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ON THE DYNAMICS OF QUASI-CONTRACTIONS

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1. INTRODUCTION.

Let (E_0, d_0, F_0) and (E_1, d_1, F_1) be two pointed complete metric spaces and $E_0 \vee E_1 = (E_0 \cup E_1)/(F_0 = F_1)$ their **join** naturally equiped with the distance d such that:

$$d(M_0, M_1) = d_0(M_0, F_0) + d_1(M_1, F_1) \quad \text{if} \quad (M_0, M_1) \in E_0 \times E_1.$$

Definition 1: A **quasi-contraction** is a map f from the disjoint union $E_0 \cup E_1$ to $E_0 \vee E_1$ such that $f_{|E_i}$ is a contraction.

This means that there exists a constant of contraction $k < 1$ such that : for any $(i, j) \in \{0, 1\}^2$, and $P, Q \in E_i \cap f^{-1}(E_j)$, $R \in E_i \cap f^{-1}(E_{1-j})$:

C1) $d_j(f(P), f(Q)) \leq k \cdot d_i(P, Q)$,

C2) $d_j(f(P), F_j) + d_{1-j}(f(R), F_{1-j}) \leq k.\ d_i(P, R)$.

Alternatively, with an abuse of notation, f can be considered as a map from $E_0 \cup E_1$ into itself. This point of view allows to iterate f and thus to speak about the dynamics of f. It is precisely our contention in this paper to discuss the dynamics of quasi-contractions, with due emphasis on the asymptotic aspects.

Recall that the ω-limit set ω(P) of a point P is the set of limit values of all converging subsequences extracted from its forward orbit, and that the **ω-limit set** ω(f) of a map f is the union of all ω-limit sets of points .

Then our main result (Theorem A in section 2) can be summarized as follows:

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1) $\omega(f)= \omega(F_0) \cup \omega(F_1)$

2) We find the complete list of possible types of orbits of F_0 and F_1 and in particular, using an obvious encoding in terms of sequences of 0's and 1's, provide all possible symbolic dynamics for these orbits.

3)The codes we find are those which correspond to the most even repartition of 0's and 1's, given a proportion of these bits, and we shall interpret these proportions as rotation numbers. Thus any quasi-contraction has two rotation numbers attached to it, and these are equal numbers or Farey neighbors (i.e. two irreductible rationals p_0/q_0 and p_1/q_1 such that $|p_0q_1-p_0q_1|=1$).

4) A code of F_i with rotation number p/q means that $\omega(F_i)$ is a periodic orbit with period q and p points in E_1, asymptotically stable if it does not contain any F_j.

5) A code of F_i with irrational rotation number means that $\omega(F_0) = \omega(F_1)$ is an asymptotically stable Cantor set.

The paper is organized as follows:

Section 2 contains some terminology we need and the statement of theorem A. The proof of this theorem is given in section 3, assuming a "reduction lemma" whose proof is differed to section 4.

Section 5 indicates how our main result apply to a problem in smooth dynamics (more precisely to the study of a codimension-2 bifurcation involving flows with a pair of homoclinic orbits).

Finally, section 6 relates the codes for quasi-contractions to the symbolic dynamics of rotations, thus justifying part of the chosen terminology, and describes as much of this symbolic dynamics as is needed in the proof of Theorem A.

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2. DYNAMICS OF QUASI-CONTRACTIONS.

Let $W = \{0,1\}^{Z^+}$ denote the set of infinite sequences written in the alphabet $\{0,1\}$, equiped with its standard topology, and let W^2 stand for the cartesian product $W \times W$ equiped with the product topology. We define the following two inflation rules :

$$\mathbf{0}: \begin{cases} 0 & \to & 01 \\ 1 & \to & 1 \end{cases} \qquad \mathbf{1}: \begin{cases} 0 & \to & 0 \\ 1 & \to & 10 \end{cases}$$

where **0** and **1** should in fact be considered as extended to self maps of W or W^2 according to the context. Then , we have the:

Definition 2: Consider the semi-groupe **SG** acting on W^2 freely generated by the two inflation rules **0** and **1** , and consider in W^2 the finite subset:

$$W^2_0 = \{(0^\infty, 10^\infty), (010^\infty, 10^\infty), (01^\infty, 1^\infty), (01^\infty, 101^\infty), (0^\infty, 1^\infty)\}.$$

A pair (w, w') $\in W^2$ is **quasi-rotation compatible** if it belongs to the closure of the set $A = SG(W^2_0)$ of all orbits under **SG** of points in W^2_0.

Remark: In the proof of Theorem A, we will encounter n^{th} iterates of **0** and **1** which read respectively:

$$\mathbf{0}_n: \begin{cases} 0 & \to & 01^n \\ 1 & \to & 1 \end{cases} \qquad \mathbf{1}_n: \begin{cases} 0 & \to & 0 \\ 1 & \to & 10^n \end{cases}$$

and more precisely their respective inverse maps (deflation rules) $\underline{\mathbf{0}}_n$ and $\underline{\mathbf{1}}_n$ (or $\underline{\mathbf{0}}$ and $\underline{\mathbf{1}}$ when n=1) .

Clearly, the set of all $\mathbf{0}_n$'s and $\mathbf{1}_n$'s generate the same semi-group $A= SG(W^2_0)$ as before. Also, each time we use $\mathbf{0}_n$'s or $\mathbf{1}_n$'s in the proof of Theorem A, the reader could modify the details in order to use only **0**'s or **1**'s (or their respective inverse maps).

Definition 3: For a quasi-rotation compatible pair (w, w'), the **rotation number** of the sequence w (respectivly w') is the asymtotic proportion of "1" in the sequence w (respectively w').

Remark: As a consequence of the results reported in section 6, it is clear that these limits always exist for quasi-rotation compatible pairs, and deserve their name.

Definition 4 : Let (E_0, d_0, F_0) and (E_1, d_1, F_1) be two pointed complete metric spaces, and let f: $E_0 \cup E_1 \to E_0 \cup E_1$ be a quasi-contraction . The **address** of a point P in $E_0 \cup E_1$, denoted by a(P), is 0 or 1 according to wether P belongs to E_0 or E_1. The **itinerary** of a point P in $E_0 \cup E_1$, denoted by **I**(P) , is the element of **W** defined by:

$$\mathbf{I}(P) = (\, a(P), a(f(P)), a\,(f^2(P)), \ldots\ldots).$$

We shall sometimes use the notation $\mathbf{I}_f(P)$, in order to display the map f whose dynamics is considered.

We can now state our main result:

Theorem A:

Let (E_0, d_0, F_0) and (E_1, d_1, F_1) be two pointed complete metric spaces, and $f: E_0 \cup E_1 \rightarrow E_0 \cup E_1$ a quasi-contraction . Then:

1) $\omega(f) = \omega(F_0) \cup \omega(F_1)$, and $\omega(F_i)$,$i \in \{0,1\}$ is asymptotically stable if it contains neither F_0 nor F_1.

2) The pair $(I(F_0), I(F_1))$ of symbolic sequences is quasi-rotation compatible. In particular the rotation numbers of $I(F_0)$ and $I(F_1)$ are the same real number or a pair of rational numbers which are Farey neighbors.

3) Assume that for $i \in \{0,1\}$, $I(F_i)$ has rotation number p_i/q_i, with $(p_i,q_i)=1$, then:

- if $p_0/q_0 \neq p_1/q_1$, $\omega(f)$ consists in two periodic orbits $\omega(F_0)$ and $\omega(F_1)$ with respective periods q_0 and q_1 ,

- if $p_0/q_0 = p_1/q_1$ $\omega(F_0)=\omega(F_1)$ and either $\omega(f)$ is a single periodic orbit with period $q=q_0=q_1$, or there is no periodic orbit, in which case $\exists\ j \in \{0,1\}$ such that $f^q(F_{1-j}) = F_j$.

