BOL. SOC. BRAS. MAT., VOL. 19, Nº 1 (1988), 39-59

A STABILITY THEOREM FOR ENTROPY SOLUTIONS OF INITIAL VALUE PROBLEMS FOR FIRST ORDER QUASILINEAR HYPERBOLIC SYSTEMS IN SEVERAL SPACE VARIABLES

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Abstract. In this paper we prove an uniqueness and stability theorem for the solutions of Cauchy problem for the systems

$$\frac{\partial}{\partial t}u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f^i(x,t,u) = g(x,t,u),$$

where u is a vector function $(u_1(x,t),\dots,u_r(x,t))$, $f^i = (a_1^i(x,t,u),\dots,a_r^i(x,t,u))$, $i = 1,\dots,n$, $g = (g_1(x,t,u),\dots,g_r(x,t,u))$, $x \in \mathbb{R}^n$ and $t \geq 0$. We use the concept of entropy solution introduced by Kruskov and improved by Lax, Dafermos and others autors. We assume that the Jacobian matrices f_u^i are symmetric and the Hessian $(a_j^i)_{uu}$ $(i=1,\dots,n;\ j=1,\dots,r)$ are positive. We obtain uniqueness and stability in L^2_{loc} within the class of those entropy solutions which satisfy

$$\frac{u_j(\text{—},x_i,\text{—},t)-u_j(\text{—},y_i,\text{—},t)}{x_i - y_i} \geq -K(t),$$

$(i=1,\dots,n;\ j=1,\dots,r)$ for $(\text{—},x_i,\text{—},t)$, $(\text{—},y_i,\text{—},t)$ on a compact set $D \subset \mathbb{R}^n \times (0,\infty)$ and a function $K(t) \in L^1_{loc}([0,\infty))$ depending on D. Here we denote by $(\text{—},x_i,\text{—},t)$ and $(\text{—},y_i,\text{—},t)$ two points whose coordinates only differ in

Recebido em 20/08/88.

the i-th space variable. At the end we relax the hypotheses of symmetry and convexity on the system and give a theorem of uniqueness and stability for entropy solutions which are locally Lipschitz continuous on a strip $R^n \times [0,T]$.

1. Introduction

In [4] O.A. Oleinik estabilished a uniqueness theorem for a rather general class of weak solutions of quasilinear equations of the form

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\phi(x,t,u) + \psi(x,t,u) = 0,$$

where the function $\phi(x,t,u)$ was supposed to be convex in u, i.e., $\phi_{uu} \geq 0$. In [2] A.E. Hurd gave a generalization of Oleinik's uniqueness result to systems subject to symmetry and convexity conditions in the case $\psi \equiv 0$. He used a variation of Holmgren's method which was also employed by Oleinik. Here we use the concept of entropy solution introduced by Kruskov [3] and, with a help of a simple observation due to Dafermos, which was mentioned by Di Perna in [1], we extend the Hurd's result to systems in several space variables including the source terms and obtain, further, stability in L^2_{loc}. We use an adaptation of the method employed by Kruskov to prove the uniqueness theorem of the referred work.

In §3. We make some comments and give a general theorem of uniqueness and stability of the locally Lipschitz continuous entropy solutions of the Cauchy problems for such systems without assumptions of symmetry and convexity, which can be proved using the same demonstration that we give to prove the main theorem. A result in this direction was given by Di Perna in [1].

2. Preliminaries and the Stability Theorem

In $\pi = \{(x,t) : x \in R^n,\ 0 \leq t < \infty\}$ we consider the quasilinear system of r equations

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x_i} f^i(x,t,u) = g(x,t,u) \tag{2.1}$$

for the vector function $u(x,t) = (u_1(x,t),\ldots,u_r(x,t))$ where

$$f^i(x,t,u) = (a^i_1(x,t,u),\ldots,a^i_r(x,t,u))$$

$i=1,\ldots,r$ (in (2.1) and in what follows if it appears two indices i in a monomial, then summation is taken from 1 to n).

The Cauchy problem, then, is stated by setting for (2.1) and initial condition

$$u(x,0) = u_0(x). \tag{2.2}$$

We say that $(a(x,t,u), b^1(x,t,u),\ldots,b^n(x,t,u))$ is an entropy vector for (2.1) if:

$$a_{uu}(x,t,u) \geq 0 \quad \text{(convexity)}; \tag{2.3}$$

$$b^i_u(x,t,u) = a_u(x,t,u)\, f^i_u(x,t,u),\quad i=1,\ldots,n \tag{2.4}$$

(compatibility equations).

