function g. Since ω and $\phi^*\omega$ define transversal measures invariant by the holonomy pseudogroup of F and the space of such measures is one-dimensional [2;2.3.6] then g is a constant c. Any lifting of ϕ to R^3 has the form $A + \beta$ with $A \in SL(3,Z)$ and β 3-periodic with periods in Z^3. Since ϕ is C^1-homotopic to the diffeomorphism of T^3 induced by A, the map $\phi^* : H_{DR}(T^3) \to H_{DR}(T^3)$, in the canonical base, is represented by the matrix ${}^tA^{-1} = [b_{ij}]$. From $\psi^*[\omega] = c[\omega]$ one gets the equations

2.8.j $$b_{j1}a_1 + b_{j2}a_2 + b_{j3} = ca_j, \quad 1 \leq j \leq 2$$

2.8.3 $$b_{31}a_1 + b_{32}a_2 + b_{33} = c$$

Since $\alpha = (a_1,a_2)$ is Liouville, then it follows from 2.8.3 that c is either a Liouville number or a rational and as c is an eigenvalue of the integer matrix $B = {}^tA^{-1}$, then c is an algebraic number. Therefore c is rational Since $\{a_1,a_2,1\}$ is linearly independent over Q, then it follows from 2.8.1, 2.8.2 and 2.8.3 that $B = cI$ and as $\det B = 1$ then $c = \pm 1$ and $A = \pm I$. This shows part i). Denote with the same letters the lifting of objects from T^3 to R^3. Assume for example that ϕ is homotopic to the identity, then $\phi(x_1,x_2,x_3) = (x_1 + \beta_1, x_2+\beta_2, x_3+\beta_3)$. Since ϕ preserves the foliation F, ϕ_*E_1 and ϕ_*E_2 are tangent to F, where $E_1 = \partial/\partial x_1 - a_1\partial/\partial x_3$ and $E_2 = \partial/\partial x_1 - a_2\partial/\partial x_3$, i.e., the functions β_1, β_2 and β_3 satisfy

2.8.4 $$E_j(a_1\beta_1 + a_2\beta_2 + \beta_3) = 0, \quad \text{for } j = 1, 2.$$

The leaves of F are dense, so 2.8.4 implies that $a_1\beta_1 + a_2\beta_2 + \beta_3 = C$, for some constant C. In other words,

$$\phi(x_1,x_2,x_3) = (x_1 + \beta_1, x_2 + \beta_2, x_3 + C - (a_1\beta_1 + a_2\beta_2)).$$

Define $H: T^3 \times R \to T^3$ by

$$H(x_1,x_2,x_3,t) = (x_1+\lambda\beta_1, x_2+\lambda\beta_2, x_3 + C - \lambda(a_1\beta_1 + a_2\beta_2))$$

where $\lambda: R \to R$ is a C^∞-function with $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = 1$ for $t \geq 1$. It is clear that H gives an integrable homotopy between the translation of T^3 induced by $(x_1,x_2,x_3) \to (x_1,x_2,x_3+C)$ and ϕ.

2.9 Theorem. Let $\omega = a_1dx_1 + \ldots + a_pdx_p + dx_{p+1}$, $\{1,a_1,\ldots,a_p\}$ linearly independent over Q, be a Liouville type form and F the foliation on T^{p+1} defined by ω. There are infinitely many C^1-conjugation classes in $\underset{\sim}{A}^\infty(R^p,T^{p+1},F)$.

Proof. To simplify we give a proof in T^3. By 2.6 i) the co-frames $\{\mu_n, \varepsilon_2\}$ are d_F-closed and thus define an infinite sequence of actions $\underset{\sim}{F}_n$ in $\underset{\sim}{A}^\infty(R^2,T^3,F)$. We show that any two actions $\underset{\sim}{F}_i$ and $\underset{\sim}{F}_j$, $i < j$ are not C^1-conjugated. In fact, if a C^1-diffeomorphism conjugates $\underset{\sim}{F}_i$ and $\underset{\sim}{F}_j$, then

2.9.1 $$\phi^*\mu_i = b\mu_j + c\varepsilon_2$$

and by 2.8.3 ϕ is integrably homotopic to a translation τ or $\tau \circ inv$. Let H be an integrable homotopy between ϕ and the translation τ, for example. As ϕ conjugates the C^∞ actions $\underset{\sim}{F}_i$ and $\underset{\sim}{F}_j$ then $H^*\mu_i$ is a C^1 1-form along $T(F\times R)$. By [4,II] we have

2.9.2 $$\phi^*\mu_i - \tau^*\mu_i = J_1^*(H^*\mu_i) - J_0^*(H^*\mu_i)$$

where $J_i: T^3 \to T^3\times R$, $J_i(x) = (x,i)$, $i = 0,1$ are the inclusion maps. If k is the homotopy operator for the foliated cohomology, then

2.9.3 $$J_1^* - J_0^* = d_F \circ k + k \circ d_{F\times R}$$

From 2.9.1, 2.9.2 and 2.9.3 we get

$$b\mu_j + c\varepsilon_2 - \tau^*\mu_i = d_F k(H^*\mu_i)$$

where $h = K(H^*\mu_i)$ is a C^1-function, contradicting 2.6 ii). Thus $\underset{\sim}{F}_i$ is not C^1-conjugated to $\underset{\sim}{F}_j$.

2.10 Remark. Let F be the linear foliation on T^{p+1} defined by the 1-form $\omega = a_1 dx_1 + \ldots + a_p dx_p + dx_{p+1}$. *If* ω *is diophantine type then the foliated cohomology group* $H^1(T^{p+1}, F)$ *is isomorphic to* R^p. In fact, let $\mu = f_1\varepsilon_1 + \ldots + f_1\varepsilon_p$ be a d_F-closed C^∞ 1-form along F. Thus $E_i f^j = E_j f^i$ for all $1 \le i, j \le p$. Consider the linear 1-form $\mu_0 = c_1\varepsilon_1 + \ldots + c_p\varepsilon_p$ where c_j is the constant term of the Fourier series of f^j for $1 \le j \le p$. The same arguments used in the proof of 2.3 can be adapted to show that there exists a C^∞-function h on T^{p+1} such that $\mu - \mu_0 = d_F h$, i.e., $E_j h = f_j - c_j$, $1 \le j \le p$. If ω is Liouville type then the cohomology classes of the sequence (μ_n) constructed in 2.6 are linearly independent in $H^1(T^{p+1}, F)$, showing that if ω *is Liouville type, then* $H^1(T^{p+1}, F)$ *has infinite dimension.*

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Pontifícia Univ. Católica do Rio de Janeiro
Departamento de Matemática
Rua Marquês de São Vicente, 225
22.451 - Rio de Janeiro-RJ