4) In the irrational case, $\omega(F_0) = \omega(F_1)$ is a Cantor set which contains F_0 and F_1.

5) For any quasi-rotation compatible pair (w,w'), there is a quasi-contraction which satifies $(I(F_0), I(F_1)) = (w,w')$; furthermore, all possibilities in 2) and 3) can be realized.

Remark 1 :

If we drop condition C2 in the definition of a quasi-contraction, even conclusion 1) of Theorem A fails to be true. One can construct counter examples by restricting $f : x \rightarrow 2x$ [mod1] to any complete set S of its orbits since then $f^{1}|_S \rightarrow S$ satisfies condition C1.

Remark 2 :

The set of codes for all points in $E_0 \cup E_1$ is far less restricted than the ones of $\omega(f)$. For instance, if E_i is a ball in R^2 with radius 1 and center F_i for $i\in\{0, 1\}$, one can arange so that the set of all codes for all points in $E_0 \cup E_1$ be $\{0, 1\}^{Z+}$, when $I(F_0) = 0^\infty$ and $I(F_0) = 1^\infty$ ([H], [T2]).

3. PROOF OF THEOREM A.

Proof of statement 1:

The first part of statement 1 is a simple consequence of the following:

Lemma 1:

Let $i \in \{0,1\}$ and $P \in E_i$, then:
$\forall n \geq 0, \exists m \geq 0$ and $(j,k) \in \{0,1\}^2$ such that both $f^n(P)$ and $f^m(F_j)$ belong to E_k and $d_k(f^n(P), f^m(F_j)) \leq k^n.d_i(P, F_i)$.

Proof of lemma 1:

The proof proceeds by induction on n.

For n=0, the result is obvious.

Suppose the result is true for $n=n_0$ with $m=m_0$ and $(j,k)=(j_0, k_0)$, then there are two possibilities:

a) $\exists k_1 \in \{0,1\}$ such that $(f^{n_0+1}(P), f^{m_0+1}(F_{j_0})) \in E_{k_1}{}^2$.

Hence using **C1**:

$$d_{k_1}(f^{n_0+1}(P), f^{m_0+1}(F_{j_0})) \leq \mathbf{k}.d_{k_0}(f^{n_0}(P), f^{m_0}(F_j)) \leq \mathbf{k}^{n_0+1}.d_i(P, F_i).$$

b) $\exists k_1 \in \{0,1\}$ such that $f^{n_0+1}(P) \in E_{k_1}$ and $f^{m_0+1}(F_{j_0}) \in E_{1-k_1}$.

Hence using **C2**:

$$d_{k_1}(f^{n_0+1}(P), F_{k_1}) \leq \mathbf{k}.d_{k_0}(f^{n_0}(P), f^{m_0}(F_j)) \leq \mathbf{k}^{n_0+1}.d_i(P, F_i).$$

Q.E.D. (lemma 1).

For the second part of statement 1, assuming $\omega(F_i) \cap \{F_0, F_1\} = \varnothing$, let:

$$\delta = \min\{d_0[(\omega(F_i) \cap E_0), F_0], d_1[(\omega(F_i) \cap E_1), F_1]\}.$$

Then, for any $P \in \omega(F_i)$, any ball with center P and radius $\delta' < \delta$ has its successive images contracted to the orbit of P.

Q.E.D. (statement 1).

Reduction lemma:

Under the hypotheses of Theorem A, there exist two closed sets $E_0^{(1)} \subset E_0$ and $E_1^{(1)} \subset E_1$ such that:

a) $F_0 \in E_0^{(1)}$ and $F_1 \in E_1^{(1)}$,

b) $f(E_0^{(1)} \cup E_1^{(1)}) \subset E_0^{(1)} \cup E_1^{(1)}$,

c) $\exists (i_1, j_1) \in \{0,1\}^2$ such that $f(E_{i_1}^{(1)}) \subset E_{j_1}^{(1)}$.

This "reduction lemma" is the main building block of the proof of Theorem A. Its proof is given in section **4**. Remark that **a)** and **b)** not only tell us that f restricted to $E_0^{(1)} \cup E_1^{(1)}$ is again a quasi-contraction, but also that $E_0^{(1)} \cup E_1^{(1)}$ contains $\omega(F_0) \cup \omega(F_1)$.

Lemma 2:

Under the hypotheses of Theorem A, if $f(F_i) \in E_i$, then:
$\forall n \geq 0, f^n(F_i) \in E_i$.

We shall use lemma 2 in the present section, and later again as a step in the proof of the reduction lemma; its proof is also reported in section **4**.

Proof of statement 2:

The proof will be organized according to the two possibilities in statement c) of the reduction lemma, that we call "case α" when $i_1 = j_1$, and "case β" when $i_1 \neq j_1$.

α) $\exists\, i_1 \in \{0,1\}$ **such that** $f(E_{i_1}^{(1)}) \subset E_{i_1}^{(1)}$

This is by far the simplest case :

- $I(F_{i_1})$ is obviously the constant sequence : $I(F_{i_1}) = (i_1,i_1,i_1,......)$.

- There are precisely two possibilities for $f(F_{1-i_1})$:

- either $f(F_{1-i_1}) \in E_{i_1}^{(1)}$, in which case:

$$I(F_{1-i_1}) = (1-i_1,i_1,i_1,......),$$

- or $f(F_{1-i_1}) \in E_{1-i_1}^{(1)}$, in which case:

$$I(F_{1-i_1}) = (1-i_1,1-i_1,1-i_1,......),$$

as a consequence of Lemma 2.

In both cases, one recognizes quasi-rotation compatible pairs with a pair of rotation numbers in $\{0,1\}^2$.

This concludes the proof of statement 2 for the case α.

β) $\exists\, i_1 \in \{0,1\}$ **such that** $f(E_{i_1}^{(1)}) \subset E_{1-i_1}^{(1)}$.

We shall first split the case β according to two complementary further specifications called respectively β.1 and β.2.

β.1: $\forall\, m > 0,\ f^m(E_{i_1}^{(1)}) \in E_{1-i_1}^{(1)}$.

Then obviously :

$$I(F_{i_1}) = (i_1,1-i_1,1-i_1,......).$$

Furthermore,

- either $f(F_{1-i_1}) \in E_{1-i_1}^{(1)}$, in which case (using lemma 2) :

$$I(F_{1-i_1}) = (1-i_1,1-i_1,1-i_1,......),$$

- or $f(F_{1-i_1}) \in E_{i_1}^{(1)}$, leading to:

$$I(F_{1-i_1}) = (1-i_1,i_1,1-i_1,1-i_1,.....),$$

so that statement 2 is proved for the case **β.1.**

β.2: $\exists\, m > 0,\ f^m(E_{i_1}^{(1)}) \cap E_{i_1}^{(1)} \neq \varnothing$.

Then we set :

$$m_1 = \min \{ m > 0 \mid f^m(E_{i_1}^{(1)}) \cap E_{i_1}^{(1)} \neq \varnothing \},$$

and we define a new map :

$$f^{(1)}: E_0^{(1)} \cup E_1^{(1)} \to E_0^{(1)} \cup E_1^{(1)},$$

by:

$$\begin{cases} f^{(1)}(P) = f(P) & \text{if } P \in E_{1-i_1}^{(1)} \\ f^{(1)}(P) = f(P) & \text{if } P \in E_{1-i_1}^{(1)} \end{cases},$$

and one readily verifies that $f^{(1)}$ is a quasi-contraction with spaces $(E_0^{(1)}, d_0, F_0)$ and $(E_1^{(1)}, d_1, F_1)$.