It is easy to see that if $u(x,t)$ is a smooth function satisfying (2.1), then $u(x,t)$ satisfies also the additional equation:

$$\frac{\partial}{\partial t}a(x,t,u) + \frac{\partial}{\partial x_i} b^i(x,t,u) = c(x,t,u) \tag{2.5}$$

for any entropy vector for (2.1), $(a(x,t,u), b^1(x,t,u),\dots,b^n(x,t,u))$, where

$$c(x,t,u) = a_u(x,t,u)g(x,t,u) - a_u(x,t,u)f^i_{x_i}(x,t,u) + a_t(x,t,u) + b^i_{x_i}(x,t,u). \tag{2.6}$$

Dafermos observed (this fact was mentioned in [1]) that if a system like (2.1) has associated to it an entropy vector, then it has associated to it also an r-parameter family of entropy vectors $(\alpha(x,t,u,v), \beta^1(x,t,u,v),\dots,\beta^n(x,t,u,v))$, $v \in \mathbb{R}^r$, which are obtained from the formulas:

$$\alpha(x,t,u,v) = a(x,t,u) - a(x,t,v) - a_u(x,t,v)(u-v), \tag{2.7}$$

$$\beta^i(x,t,u,v) = b^i(x,t,u) - b^i(x,t,v) - a_u(x,t,v)(f^i(x,t,u) - f^i(x,t,v)). \tag{2.8}$$

For this family of entropy vectors, whenever $u(x,t)$ is a smooth function satisfying (2.1) we then have the equations:

$$\frac{\partial}{\partial t}\alpha(x,t,u(x,t),v) + \frac{\partial}{\partial x_i}\beta^i(x,t,u(x,t),v) = \gamma(x,t,u(x,t),v), \quad v \in \mathbb{R}^r, \text{ with}$$

$$\gamma(x,t,u,v) = \alpha_u(x,t,u,v)g(x,t,u) - \alpha_u(x,t,u,v)f^i_{x_i}(x,t,u) + \alpha_t(x,t,u,v) + \beta^i_{x_i}(x,t,u,v). \tag{2.9}$$

Conditions (2.3) and (2.4) are satisfied by the trivial entropy vectors

$$\pm(u_j, a^1_j(x,t,u),\dots,a^n_j(x,t,u)),$$

$j=1,\dots,r$, which, however, do not give any additional equation (they, in fact, give exactly the r equations of system (2.1)) and generate the trivial r-parameter family of entropy vectors (all of them zero).

We say that $(a(x,t,u), b^1(x,t,u),\dots,b^n(x,t,u))$ is a *genuine entropy vector* if

$$a_{uu}(x,t,u) > 0 \quad \text{(strict convexity)}$$

for all $x \in \mathbb{R}^n$ $t > 0$, $u \in \mathbb{R}^r$.

Following Kruskov we say that a bounded measurable vector function $u(x,t)$ is an *entropy solution* of the Cauchy problem (2.1), (2.2) in $\pi_T = \mathbb{R}^n \times [0,T]$ if it satisfies the conditions bellow:

(D1) for any entropy vector $(a(x,t,u), b^1(x,t,u),\dots,b^n(x,t,u))$ for the system (2.1) and every $\Phi \in C^\infty_0(\mathring{\pi}_T)$ (we denote by $\mathring{\pi}_T$ the interior of π_T) with $\Phi \geq 0$, we have

$$\iint_{\pi_T} \{a(x,t,u(x,t))\Phi_t + b^i(x,t,u(x,t))\Phi_{x_i} + c(x,t,u(x,t))\Phi\}\,dxdt \geq 0; \tag{2.10}$$

(D2) there exists a set η of zero measure on $[0,T]$ such that for $t \in [0,T]\setminus\eta$ the function $u(x,t)$ is defined almost everywhere in $\mathbb{R}^n$, and for any $X > 0$ we have

$$\lim_{\substack{t\to 0 \\ t \in [0,T]\setminus\eta}} \int_{|x|\leq X} |u(x,t) - u_0(x)|\,dx = 0. \tag{2.11}$$

As was observed in [3] if we put in (2.10) the trivial entropy vectors

$$\pm(u_j, a_j^1(x,t,u), \ldots, a_j^n(x,t,u))$$

$(j=1,\ldots,r)$ we obtain the usual integral identity

$$\iint_{\pi_T} \{u(x,t)\Phi_t + f^i(x,t,u(x,t))\Phi_{x_i} + g(x,t,u(x,t))\Phi\}\,dxdt = 0.$$

We note further that if for a given system like (2.1) we have a genuine entropy vector, we then also have for it an r-parameter family $(\alpha(x,t,u,v), \beta^1(x,t,u,v), \ldots, \beta^n(x,t,u,v))$, $v \in \mathbb{R}^r$, of genuine entropy vectors wich are define by (2.7), (2.8). Hence, if $u(x,t)$ is an entropy solution of the Cauchy problem (2.1), (2.2) and we have a genuine entropy vector for (2.1) then $u(x,t)$ must satisfy the integral inequality

$$\text{(2.12)}\qquad \iint_{\pi_T} \{\alpha(x,t,u(x,t),v)\Phi_t + \beta^i(x,t,u(x,t),v)\Phi_{x_i} + \gamma(x,t,u(x,t),v)\Phi\}\cdot dxdt \geq 0,$$

for any $v \in \mathbb{R}^r$ and $\Phi \in C_0^\infty(\pi_T)$, $\Phi \geq 0$.