Consequently, the reduction lemma applies and there exist two closed sets:

$$E_0^{(2)} \subset E_0^{(1)} \text{ and } E_1^{(2)} \subset E_1^{(1)},$$

such that:

a) $F_0 \in E_0^{(2)}$ and $F_1 \in E_1^{(2)}$,
b) $f^{(1)}(E_0^{(2)} \cup E_1^{(2)}) \subset E_0^{(2)} \cup E_1^{(2)}$,
c) $\exists\, (i_2, j_2) \in \{0,1\}^2$ such that $f^{(1)}(E_{i_2}^{(2)}) \subset E_{j_2}^{(2)}$.

The knowledge of the dynamics of $f^{(1)}$ encompasses whatever we want to know about the dynamics of f. More precisely, we have the following two properties :

P_1: $\omega_{f^{(1)}}(F_i) \subset \omega_f(F_i) = \cup_{m_1 > m \geq 0} f^m(\omega_{f^{(1)}}(F_i))$; $i \in \{0,1\}$.

P_2: $I_{f^{(1)}}(F_i) = (a_0^{(1)}, a_1^{(1)}, a_2^{(1)}, \ldots\ldots) \Rightarrow I_f(F_i) = (a_0, a_1, a_2, \ldots\ldots)$

where:

$$a_k = \begin{cases} a_k^{(1)} & \text{if } a_1^{(1)} = 1 - i_1 \\ (a_k^{(1)}, b_1, .., b_{m_1-1}) \text{ with } b_n \equiv 1 - a_k^{(1)} \text{ for } n \in \{1; ..; m_1 - 1\} & \text{if } a_k^{(1)} = i_1. \end{cases}$$

The construction of $f^{(1)}$ out of f thus appears as the first step of an inductive process which after n steps would leave us with a map :

$$f^{(n)}: E_0^{(n)} \cup E_1^{(n)} \to E_0^{(n)} \cup E_1^{(n)}.$$

We shall set $f^{(0)} = f$.

Now we can split further case β.2 according to wether all the successive $f^{(n)}$'s fall in case β.2 (β.2.2) , or one find one of them in either the case α or the case β.1 (β.2.1).

β.2.1: $\exists\, n_0$ such that $f^{(n_0)}$ falls in case α or β.1.

In this case the inductive process stops and statement 2 is satisfied for $f^{(n_0-1)}$ since we are is the same situation as in case α or as in case β.1. Using P_2 and the definition of quasi-rotation compatible pairs, this concludes the proof of statement 2 for f in case β.2.1.

β.2.2: $\forall\, n > 0$, $f^{(n)}$ falls in case β.2.

Then the inductive process never stops, and we are left with the last alternative:

β.2.2.1: $\exists\, n_0 > 0, \exists\, i \in \{0, 1\}$ such that $\forall\, n \geq n_0$, $i_n = i$.

In this case , the pair $(I(F_0), I(F_1))$ belongs to the closure of the set $A = SG(W^2{}_0)$ (see definition 1 of section 2) and thus is quasi-rotation compatible.

Since the inflation rule which relates $I_{f^{(m)}}(F_i)$ to $I_{f^{(m-1)}}(F_i)$, $i \in \{0,1\}$, is eventually constant (i.e.**0** or **1** according to wether i_{n_0} is 0 or 1), we get a single rational rotation number (which in fact is 0 or 1 according to wether i_{n_0} is 0 or 1: see section 6).

β.2.2.2. $\forall\, n_0 > 0, \exists\, n \geq n_0$ **such that** $f^{(n-1)}(E_0^{(n)}) \subset E_1^{(n)}$ **and** $f^{(n)}(E_1^{(n+1)}) \subset E_0^{(n+1)}$.

In this case again, the pair $(I(F_0), I(F_1))$ belongs to the closure of the set $A = SG(W^2_0)$ and thus is quasi-rotation compatible. Since the inflation rule which relates $I_{f^{(m)}}(F_i)$ to$I_{f^{(m-1)}}(F_i)$, $i \in \{0,1\}$, is not eventually constant, we get a single irrational rotation number (see section 6).

Q.E.D. (statement 2).

__Remark 3 :__ Irrational rotation numbers correspond to and only to the case β.2.2.2..

__Proof of statement 3:__

Using statement 2, the proof of this statement needs essentially manipulations on ordinary contractions; details are left to the reader.

Q.E.D. (statement 3).

__Proof of statement 4:__

From Remark 3 and the definition of quasi-rotation compatible pairs, we know that for the irrational case, $I(f^2(F_0))=I(f^2(F_1))$; so that $\omega(F_0) = \omega(F_1)$ is a simple consequence of **C1**.

It remains to prove that $\omega(f)$ is :

-α) closed,

-β) perfect,

-γ) totally discontinuous.

- α) is a general feature of ω-limit sets.

- For β) , one has to prove that any point in $\omega(f)$ is an accumulation point of $\omega(f)$. This will be true if the F_i's are themselves accumulation points of $\omega(f)$. Thus let us consider a point F_i , and for any $n_0>0$ such that $f^{n_0}(F_i)) \in E_i$, let :

$$0< \delta_{n_0} = d_i(f^{n_0}(F_i) , F_i).$$

Since $I(F_i)$ is not periodic, thanks to **C1** and **C2**, there exists $n>n_0$ such that

$$0< d_i(f^n(F_i) , F_i) \leq k.\delta_{n_0}.$$

This proves β) .

- For γ), one has to prove that the connected components of $\omega(f)$ are reduced to a single point. Let us then notice that, for any $\varepsilon > 0$, there exists $N_\varepsilon>0$ such that, for all $n \geq N_\varepsilon$, $\omega(f^{(n)}) \subset B_0(F_0,\varepsilon) \cup B_1(F_1,\varepsilon)$, where $B_i(P, r)$ is the closed ball in (E_i, d_i) with center P and radius r, because the upper bound on the diameters of the sets $\omega(f^{(p)}) \cap E_0$ and $\omega(f^{(p)}) \cap E_1$ gets improved by a factor **k** at each induction step. Hence using **P1**, one has:

$$\omega(f) \subset \bigcup_{m \geq 0} \left(B_0(f^m(F_0),k^m \varepsilon) \cup B_1(f^m(F_1),k^m \varepsilon)\right).$$

If now K is a connected component of $\omega(f)$ with diameter δ, it is possible to cover it with a subset of this set of balls, so that we get the estimate:

$$\delta \leq 2\varepsilon/(1-k),$$

and since ε can be chosen arbitrarily small, we get δ =0.

Q.E.D. (statement 4).

__Proof of statement 5:__

It is in fact easy to construct all examples with $\omega(f)$ indecomposable, just using piecewise linear contractions with a single dicontinuity ([G,T],[T1]).

The cases where ω(f) consists in two periodic orbits can also be obtained from maps on the interval if one accepts maps which are not piecewise monotone, i.e. have infinitely many segments of monotonicity (see [G,G,T.1] for flows whose first-return map is approximated by such a one-dimensional map).

It is however easier to construct abstract examples starting from an easily constructed example for each code in the set $W^2{}_0$, and using constructions which imitate the inflations $\mathbf{0_n}$ and $\mathbf{1_n}$. Details are left to the reader.

In Figure1a, we illustrate the meaning for f of the inverse $\underline{\mathbf{0}}_n$ of $\mathbf{0_n}$, and in figure 1b, the construction corresponding to $\mathbf{0_n}$.

Q.E.D. (statement 5).