We now pass to the assumptions wich will be made about system (2.1) for the statement of the main result of the present work:

(A1) The functions $a_j^i(x,t,u)$ possess derivatives

$$\partial a_j^i/\partial u_k, \quad \partial a_j^i/\partial x_\ell, \quad \partial^2 a_j^i/\partial x_\ell \partial u_k, \quad \partial^2 a_j^i/\partial u_k \partial u_\ell,$$

which are bounded on bounded subsets of the (x,t,u)-space.

(A2) Let

$$\frac{\partial a_j^i}{\partial u_k}(x,t,u) = a_{jk}^i(x,t,u).$$

Then, if u is bounded, i.e., $\sum u_i^2 \leq M^2$, there exists a constant c, depending only on M, such that

$$-c\sum_{j=1}^r \xi_j^2 \leq \sum_{j,k=1}^r a_{jk}^i(x,t,u)\xi_j\xi_k \leq c\sum_{j=1}^r \xi_j^2,$$

$i=1,\ldots,n$, for all vectors $\xi = (\xi_1,\ldots,\xi_r)$.

(A3) (Symmetry). For all x,t and u,

$$a_{jk}^i(x,t,u) = a_{kj}^i(x,t,u) \qquad (j,k=1,\ldots,r),$$

$i=1,\ldots,n$.

(A4) (Convexity). For all x,t and u, and each $\ell=1,\ldots,r$ and $i=1,\ldots,n$, we have

$$\text{(2.13)}\qquad \sum_{j,k}^r \frac{\partial a_{\ell k}^i}{\partial u_j}(x,t,u)\xi_j\xi_k \geq 0,$$

for all vectors $\xi = (\xi_1,\ldots,\xi_r)$.

Assumption (A3) guarantees, for system (2.1), the existence of a genuine entropy vector $(a(x,t,u), b^1(x,t,u),\ldots,b^n(x,t,u))$ with

$$\text{(2.14)}\qquad a(x,t,u) = a(u) = \frac{1}{2}|u|^2 = \frac{1}{2}(u_1^2+\ldots+u_r^2),$$

$$\text{(2.15)}\qquad b^i(x,t,u) = \sum_{j=1}^r u_j a_j^i - \Phi^i, \qquad i=1,\ldots,n,$$

where the Φ^i are functions satisfying

$$\frac{\partial}{\partial u_j}\Phi^i = a_j^i, \qquad i=1,\ldots,n; \quad j=1,\ldots,r.$$

We will prove the stability of the entropy solutions of the Cauchy problems for (2.1), relatively to the initial data (in one sense that will be made clear below), and consequently the uniqueness of such solutions, within the class of those entropy solutions which satisfy the following condition, which is an adaptation of that introduced by Oleinik[1] in [4]:

(*) Given any compact set $D \subseteq \pi$, there exists a corresponding function $K(t) \in L^1_{loc}([0,\infty))$ such that

$$\frac{u_j(\text{---},x_i,\text{---},t)-u_j(\text{---},y_i,\text{---},t)}{x_i-y_i} \geq -K(t) \tag{2.16}$$

($j=1,\ldots,r$; $i=1,\ldots,n$) holds a.e. for $(\text{---},x_i,\text{---},t)$ and $(\text{---},y_i,\text{---},t)$ on D, where we represent by $(\text{---},x_i,\text{---},t)$ and $(\text{---},y_i,\text{---},t)$ points which only differ on the i-th space variable.

2.1 Theorem: *Assume that* (2.1) *satisfy* (A1)-(A4). *Let* $u(x,t)$, $v(x,t)$ *be two entropy solutions of* (2.1), (2.2), *on* $\pi_T = \mathbb{R}^n \times [0,T]$, *satisfying condition* (*) *and*

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x)$$

with u_0 *and* v_0 *bounded measurable functions on* $\mathbb{R}^n$. *Then, for all* $X > 0$ *there exists a function* $c(t) \in L^1([0,T])$ *and a positive constant* K *such that*

$$\int_{|x|\leq X} |u(x,t)-v(x,t)|^2 dx \leq e^{\int_0^t c(s)ds} \int_{|x|\leq X+Kt} |u_0(x)-v_0(x)|^2 dx,$$

for almost all $t \in [0,T]$.