FIGURES 1-a and 1-b.come here.

See the end of the paper

4. PROOF OF THE REDUCTION LEMMA.

Let us first recall the statement of the:

Reduction lemma:

Under the hypotheses of Theorem A, there exist two closed sets $E_0^{(1)} \subset E_0$ and $E_1^{(1)} \subset E_1$ such that:

a) $F_0 \in E_0^{(1)}$ and $F_1 \in E_1^{(1)}$,

b) $f(E_0^{(1)} \cup E_1^{(1)}) \subset E_0^{(1)} \cup E_1^{(1)}$,

c) $\exists\ (i_1, j_1) \in \{0,1\}^2$ such that $f(E_{i_1}{}^{(1)}) \subset E_{j_1}{}^{(1)}$.

The proof of this lemma (called RL hereafter) is rather complicated and needs some explanations. It will be done in five steps and we now describe this sheme.

First step :

We prove the reduction lemma for all quasi-contractions with constant of contraction **k** smaller than 1/2. This result will be called the "1/2 reduction lemma" and noted (1/2)RL. There is a big gap in the level of difficulties between proving the (1/2) RL and the RL which cannot be attaqued with the same rough techniques.

Second step :

We give a weaker version of the reduction lemma , called the "pre-reduction lemma" and noted PRL which differs from the RL only by the fact that we do not ask to the sets $E_0^{(1)} \subset E_0$ and $E_1^{(1)} \subset E_1$ to be closed.

In this second step, we prove the PRL for all the quasi-contractions but the ones which are in one of the two following configurations noted (*) :

$$\begin{cases} f(F_0)\in E_1 \text{ and } f^m(F_0)\in E_1 \Rightarrow f^{m+1}(F_0)\in E_0 \\ f(F_1)\in E_0;\ f^2(F_1)\in E_0 \text{ and } f^3(F_1)\in E_1 \\ \text{and } \exists\, p>0:\ f^{2p}(F_0)\in E_0 \text{ and } f^{2p+1}(F_0)\in E_0. \end{cases}$$

and the symmetric configuration obtained by exchanging 0 and 1. In general, a configuration is a set of constraints on the itineraries of the orbits of F_0 and F_1 under a quasi-contraction.

Third step :

We prove that if the reduction lemma is true for all quasi-contractions with constant of contraction smaller than a given $k_0 < 1$, then the pre-reduction lemma is also true for all quasi-contractions with constant of contraction smaller than $(k_0)^{1/2}$(even the ones in configuration (*) which are the ones where the result is not yet known at the preceding stage). In a symbolic way, we have :

$$(k_0).RL \Rightarrow (k_0)^{1/2}.PRL$$

Fourth step :

We prove that if the pre-reduction lemma is true for all quasi-contractions with constant of contraction smaller than a given $k_0 < 1$, then the reduction lemma is also true for all quasi-contractions with constant of contraction smaller than k_0. In symbols:

$$(k_0).PRL \Rightarrow (k_0).RL.$$

Fifth step :

We conclude by remarking that applying again successively step3, step 4, step3, step 4 .... we get the proof of the reduction lemma.

Now, we can proceed to prove the reduction lemma step by step.

FIRST STEP :

Lemma 3 ((1/2)RL) :

The reduction lemma holds true for quasi-contractions with **k<1/2**.

Proof of (1/2)RL:

Let f be a quasi-contraction with **k**<1/2, we set :

$$\delta = \max\{d_j(f(F_i), F_j),\ f(F_i) \in E_j,\ j\in \{0, 1\}\} = d_{j_0}(f(F_{i_0}), F_{j_0}).$$

Then, the reduction lemma holds with:

$$E_i^{(1)} = B_i(F_i, \delta/(1\text{-}\mathbf{k})).$$

If $\delta = 0$ the result is obvious and we asume now that $\delta \neq 0$. Notice that F_i belongs to $E_i^{(1)}$ from the definition.

Let P belong to $E_i^{(1)}$, and f(P) belong to E_j.

If $f(F_i)$ does not belong to E_j then, thanks to C2:

$$d_j(f(P), F_j) \le \mathbf{k}.d_i(P, F_i) \le \mathbf{k}.\delta/(1\text{-}\mathbf{k}),$$

thus f(P) belongs to $E_j^{(1)}$.

If $f(F_i)$ belongs to E_j , then, thanks to **C1**:

$$d_j(f(P), f(F_i)) \le \mathbf{k}.d_i(P, F_i) \le \mathbf{k}.\delta/(1-\mathbf{k}),$$

but:

$$d_j(f(F_i), F_j) \le \delta,$$

and $$d_j(f(P), F_j) \le \delta + \mathbf{k}.\delta/(1-\mathbf{k}) \le \delta/(1-\mathbf{k}),$$

thus f(P) belongs to $E_j^{(1)}$.

So far, we have proved that:

$$f(E_0^{(1)} \cup E_1^{(1)}) \subset E_0^{(1)} \cup E_1^{(1)}.$$

Now, let us prove that:

$$f(E_{i_0}) \subset E_{j_0}.$$

Asume that P is in E_{i_0} and f(P) is in E_{1-j_0}, then, using C2:

$$\delta = d_{j_0}(f(F_{i_0}), F_{j_0}) \le \mathbf{k}.\delta/(1-\mathbf{k}).$$

But this inequality is imposible if $\delta \neq 0$ and **k**<1/2.

Q.E.D.((1/2)RL).

This restricted version of the reduction lemma is much simpler than the general case, in that $E_0^{(1)}$ and $E_1^{(1)}$ can be defined directly as closed balls when **k**<1/2. This will not any more be true for larger values of **k**, which motivates the nature of the following step.

<u>SECOND STEP :</u>

We now give the formulation of the "pre-reduction lemma".

<u>Lemma 4 (PRL):</u>

Under the hypotheses of Theorem A, there exist two sets $G_0 \subset E_0$ and $G_1 \subset E_1$ such that:

a) $F_0 \in G_0$ and $F_1 \in G_1$,

b) $f(G_0 \cup G_1) \subset G_0 \cup G_1$,

c) $\exists\,(i_1, j_1) \in \{0,1\}^2$ such that $f(G_{i_1}) \subset G_{j_1}$.

In this second step of the proof of the RL, we shall not prove completely the PRL, but only most of it. More precisely, we shall prove the same conclusion assuming we are not in the pair of configurations noted (*), and defined previously. This restricted statement is what we call "most of PRL".

<u>Proof of "most of PRL":</u>

We shall eventually prove the "pre-reduction lemma" with:

$$G_i = E_i \cap (\cup_{m\ge 0}\{f^m(F_0)\} \cup \cup_{n \ge 0}\{f^n(F_1)\}) \text{ ,} i \in \{0,1\}.$$

Defining now $(j_0, j_1) \in \{0,1\}^2$ by $F_0 \in E_{0,j_0}$ and $F_1 \in E_{1,j_1}$, we will begin to organize the proof of PRL according to the four possible choices corresponding to the set of values of the pair (j_0, j_1). As the argument will develop, we will be obliged to isolate the (*) configurations, and finish only at this stage the annunced proof of what we call "most of the PRL".

The following obvious result will be useful in many instances:

Lemma 5 (Coincidence lemma):

Let $\{(E_\alpha,d_\alpha)\}_{\alpha\in A}$ be a collection of metric spaces, P a finite subset of $\cup_{\alpha\in A}(E_\alpha\times E_\alpha)$, and $k\in[0,1[$. Assume that there exists a map:

$$g:\quad P \to P, \qquad (x,y)\in E_\beta^2 \to (x',y')\in E_\gamma^2$$

such that:

$$d(x,y)\leq k.d(x,y).$$

Then there exists $\lambda\in A$ and $z\in E_\lambda$ such that $(z,z)\in P$.

In the sequel, the reader will find many claims that "the reduction lemma holds true with a given set P". What is meant is that for the given P, there exist a g as above so that the conclusion of the coincidence lemma holds true.

Recall the statement of

Lemma 2:

Under the hypotheses of theorem A, if $f(F_i)\in E_i$ then $\forall n\geq 0$, $f^n(F_i)\in E_i$.

Proof of lemma 2:

Assume the lemma is false. Then, there exists a first m with $f^m(F_i)\in E_i$ and $f^{m+1}(F_i)\in E_{1-i}$.

Remark: from the hypothesis we know that $m\geq 1$.

Claim: the coincidence lemma holds true with the set P constructed with $A=\{i\}$, and defined by $P=\{(a,b)|\ a=f^p(F_i),\ b=f^q(F_i),\ p\neq q,\ 0\leq p\leq m, 0\leq q\leq m\}$.

Proof of the claim:

There are only two cases to be considered:

a) $p<m, q<m$: then $(f^{p+1}(F_i),f^{q+1}(F_i))\in P$; we set:

$$g(f^p(F_i),f^q(F_i))\equiv(f^{p+1}(F_i),f^{q+1}(F_i)),$$

and we have:

$$d_i(f^{p+1}(F_i),f^{q+1}(F_i))\leq k.d_i(a,b).$$

b) $p=m, q<m$: then $(f^{q+1}(F_i),F_i)\in P$; we set:

$$g(f^q(F_i),F_i)\equiv(f^{q+1}(F_i),F_i),$$

and we have:

$$d_i(f^{q+1}(F_i),F_i)\leq k.d_i(a,b).$$

Q.E.D.(Claim).

Hence, from the coincidence lemma, we know that there exist:

$$0\leq m_1\leq m,\ 0\leq m_2\leq m, m_1<m_2 \text{ such that } f^{m_1}(F_i)=f^{m_2}(F_i).$$

Then $f^{m_1+m+1-m_2}(F_i)=f^{m+1}(F_i)$, and since $m+m_1-m_2<m$, we get that $f^{m+1}(F_i)\in E_i$, a contradiction.

Q.E.D.(lemma 2).

The case $(i_0,i_1)=(0,1)$.

Lemma 6 (PRL true in the case $(j_0,j_1)=(0,1)$):

Under the hypotheses A,PRL holds true for the configuration $(j_0,j_1)=(0,1)$.

Proof of lemma 6:

From lemma 2, we know that:

$$\cup_{n\geq 0}\{f^n(F_i)\} \subset E_i,$$

thus $f(G_i)\subset G_i$, for $i\in\{0,1\}$. This takes care of the case $(j_0,j_1)=(0,1)$.

Q.E.D.(lemma 6).

The cases $(j_0,j_1)=(0,0)$ and $(j_0,j_1)=(1,1)$.

Lemma 7 (PRL true in the cases $(j_0,j_1)=(0,0)$ and $(j_0,j_1)=(1,1)$):

Under the hypotheses of theorem A, PRL holds true for the configurations $(j_0,j_1)=(0,0)$ and $(j_0,j_1)=(1,1)$.

Proof of lemma 7:

Since these two configurations are obviously equivalent, we shall only consider the case when $(j_0,j_1)=(0,0)$.

From lemma 2, we know that:

$$\cup_{n\geq 0}\{f^n(F_0)\} \subset E_0.$$

Consequently, we can use the following more precise definitions:

$$G_0=E_0\cap(\cup_{p\geq 0}\{f^p(F_0)\} \cup \cup_{q\geq 0}\{f^q(F_1)\}),$$
$$G_1=E_1\cap(\cup_{n\geq 0}\{f^n(F_1)\}).$$

We shall show that $f(G_1)\subset G_0$, i.e. F_1 does not have two consecutive images, say $f^m(F_1)$ and $f^{m+1}(F_1)$, in E_1. This is obviously true if $G_1=\{F_1\}$.

Remark: notice that the m above should satisfy $m\geq 2$ since $f(F_1)\in E_0$ from the hypothesis.

Claim: the coincidence lemma holds true with the set P constructed with $A=\{1\}$, and defined by $P=\{(a,b)\in E_1{}^2 | a=f^p(F_1), b=f^q(F_1), p\neq q, 0\leq p\leq m, 0\leq q\leq m\}$.

Proof of the claim:

We distinguish two case, a) and b).

a) if $p<q<m$, there are two possibilities:

α) $\forall i\in\{1,2,...m-q\}$, $\exists j(i)\in\{0,1\}$ with:

$$(f^{p+i}(F_1),f^{q+i}(F_1))\in E_{j(i)}\times E_{j(i)}.$$

Then we have:

$$d_1(f^{p+m-q}(F_1),f^m(F_1))\leq k^{m-q}.d_1(a,b).$$

β) $\exists i_0\in\{1,2,...m-q\}$ such that:

$$\forall i:0\leq i< i_0,\ \exists j(i)\in\{0,1\} \text{ with: } (f^{p+i}(F_1),f^{q+i}(F_1))\in E_{j(i)}\times E_{j(i)},$$
$$\exists j(i_0)\in\{0,1\} \text{ with: } (f^{p+i_0}(F_1),f^{q+i_0}(F_1))\in E_{j(i_0)}\times E_{1-j(i_0)}.$$

Then we define r by:

$$r=\begin{cases} p+i_0 & \text{if} \quad j(i_0)=1 \\ q+i_0 & \text{if} \quad 1-j(i_0)=1 \end{cases},$$

and from C1 and C2 we have:

$$d_1(f^r(F_1),F_1) \leq k^{i_0}.d_1(a,b).$$

b) p<q=m:

from C2 we have:

$$d_0(f^{p+1}(F_1),F_0) \leq k.d_1(a,b).$$

combining C1 and C2 and the fact that:

$$\cup_{n\geq 0}\{f^n(F_0)\}\in E_0,$$

we have:

$$d_1(f^{p+s}(F_1),F_1) \leq k^s.d_1(a,b),$$

where $f^{p+s}(F_1)$ is the next element of the orbit of F_1 which belongs to E_1 after $f^p(F_1)$.

Q.E.D.(Claim).

Hence, from the coincidence lemma, we know that there exist:

$0\leq m_1\leq m$, $0\leq m_2\leq m, m_1<m_2$ such that $f^{m_1}(F_1)=f^{m_2}(F_1)\in E_1$.

It follows that $f^{m_1+m-m_2}(F_1)\in E_1$. Since $f^{m_1+m+1-m_2}(F_1)\in E_0$, we get that $f^{m+1}(F_1)\in E_0$, a contradiction.

Q.E.D.(lemma 7).

Lemma 8:

Under the hypotheses of theorem A, if there exists $n\geq 1$ and $i\in\{0,1\}$ such that:

$$f^n(F_i)\in E_{1-i} \quad \text{and} \quad f^{n+1}(F_i)\in E_{1-i}, \qquad (**)$$

then:

$$f^1(F_i)\in E_{1-i} \quad \text{and} \quad f^2(F_i)\in E_{1-i}.$$

Proof of lemma 8:

Let m be the smallest n such that (**) holds true. If m=1, the lemma is proved, and for m infinite, there is nothing to prove. We shall thus suppose that m>1 is finite.

Claim: for m>1 and finite, the coincidence lemma holds true with the set P constructed with A={1-i}, and defined by:

$$P=\{(a,b)\in E_{1-i}{}^2 \mid a=f^p(F_i),\ b=f^q(F_i),\ p\neq q,\ 0\leq p<q\leq m\}.$$

The proof of the claim and the conclusion of the proof of lemma 8 go along the same lines as before. Details are left to the reader.