Remembering the definitions of $\alpha(x,t,u,v)$, $\beta^i(x,t,u,v)$ and $\gamma(x,t,u,v)$, in (2.7), (2.8) and (2.9), respectively, we set

[1] In fact condition (2.16) is the reverse of Oleinik's condition but this is only a matter of choose of referentials.

$$\alpha^*(x,t,u,v) = \alpha(x,t,u,v) + \alpha(x,t,v,u), \tag{2.17}$$

$$\beta^{*i}(x,t,u,v) = \beta^i(x,t,u,v) + \beta^i(x,t,v,u), \tag{2.18}$$

$$\gamma^*(x,t,u,v) = \gamma(x,t,u,v) + \gamma(x,t,v,u). \tag{2.19}$$

We also define the functions

$$\varepsilon^{*^0}(x,t,u,v) = \alpha(x,t,u,v) - \alpha(x,t,v,u), \tag{2.20}$$

$$\varepsilon^{*^i}(x,t,u,v) = \beta^i(x,t,u,v) - \beta^i(x,t,v,u). \tag{2.21}$$

As we have already said, for a system satisfying (A3) we have a genuine entropy vector which is given by (2.14) and (2.15). So, in this case we have

$$\alpha(x,t,u,v) = \alpha(u,v) = \frac{1}{2}|u|^2 - \frac{1}{2}|v|^2 - v(u-v)$$

$$= \frac{1}{2}|u-v|^2,$$

and by the definitions

$$\alpha^*(x,t,u,v) = \alpha^*(u,v) = |u-v|^2,$$

$$\varepsilon^{*^0}(x,t,u,v) \equiv 0,$$

for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$, $u,v \in \mathbb{R}^r$.

In the proof of Theorem 2.1 we will need the following:

2.2 Lemma: (a) *Let* $D \subseteq \pi_T$ *be a compact set. Then there exists* $K > 0$ *such that*

$$(2.22)\qquad \frac{|\beta^{*i}(x,t,u,v)|}{\alpha^*(x,t,u,v)} \le K$$

for $(x,t) \in D$ *and* u,v *in a bounded set of the* (u,v)*-space.*

(b) *for each* $j=1,\dots,r$ *and* $i=1,\dots,n$ *we have*

$$(2.23)\qquad \frac{\partial}{\partial u_j}\varepsilon^{*i}(x,t,u,v) \ge 0,$$

$$(2.24)\qquad \frac{\partial}{\partial v_j}\varepsilon^{*i}(x,t,u,v) \le 0.$$

Proof. Assertion (a) is a general fact for systems (2.1) satisfying only (A1) and (A2) and for wich we have a genuine entropy vector. By definition, we have

$$\beta^{*i}(x,t,u,v) = (a_u(x,t,u) - a_v(x,t,v))(f^i(x,t,u) - f^i(x,t,v)),$$

$$\alpha^*(x,t,u,v) = (a_u(x,t,u) - a_v(x,t,v))(u-v).$$

By the strict convexity of $\underset{\sim}{a}$ there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1|u-v|^2 \le {}^t(u-v)a_{uu}(x,t,u^*)(u-v) \le c_2|u-v|^2,$$

for all u^* in a bounded subset of $\mathbb{R}^r$ and $(x,t) \in D$. From this and (A2) we get trivially (2.22). We now prove assertion (b). We have

$$\beta^i(x,t,u,v) = b^i(x,t,u) - b^i(x,t,v) - a_v(v)(f^i(x,t,u) - f^i(x,t,v))$$

and then

$$\varepsilon^{*i}(x,t,u,v) = 2(b^i(x,t,u) - b^i(x,t,v)) - (a_u(u)+a_v(v))(f^i(x,t,u)-f^i(x,t,v)).$$

So, it follows that

$$\begin{aligned}\varepsilon_u^{*i}(x,t,u,v) &= 2b_u^i(x,t,u) - a_{uu}(u)(f^i(x,t,u)-f^i(x,t,v)) - (a_u(u)+a_u(v))f_u^i(x,t,u)\\ &= a_u(u)f_u^i(x,t,u) - a_v(v)f_u^i(x,t,u) - a_{uu}(u)(f^i(x,t,u)-f^i(x,t,v))\\ &= (a_u(u)-a_v(v))f_u^i(x,t,u) - a_{uu}(u)(f^i(x,t,u)-f^i(x,t,v)),\end{aligned}$$

where in the second equality we have used the compatibility equation (2.4). We now use the fact that $a(u) = \frac{1}{2}|u|^2$ and, hence,

$$a_u(u) = u, \qquad a_{uu}(u) = I.$$

So, we have

$$\varepsilon^{*i}(x,t,u,v) = {}^t(u-v)f_u^i(x,t,u) - f^i(x,t,u) + f^i(x,t,v).$$

Then, using (A3) (symmetry) and (A4) (convexity) and remembering Taylor's formula, we get

$$\frac{\partial}{\partial u_j}\varepsilon^{*i}(x,t,u,v) = f_{ju}^i(x,t,u)(v-u) - f_j^i(x,t,u) + f_j^i(x,t,v) \ge 0.$$

Analogously we prove (2.24).