Q.E.D.(claim and lemma 8).

Remark: In the rest of the paper, we shall no longer give the proofs for claims such as the one above, nor give details on the way to use them.

For G_0 and G_1 violating the PRL, it is evidently necesary that two successive points of the orbit of F_0 or F_1 be in the same E_i since otherwise:

$$f(G_0)\subset G_1 \text{ and } f(G_1)\subset G_0.$$

Thus we have to examine three configurations:

A: $\exists$ m,n≥0 such that:

$$f^m(F_0)\ ,\ f^{m+1}(F_0)\in E_1\ ,$$

and

$$f^n(F_1)\ ,\ f^{n+1}(F_1)\in E_0\ .$$

B: $\exists$ m,n≥0 such that:

$$f^m(F_0)\ ,\ f^{m+1}(F_0)\in E_0\ ,$$

and

$$f^n(F_1)\ ,\ f^{n+1}(F_1)\in E_1\ ,$$

and one is not in a configuration C as defined hereafter.

C: $\exists$ i∈{0,1}, and m,n≥0 such that:

$$f^m(F_i)\ ,\ f^{m+1}(F_i)\in E_i\ ,$$

and

$$f^n(F_i)\ ,\ f^{n+1}(F_i)\in E_{1-i}\ .$$

Since we have already treated the cases where (j_0,j_1) is (0,1),(0,0) and (1,1), it would only remain, in order to prove the PRL, to show that none of the configurations **A**, **B**, and **C** as defined above, can occur in the case (j_0,j_1)=(1,0). We shall prove this completely for the configurations **A** and **B**, and it is in the study of the configurations **C** that we shall isolate the (*) configuration described at the beginning of this section.

Impossibility of the configuration A in the case (j_0,j_1)=(1,0).

Lemma 9 :

Under the hypotheses of theorem A, one cannot find two integers m and n such that:

$$(\alpha)\quad f^m(F_0)\ ,\ f^{m+1}(F_0)\in E_1,$$

and

$$(\beta)\quad f^n(F_1)\ ,\ f^{n+1}(F_1)\in E_0.$$

Proof of lemma 9:

From lemma 8, we already know that if two such integers m and n exist, then m=1 and n=1 satisfy the same conditions. Thus it only remains to prove the impossibility of:

$$(\alpha')\quad f(F_0)\ ,\ f^2(F_0)\in E_1,$$

and

$$(\beta')\quad f(F_1)\ ,\ f^2(F_1)\in E_0.$$

If (α') and (β') hold simultaneously, we deduce using C2 that:

(α') implies:

$$d_0(f(F_1),F_0)\le k.d_1(f(F_0),F_1),$$

and (β') implies:

$$d_1(f(F_0),F_1)\le k.d_0(f(F_1),F_0).$$

It follows that $f(F_1)=F_0$ which is impossible since $f(F_0)\in E_1$ and $f^2(F_1)\in E_0$.

Q.E.D.(lemma 9).

Impossibility of the configuration B in the case $(i_0,i_1)=(1,0)$.

Lemma 10 :

Under the hypotheses of theorem A, $f(F_0)\in E_1$, $f(F_1)\in E_0$, one cannot find two even positive integers m_0 and m_1 such that:

(α) $f^i(F_0)\in E_0$, $i< m_0$, i even, $f^i(F_0)\in E_1$, $i< m_0$, i odd,

(β) $f^j(F_1)\in E_1$, $j< m_1$, j even, $f^j(F_1)\in E_0$, $j< m_1$, j odd,

(γ) $f^{m_0}(F_0)\in E_0$, $f^{m_0+1}(F_0)\in E_0$,

(δ) $f^{m_1}(F_1)\in E_1$, $f^{m_1+1}(F_1)\in E_1$,

Proof of lemma 10:

Assume lemma 10 is false.

Claim: the coincidence lemma holds true with the set P constructed with A={1-i}, and defined by:

$P=\{(a,b)\in E_0{}^2\cup E_1{}^2 | \exists\ i_0\in\{0,1\}^2, p\leq m_i,\ q\leq m_j\ p\neq q,$ such that $a=f^p(F_i),\ b=f^q(F_j)\}$.

The proof of the claim and the conclusion of the proof of lemma 10 go along the same lines as before. Details are left to the reader.

Q.E.D.(claim and lemma 10).

By exchanging if necessary the indices 0 and 1, it only remains for PRL, to prove that configuration C cannot occur if:

$$H: \begin{cases} f(F_0)\in E_1 \text{ and } f^m(F_0)\in E_1 \Rightarrow f^{m+1}(F_0)\in E_0 \\ f(F_1)\in E_0\ ;\ f^2(F_1)\in E_0 \end{cases},$$

In fact, we will need to split the configuration C according to the more precise configurations:

$$H_1: \begin{cases} f(F_0)\in E_1 \text{ and } f^m(F_0)\in E_1 \Rightarrow f^{m+1}(F_0)\in E_0 \\ f(F_1)\in E_0;\ f^2(F_1)\in E_0 \text{ and } f^3(F_1)\in E_0 \end{cases},$$

or:

$$H_2: \begin{cases} f(F_0)\in E_1 \text{ and } f^m(F_0)\in E_1 \Rightarrow f^{m+1}(F_0)\in E_0 \\ f(F_1)\in E_0;\ f^2(F_1)\in E_0 \text{ and } f^3(F_1)\in E_1 \end{cases},$$

but first, we need to proceed with a pair of lemmas which will be usefull in both cases.

Lemma 11:

Under the hypotheses of theorem A and assuming **H**, let j>1 be the smallest integer i greater than one, if any, such that $f^i(F_0)\in E_1$, and let j=2+n, n≥0. Let then p>1 be such that $f^p(F_0)\in E_1$; we know that $f^{p+1}(F_0)\in E_0$ and we denote m(p) the smallest positive integer m such that:

$$f^{p+1+m}(F_0)\in E_1,$$

if such an m exists, or m(p)=+∞ in the other case, then:

$$\forall p>1,\ m(p)\geq n.$$

Proof of lemma 11:

The case $m(p)=+\infty$ is obvious and we now consider that m(p) is finite. If the lemma is false, there exists a first $p=p_0$ such that $m(p)<n$. Then, one gets a contradiction and the conclusion of the proof of lemma 11 using:

Claim: the coincidence lemma holds true with the set P constructed with A={1}, and defined by:

$$P=\{(a,b)\in E_1{}^2 \mid a=f^q(F_0),\ b=f^r(F_0),\ 0\leq q<r\leq p_0\}.$$

Q.E.D.(claim and lemma 11).

Lemma 12:

Under the hypotheses of theorem A and assuming H, assume furthermore that for $k<r$,

$$f^k(F_1)\in E_1 \Rightarrow f^{k+1}(F_1)\in E_0.$$

Denote by $n\geq 2$ the number such that $f^s(F_1)\in E_0$ if $s\in\{1,2,...,n\}$, and $f^{n+1}(F_1)\in E_1$. Let j, $o<j<r$, be such that $f^j(F_1)\in E_1$. Finally, let m(j) be such that $f^t(F_1)\in E_0$ for $t\{j+1,...,j+m(j)\}$, and $f^{j+m(j)+1}(F_1)\in E_1$. Then $m(j)\leq n$.

Proof of lemma 12:

If the lemma is false, there exists a first $j=j_0<r$ such that $m(j_0)>n$. The contradiction which allows to conclude the proof of this lemma then comes from the:

Claim: the coincidence lemma holds true with the set P constructed with A={1}, and defined by:

$$P=\{(a,b)\in E_1{}^2 \mid a=f^p(F_1),\ b=f^r(F_1),\ 0\leq p<q\leq j_0\}.$$

Q.E.D.(claim and lemma 12).

Impossibility of case C assuming H_1:

Lemma 13:

Under the hypotheses of theorem A and assuming H_1, one has:

$$f^3(F_0)\in E_0.$$

Proof of lemma 13:

The lemma follows from the:

Claim: the coincidence lemma holds true with the set P constructed with A={0}, and defined by:

$$P=\{(f(F_1),F_0),(f^2(F_1),F_0)\}.$$

Q.E.D.(claim and lemma 13).

Lemma 14:

Under the hypotheses of theorem A and assuming H_1, if $f^m(F_0)\in E_1$ (in which case one knows that $f^{m+1}(F_0)\in E_0$), then:

$$f^{m+2}(F_0)\in E_0.$$

Proof of lemma 14:

This is a direct consequence of lemmas 11 and 13.

Q.E.D.(lemma 14).

One then has the following:

Lemma 15 (impossibility of case C assuming H_1):

Under the hypotheses of theorem A and assuming H_1, then for all $k\geq 0$:

$$f^k(F_1)\in E_1 \Rightarrow f^{k+1}(F_1)\in E_0.$$

Proof of lemma 15:

If the lemma is false, there exists a first $r=r_0$ such that:

$$f^{k_0}(F_1)\in E_1 \quad \text{and} \quad f^{k_0+1}(F_1)\in E_1.$$

One then concludes thanks to the:

Claim: the coincidence lemma holds true with the set P constructed with A={1}, and defined by:

$$P=\{(F_1,f^i(F_1))\in E_1{}^2 \mid i\le r_0\}\cup\{(f(F_0),f^j(F_1))\in E_1{}^2 \mid j\le r_0\}.$$

Q.E.D.(claim, lemma 15 and PRL assuming H_1).

An easy case for the impossibility of case C assuming H_2:

We shall now prove the impossibility of case C assuming H_2 and the supplementary hypothesis:

$$H_2{}' : \forall n\ge 0,\ f^{2n}(F_0)\in E_0 \quad \text{and} \quad f^{2n+1}(F_0)\in E_1.$$

Lemma 16:

Under the hypotheses of theorem A, case C is impossible assuming H_2 and $H_2{}'$.

Proof of lemma 16:

We have to show that there is no p>0 such that:

$$f^p(F_1)\in E_1 \quad \text{and} \quad f^{p+1}(F_1)\in E_1.$$

Because of $H_2{}'$, $\{f^{2n}(F_0)\}_{n\ge 0}$ and $\{f^{2n+1}(F_0)\}_{n\ge 0}$ are two Cauchy sequences, respectively in E_0 and E_1, which converge respectively to $L_0\in E_0$ and $L_1\in E_1$.

Since f is a quasi-contraction:

$$\forall n\ge 0,\ d_0(f^2(F_1),F_0)\le k.d_0(f(F_1),f^{2n+2}(F_0))\le$$
$$\le k^2.d_1(F_1,f^{2n+1}(F_0))\le k^3.d_0(f(F_1),f^{2n}(F_0)).$$

Hence $f(F_1)=F_0=L_0'$, so that the orbit of F_1 cannot have two consecutive points in E_1.

Q.E.D.(lemma 16).

So far we have succeded to eliminate all that could contradict the PRL, except for the (*) configuration.

Q.E.D.(step 2).

STEP 3:

Lemma 17 :

If for some $k_0\in[0,1[$, the reduction lemma holds true for $k< k_0$, then the PRL holds true for all $k< k_0{}^{1/2}$.

Proof of lemma 17:

Let f be a quasi-contraction with $k< k_0{}^{1/2}$. If f does not satisfiy the PRL, then f necessarily presents a configuration (*), for instance (with the roles of 0 and 1 as chosen at the begining of the chapter):

$$\begin{cases} f(F_0)\in E_1 \text{ and } f^m(F_0)\in E_1 \Rightarrow f^{m+1}(F_0)\in E_0 \\ f(F_1)\in E_0;\ f^2(F_1)\in E_0 \text{ and } f^3(F_1)\in E_1 \\ \text{and } \exists\, p>0:\ f^{2p}(F_0)\in E_0 \text{ and } f^{2p+1}(F_0)\in E_0 \end{cases}$$

and there exists a first $n=n_0$ such that:

$$f^{n_0}(F_1)\in E_1 \text{ and } f^{n_0+1}(F_1)\in E_1.$$

Let then define:

$$H^0=\cup_{m\geq 0}\{f^m(F_0)\} \cup \cup_{n_0\geq n\geq 0}\{f^n(F_1)\},$$

and

$$G_0^{(1)}=E_0 \cap H^0,\quad G_1^{(1)}=E_1\cap H^0.$$

We define a map:

$$g: G_0^{(1)}\cup G_1^{(1)} \rightarrow G_0^{(1)}\cup G_1^{(1)},$$

by:

$$g(P)=f(P) \quad \text{if} \quad P\neq f^p(F_1),$$
$$g(f^p(F_1))=F_0.$$

g satisfies conditions C1 and C2. Hozever, because $G_0^{(1)}$ and $G_1^{(1)}$ need not be closed, one cannot insure that g is a quasi-contraction. Thus we extend g to a map:

$$\overline{g}: \overline{G}_0^{(1)}\cup\overline{G}_1^{(1)} \rightarrow \overline{G}_0^{(1)}\cup\overline{G}_1^{(1)},$$

by setting:

$$\overline{g}\ (\lim_{P_i\in G_j^{(1)}}(P_i))=\lim{}^*(g(P_i)),$$

where lim* accounts for the following minor difficulty:

- if $\lim(g(P_i))$ does not depend on the defining sequence, then $\lim^*\equiv\lim$.
- in the other case, i.e. when $\lim(g(P_i))$ can be F_0 or F_1 according to the defining sequence for $\lim(P_i)$, we set $\lim^*(g(P_i))$ to be F_1 if the defining sequences are in $G_0^{(1)}$ or F_0 if the defining sequences are in $G_1^{(1)}$.

Of course, this difficulty disapears if one consider quasi-contractions as maps from the union of two spaces to their join.

$\overline{g}$ is a quasi-contraction on $\overline{G}_0^{(1)}\cup\overline{G}_1^{(1)}$ and thus satisfies lemma 11 and lemma 12. It follows that:

$$P\in \overline{G}_0^{(1)} \Rightarrow \begin{cases} \text{either } \overline{g}(P)\in\overline{G}_0^{(1)} \text{ and } \overline{g}^2(P)\in\overline{G}_1^{(1)} \\ \text{or } \overline{g}(P)\in\overline{G}_1^{(1)} \end{cases},$$

and we have also:

$$P\in \overline{G}_1^{(1)} \Rightarrow \overline{g}(P)\in \overline{G}_0^{(1)}.$$

Notice that, by construction, g satisfies the PRL. The proof of lemma 17 will be obtained by proving that such a map cannot exist.