Proof of Theorem 2.1: We suppose initially that f^i and g do not depend on x,t. Let the smooth function $\zeta(x,t;y,\tau) \geq 0$ be of compact support on $\overset{\circ}{\pi}_T \times \overset{\circ}{\pi}_T$. In inequality (2.11) we set $v = v(y,\tau)$ and $\phi = \zeta(x,t;y,\tau)$, for a fixed point $(y,\tau) \in \pi_T$, and we then integrate over π_T, in the variables (y,τ):

$$(2.25) \qquad \iiiint_{\pi_T\times\pi_T} \{\alpha(u(x,t),v(y,\tau))\zeta_t+\beta^i(u(x,t),v(y,\tau))\zeta_{x_i} + \gamma(u(x,t),v(y,\tau))\zeta\}dxdtdyd\tau \geq 0.$$

In exactly the same way, starting from integral inequality (2.11) for the function $v(y,\tau)$ written in the variables (y,τ), for $v = u(x,t)$ and $\phi = \zeta(x,t;y,\tau)$ we integrate over π_T, in the variables (x,t), to obtain the inequality

$$(2.26) \qquad \iiiint_{\pi_T\times\pi_T} \{\alpha(v(y,\tau),u(x,t))\zeta_\tau+\beta^i(v(y,\tau),u(x,t))\zeta_{y_i} + \gamma(v(y,\tau),u(x,t))\zeta\}dyd\tau dxdt \geq 0.$$

Adding (2.25) and (2.26) twice and rearranjing the factors under the integral signs, we obtain

$$(2.27) \qquad \iiiint_{\pi_T\times\pi_T} \{\alpha^*(u(x,t),v(y,\tau))(\zeta_t+\zeta_\tau) + \beta^{*i}(u(x,t),v(y,\tau))(\zeta_{x_i}+\zeta_{y_i}) + 2\gamma^*(u(x,t),v(y,\tau))\zeta + \varepsilon^{*i}(u(x,t),v(y,\tau))(\zeta_{x_i}-\zeta_{y_i})\}\cdot dxdtdyd\tau \geq 0.$$

Changing the roles of (x,t) and (y,τ), preserving the function $\zeta(x,t;y,\tau)$, and proceeding in the same way as above,

we get a new inequality analogous to (2.27). Then, adding it with (2.27), the following inequality is fulfilled:

$$(2.28) \qquad \iiiint_{\pi_T\times\pi_T} \{(\alpha^*(u(x,t),v(y,\tau))+\alpha^*(u(y,\tau),v(x,t)))(\zeta_t+\zeta_\tau) + (\beta^{*i}(u(x,t),v(y,\tau))+\beta^{*i}(u(y,\tau),v(x,t)))(\zeta_{x_i}+\zeta_{y_i}) + 2(\gamma^*(u(x,t),v(y,\tau))+\gamma^*(u(y,\tau),v(x,t)))\zeta + (\varepsilon^{*i}(u(x,t),v(y,t))-\varepsilon^{*i}(u(y,\tau),v(x,t)))(\zeta_{x_i}-\zeta_{y_i})\cdot dxdtdyd\tau \geq 0.$$

We now give an apropriated definition to ζ. Let δ be a smooth function with support in $[-1,1]$, and such that

$$\int_{-\infty}^{\infty} \delta(\sigma)d\sigma = 1.$$

We also require δ to be an even function, i.e.,

$$\delta(-s) = \delta(s).$$

For any $h > 0$ we set

$$\delta_h(\sigma) = h^{-1}\delta(h^{-1}\sigma).$$

Then, we define

$$\zeta(x,t;y,\tau) = \zeta^h(x,t;y,\tau) = \delta_h(\frac{t-\tau}{2})\delta_h(\frac{x_1-y_1}{2})\dots\delta_h(\frac{x_n-y_n}{2})\cdot\phi(\frac{x+y}{2},\frac{t+\tau}{2}),$$

where $\phi \geq 0$ is a test function on π_T. We see that:

$$\zeta^h_t+\zeta^h_\tau = \delta_h(\tfrac{t-\tau}{2})\delta_h(\tfrac{x_1-y_1}{2}) \dots \delta_h(\tfrac{x_n-y_n}{2})\,\phi_t(\tfrac{x+y}{2},\ \tfrac{t+\tau}{2});$$

$$\zeta^h_{x_i}+\zeta^h_{y_i} = \delta_h(\tfrac{t-\tau}{2})\delta_h(\tfrac{x_1-y_1}{2}) \dots \delta_h(\tfrac{x_n-y_n}{2})\,\phi_{x_i}(\tfrac{x+y}{2},\ \tfrac{t+\tau}{2});$$

$$\zeta^h_{x_i}-\zeta^h_{y_i} = \delta_h(\tfrac{t-\tau}{2})\delta_h(\tfrac{x_1-y_1}{2}) \dots \delta_h(\tfrac{x_i-y_i}{2}) \dots \delta_h(\tfrac{x_n-y_n}{2})\,\phi(\tfrac{x+y}{2},\ \tfrac{t+\tau}{2})$$

$$=:\ \delta_h(\tfrac{t-\tau}{2})\text{---}\,\delta_k(\tfrac{x_i-y_i}{2})\text{---}\phi(\tfrac{x+y}{2},\ \tfrac{t+\tau}{2}).$$

Let I^h_1, I^h_2 and I^h_3 be the first three terms in the left side of inequality (2.27). When we make $h \to 0$ we have

$$(2.29)\qquad I^h_1+I^h_2+I^h_3 \to 2^{n+2}\iint_{\pi_T}\{\alpha^*(u(x,t),v(x,t))\phi_t + \beta^{*i}(u(x,t),v(x,t))\phi_{x_i}+2\gamma^*(u(x,t),v(x,t))\phi\}\,dxdt.$$

To prove this we proceed as Kruskov in [3], using Lemma 2 of that paper. We now put

$$I^{ih}_4 = \iiiint_{\pi_T\times\pi_T} (\varepsilon^{*i}(u(x,t),v(y,\tau))-\varepsilon^{*i}(u(y,\tau),v(x,t)))\cdot\delta_h(\tfrac{t-\tau}{2})\text{---}\delta_h'(\tfrac{x_i-y_i}{2})\text{---}\phi(\tfrac{x+y}{2},\tfrac{t+\tau}{2})\,dxdtdyd\tau.$$

We write I^{ih}_4 as a telescope sum in the following form

$$I^{ih}_4 = \sum_{j=1}^{n}\Big\{\iiiint_{\pi_T\times\pi_T}(\varepsilon^{*i}(u(\text{---},x_j,\text{---},t),v(\text{-/-},y_j,\text{-/-},\tau)) - \varepsilon^{*i}(u(\text{---},y_j,\text{---},t),v(\text{-/-},x_j,\text{-/-},\tau)))\cdot\delta_h(\tfrac{t-\tau}{2})\text{---}\delta_h'(\tfrac{x_i-y_i}{2})\text{---}\phi(\tfrac{x+y}{2},\tfrac{t+\tau}{2})\,dxdtdyd\tau\Big\} +$$

$$+ \iiiint_{\pi_T\times\pi_T}(\varepsilon^{*i}(u(\text{---},t),v(\text{-/-},\tau)) - \varepsilon^{*i}(u(\text{---},\tau),v(\text{---},t)))\cdot\delta_h(\tfrac{t-\tau}{2})\text{---}\delta_h'(\tfrac{x_i-y_i}{2})\phi(\tfrac{x+y}{2},\ \tfrac{t+\tau}{2})\,dxdtdyd\tau.$$

We recall the fact that δ_h is an even function and, consequently, δ_h' is an odd one. Then if $j \neq i$ the integrand of the j-th term of the sum changes its sign if we replace the variable x_j by y_j and vice-versa. Analogously for the last term of the sum and the variable t and τ. Hence, we see that the unique term of the above sum wich is not zero is the i-th. So, we have

$$I^{ih}_4 = \iiiint_{\pi_T\times\pi_T}(\varepsilon^{*i}(u(\text{---},x_i,\text{---},t),v(\text{-/-},y_i,\text{-/-},\tau)) - \varepsilon^{*i}(u(\text{---},y_i,\text{---},t),v(\text{-/-},x_i,\text{-/-},\tau)))\cdot\delta_h(\tfrac{t-\tau}{2})\text{---}\delta_h'(\tfrac{x_i-y_i}{2})\text{---}\phi(\tfrac{x+y}{2},\tfrac{t+\tau}{2})\,dxdtdyd\tau$$

$$= 2\iiiint_{y_i\geq x_i}\delta_h(\tfrac{t-\tau}{2})\text{---}\delta_h'(\tfrac{x_i-y_i}{2})\text{---}\phi(\tfrac{x+y}{2},\ \tfrac{t+\tau}{2})$$

$$\cdot\Big\{\Big(\int_0^1 \varepsilon^{*i}_u(su(\text{---},x_i,\text{---},t)+(1-s)u(\text{---},y_i,\text{---},t),sv(\text{-/-},y_i,\text{-/-},\tau)+ (1-s)v(\text{-/-},x_i,\text{-/-},\tau))\,ds\Big)\cdot(u(\text{---},x_i,\text{---},t)-u(\text{---},y_i,\text{---},t)) +$$

$$+ \Big(\int_0^1 \varepsilon^{*i}_v(su(\text{---},x_i,\text{---},t)+(1-s)u(\text{---},y_i,\text{---},t),sv(\text{-/-},y_i,\text{-/-},\tau) + (1-s)v(\text{-/-},x_i,\text{-/-},\tau))\,ds\Big)\cdot(v(\text{-/-},y_i,\text{-/-},\tau)-v(\text{---},x_i,\text{-/-},\tau))\Big\}\,dxdtdyd\tau.$$

We now use condition (*) and Lemma 1.2, item (b), to obtain

$$I_4^{i,h} \leq 2 \iiiint_{y_i \geq i} dx\,dt\,dy\,d\tau \cdot \delta_h\left(\frac{t-\tau}{2}\right) \text{———} \delta_h'\left(\frac{x_i-y_i}{2}\right) \text{———} \phi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)$$

$$\cdot \{K^u(t)\Big[\sum_{k=1}^{r} \int_0^1 \frac{\partial \varepsilon^{*i}}{\partial u_k}(su(\text{—},y_i,\text{—})+(1-s)u(\text{—},x_i,\text{—}),sv(\text{—},x_i,\text{—})$$

$$+ (1-s)v(\text{—},y_i,\text{—}))\,ds\Big](y_i-x_i)$$

$$+ K^v(t)\Big[\sum_{k=1}^{r} \int_0^1 \frac{\partial \varepsilon^{*i}}{\partial v_k}(su(\text{—},y_i,\text{—}) +$$

$$+ (1-s)u(\text{—},x_i,\text{—}),sv(\text{—},x_i,\text{—}) +$$

$$+ (1-s)v(\text{—},y_i,\text{—}))\,ds\Big](x_i-y_i)\}.$$

Then, for some $K(t) \in L^1([0,T])$ and some constant $c > 0$, we have

$$I_4^{i,h} \leq \frac{c}{h^{n+1}} \iiiint_{\substack{|\frac{x-y}{2}| \leq h,\ |\frac{x+y}{2}| \leq R \\ |\frac{t-\tau}{2}| \leq h,\ 0 \leq \frac{t+\tau}{2} \leq T}} \phi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)|K(t)|$$

$$\cdot \{\sum_{\substack{k=1\\i=1}}^{\substack{i=n\\k=r}} \int_0^1 \left(\left|\frac{\partial}{\partial u_k}\varepsilon^{*i}(\text{—},\text{—})\right| + \left|\frac{\partial}{\partial v_k}\varepsilon^{*i}(\text{—},\text{—})\right|\right) ds\}\,dx\,dt\,dy\,d\tau.$$

We use again Lemma 2 of [3] to prove that the right side member of the last inequality converges to

$$(2.30) \qquad 2^{n+1}c\iint_{\pi_T} \phi(x,t)|K(t)|\{\sum_{\substack{k=1\\i=1}}^{\substack{i=n\\k=r}} \left(\left|\frac{\partial}{\partial u_k}\varepsilon^{*i}(u(x,t),v(x,t))\right|\right.$$

$$\left. + \left|\frac{\partial}{\partial v_k}\varepsilon^{*i}(u(x,t),v(x,t))\right|\right)\}\,dx\,dt.$$

We then arrive to the following integral inequality which is obtained from (2.28) joining (2.29) and (2.30):

$$\iint_{\pi_T}\{\alpha^*(u(x,t),v(x,t))\phi_t + \beta^{*i}(u(x,t),v(x,t))\phi_{x_i}$$

$$+ \tilde{K}(t)[\gamma^*(u(x,t),v(x,t)) +$$

$$+ \sum_{\substack{k=1\\i=1}}^{\substack{i=n\\k=r}} \left(\left|\frac{\partial}{\partial u_k}\varepsilon^{*i}\right| + \left|\frac{\partial}{\partial v_k}\varepsilon^{*i}\right|\right)]\phi\}\,dx\,dt \geq 0.$$

Observing that the functions between the brackets are bounded by a multiple of the square of the norm of $u-v$, we then obtain

$$(2.31) \qquad \iint_{\pi_T}\{\alpha^*(u(x,t),v(x,t))\phi_t + \beta^{*i}(u(x,t),v(x,t))\phi_{x_i}$$

$$+ c(t)|u(x,t) - v(x,t)|^2\phi\}\,dx\,dt \geq 0,$$

where $c(t)$ is a function in $L^1([0,T])$.

We now pass to the adequate definition of ϕ. We define ϕ in the same way as Kruskov in [3]. So, let $K > 0$ satisfy (2.22) in Lemma 2.2. Let η^u and η^v be the sets of measure zero which appear in (D2). Given $t_0 \in [0,T]$, define η^μ as the set of points in $[0,t_0]$ which are not Lebesgue points of the bounded measurable function

$$\mu(t) = \int_{|x| \leq X+K(t_0-t)} |u(x,t)-v(x,t)|^2\,dx.$$

We set $\eta^0 = \eta^u \cup \eta^v \cup \eta^\mu$; it is clear that η^0 has measure zero. We define

$$\alpha_h(\sigma) = \int_{-\infty}^{\sigma} \delta_h(\sigma)d\sigma$$

and take two numbers ρ and $\tau \in (0,t_0)\backslash\eta^0$, $\rho < \tau$. In (2.31) we set

$$\phi = [\alpha_h(t-\rho)-\alpha_h(t-\tau)]\chi(x,t),\ h < \min(\rho,t_0-\tau),$$

where

$$\chi = \chi_\varepsilon(x,t) = 1-\alpha_\varepsilon(|x|-X-K(t_0-t)+\varepsilon),\quad \varepsilon > 0.$$

We note that χ satisfies the relations

$$0 \equiv \chi_t+K|\chi_x| \geq \chi_t + \frac{\beta^{*i}(u,v)}{\alpha^*(u,v)}\chi_{x_i}.$$

From (2.31) we obtain the inequality:

$$\iint_{\pi_T} [\delta_h(t-\rho)-\delta_h(t-\tau)]\chi_\varepsilon(x,t)\alpha^*(u(x,t),v(x,t))dxdt$$
$$+ \iint_{\pi_T}\phi(x,t)c(t)|u(x,t)-v(x,t)|^2 dxdt \geq 0.$$

Since ρ and τ are Lebesgue points of $\mu(t)$ and $\alpha^*(u,v) = |u-v|^2$, it follows that when $h \to 0$ and $\varepsilon \to 0$ we obtain:

$$\int_{|x|\leq X+K(t_0-\tau)} |u(x,\tau)-v(x,\tau)|^2dx \leq$$
$$\leq \int_{|x|\leq X+K(t_0-\rho)} |u(x,\rho)-v(x,\rho)|^2dx$$
$$+ \int_\rho^\tau c(\tau)\left(\int_{|x|\leq X+K(t_0-t)} |u(x,t)-v(x,t)|^2dx\right)dt.$$

Applying Gronwall's Lemma we arrive to

$$\int_{|x|\leq X+K(t_0-t)} |u(x,\tau)-v(x,\tau)|^2dx \leq e^{\int_\rho^\tau c(t)dt}\int_{|x|\leq X+K(t_0-\rho)} |u(x,\rho)-v(x,\rho)|^2dx.$$

We now make $\rho \to 0$ and obtain the desired result. The case in which the functions f^i and g depends explicitly on the variables (x,t) does not present any additional difficulty and can be treated in the same way as the above. We must only to take into account also that

$$|\beta^{*i}_{x_j}(x,t,u,v)| \leq c|u-v|^2,$$

and

$$|\beta^{*i}_t(x,t,u,v)| \leq c|u-v|^2,$$

for some constant c, in each bounded subset of the (x,t,u,v)-space, and use still more times the Lemma 2 of [3]. □

3. Remarks

The crucial point of the proof of theorem 2.1 is the estimative of the terms involving the functions ε^{*i}. It is precisely there that we need to use assumptions (A3), (A4) and condition (*). These terms appear because of the lack of symmetry, in general, in the entropy vectors $(\alpha(x,t,u,v),$ $\beta^1(x,t,u,v),\dots,\beta^n(x,t,u,v))$ relatively to the variables (u,v). In the case of one equation, the entropy vectors introduced by Volpert and Kruskov, namely,

$$(|u-v|,\ \mathrm{sign}(u-v)(f^1(x,t,u)-f^1(x,t,v)),\ldots,$$

$$\mathrm{sign}(u-v)(f^n(x,t,u)-f^n(x,t,v))),$$

are symmetric with respect to the variables (u,v). But, working with systems, entropy vectors analogous to these could not be used in general.

We finally remark that the same demonstration of theorem 2.1 could be used to prove a general stability theorem for locally Lipschitz solutions of Cauchy problems for systems (2.1) which satisfies (A1) and (A2), only, and possesses a genuine entropy vector. More clearly, we have the following:

3.1 Theorem: *Assume that system* (2.1) *satisfies* (A1), (A2) *and possess a genuine entropy vector. Let* $u(x,t)$ *and* $v(x,t)$ *be two entropy solutions of the Cauchy problem for* (2.1), *in* $\pi_T = \mathbb{R}^n \times |0,T|$, *with*

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x),$$

and suppose that u *and* v *are locally Lipschitz continuous on* π_T. *Then, for each* $X>0$ *there exist positive constants* C *and* K *such that*

$$\int_{|x|\le X} |u(x,t)-v(x,t)|^2\, dx \le C \int_{|x|\le X+Kt} |u_0(x)-v_0(x)|\, dx,$$

for all $t \in [0,T]$.

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