We now define a new map:

$$\overline{g}^{(1)}: \overline{G}_0^{(1)}\cup\overline{G}_1^{(1)} \rightarrow \overline{G}_0^{(1)}\cup\overline{G}_1^{(1)},$$

by:

$$g^{(1)}(P)=\begin{cases} \overline{g}\,(P) & \text{if} \quad P\in \overline{G}_0{}^{(1)} \\ \overline{g}\,{}^2(P) & \text{if} \quad P\in \overline{G}_1{}^{(1)} \end{cases}.$$

Defining:

$$H^{(1)}=\cup_{m\geq 0}\overline{\{(g^{(1)})^m(F_0)\}}\cup\cup_{n\geq 0}\overline{\{(g^{(1)})^n(F_1)\}},$$

and

$$G_0{}^{(2)}=E_0\cap H^{(1)} \quad , \quad G_1{}^{(2)}=E_1\cap H^{(1)},$$

we remark that:

$$g^{(1)}(G_0{}^{(2)}\cup G_1{}^{(2)}) \subset G_0{}^{(2)}\cup G_1{}^{(2)},$$

and:

$$g^{(1)}(G_0{}^{(2)}) \subset G_1{}^{(2)}.$$

Notice that $G_0{}^{(2)}$ and $G_1{}^{(2)}$ are closed sets, and that $g^{(1)}$ is a quasi-contraction on $G_0{}^{(2)}\cup G_1{}^{(2)}$. This allows us to introduce a further map:

$$g^{(2)}:G_0{}^{(2)}\cup G_1{}^{(2)} \to G_0{}^{(2)}\cup G_1{}^{(2)},$$

by:

$$g^{(2)}(P)=\begin{cases} (g^{(1)})^2(P) & \text{if} \quad P\in G_0{}^{(2)} \\ (g^{(1)})(P) & \text{if} \quad P\in G_1{}^{(2)} \end{cases}.$$

One can check that $g^{(2)}$ is a quasi-contraction with constant $k^2<k_0$. As a consequence $g^{(2)}$ satisfies Theorem A.

Hence, the pair $(I_{g(2)}(F_0),I_{g(2)}(F_1))$ of symbolic sequences is quasi-rotation compatible. Furthermore, one can reconstruct the itineraries $I_g(F_0)$ and $I_g(F_1)$ using successively the two inflation rules :

$$\mathbf{0:}\begin{cases} 0 & \to & 01 \\ 1 & \to & 1 \end{cases} \qquad \mathbf{1:}\begin{cases} 0 & \to & 0 \\ 1 & \to & 10 \end{cases}$$

By definition of the set $A = SG(\ W^2{}_0)$, the pair $(I_g(F_0),I_g(F_1))$ of symbolic sequences is also quasi-rotation compatible.

Let us then remark that because we are dealing with a configuration (*), $I_g(F_0)$ reads:

$$I_g(F_0)= 010......1001... \ .$$

By construction, F_0 belongs to the orbit of F_1 under g, and going backward on this orbit startiong from F_0 yields successively $g^{-1}(F_0)\in E_1$, $g^{-2}(F_0)\in E_0$. Consequently, one gets:

$$I_g(F_1)= 10... ...01I_g(F_0),$$

which violates the minimality of $I_g(g(F_1))$ among all the shifts of $I_g(F_1)$ starting with the symbol 0 (see Theorem 2 ,§6). This contradiction concludes the proof of lemma 17.

Q.E.D.(lemma 17).

STEP 4:

Lemma 18:

If for some $k_0\in[0,1[$, the reduction lemma holds true for $k< k_0$, then the RL holds true for all $k< k_0{}^{1/2}$.

Proof of lemma 18:

We remark that in order to describe $I(F_0)$ and $I(F_1)$, we do not need the complete reduction lemma; the pre-reduction lemma is quite enough.

Consequently, if f is a quasi-contraction with contraction constant $k< k_0^{1/2}$, lemma 17and §3 tell us that f satisfies the statement 2) of Theorem A. Hence, we can split the proof of lemma 18 according to:

1st case: both $I(F_0)$ and $I(F_1)$ are eventually periodic,

2nd case: none is.

In both cases, defining G_0 and G_1 as usual, and assuming $f(G_i)\subset G_j$ for $(i,j)\in \{0,1\}^2$, we have to show that:

$$P\in \overline{G}_i \quad \Rightarrow \quad f(P)\in \overline{G}_j .$$

In the first case, if $f(P)\in \overline{G}_i$, then there are $k\in \{0,1\}$, $p>0$, $n>0$ such that:

$$\lim_{m\to\infty}(f^{p+m.n}(F_k))=P.$$

If we assume that:

$$f(P)\notin \overline{G}_j ,$$

then necessarily:

$$f(P)\in \overline{G}_{1-j}.$$

In fact,by C2 we have:

$$f(P)=F_{1-j},$$

and:

$$\lim_{m\to\infty}(f^{p+m.n+1}(F_k))=F_j.$$

This implies that there is some $q<n$ and $s\in \{0,1\}$ such that:

$$\lim_{m\to\infty}(f^{p+m.n}(F_k))=f^q(F_s).$$

Since the orbits of the F_k's belong to $G_0\cup G_1$, we have reached a contradiction.

In the second case, we know that $j=1-i$. For definiteness, let us assume $i=1$, i.e.:

$$f(G_1)\subset G_0$$

Then the pair $(I(F_0),I(F_1))$ of symbolic sequences is quasi-rotation compatible and the rotation numbers of $I(F_0)$ and $I(F_1)$ are the same irrational number.

In particular, among all points in the orbits of F_0 and F_1, F_0 yields the maximal itinerary begining with 0 (see Theorem 2,§6).

Assume now that $f(P)\notin \overline{G}_0$. Again C2 implies that $f(P)= F_1$. Furthermore, for any sequence $\{P_n\}_{n>0}$ in G_1 with:

$$\lim_{m\to\infty}P_n=P,$$

we have:

$$\lim_{m\to\infty}f(P_n)=F_0.$$

Since there is no m>0 with $f^m(F_0)\in \{F_0,F_1\}$, for any N>0, there is a N'>0 such that for all n>N', the itineraries of F_0 and P_n coincide on at least the first N symbols.

Furthermore:

- the itinerary of F_0 contains some pair of consecutive zeros, by irrationality of the rotation number,
- there is a sequence $\{Q_n\}_{n>0}$ in G_0 with:

$$\forall n>0\ ,\ f(Q_n)=P_n.$$

Let then N be such that there exists a pair of consecutive zeros among the first N symbols of $I(F_0)$. Then, for n>N we have $I(Q_n)>I(F_0)$ since the first N+2 symbols of $I(Q_n)$ are 01 followed by the first N symbols of $I(F_0)$.

This contradicts the maximality of $I(F_0)$ (see Theorem 2,§6), which concludes the proof of lemma 18.

Q.E.D.(lemma 18).

STEP 5:

Iterating ad infinitum the succession of steps 3 and 4 yields the complete proof of the reduction lemma.

Q.E.D.(RL & Theorem A).

5. APPLICATION TO BIFURCATION THEORY.

Let us consider a C^1 flow X_0 in R^3 with a critical point P of saddle-focus type and a pair of homoclinic orbits Γ_i, $i\in\{0,1\}$ bi-asymptotic to P as $t\to\pm\infty$, as represented in figure 2.

FIGURE 2.comes here.

See the end of the paper

One can look for the structure of the orbits which remain close to $\Gamma_0\cup\Gamma_1$ for any flow X which is C^1-close to X_0. Since the shapes of Γ_0 and Γ_1 are essentially irrelevant for this problem, a natural description of such orbits is to indicate how they successively follow "closely" Γ_0 and Γ_1 as time goes on.

For instance the two periodic orbits represented in figures 3-a and 3-b can be described respectively by:

$$\Gamma_0\Gamma_0\Gamma_1\Gamma_1\Gamma_0\Gamma_0\Gamma_1\Gamma_1\Gamma_0\Gamma_0\ldots\ldots,$$

and

$$\Gamma_0\Gamma_1\Gamma_0\Gamma_1\Gamma_0\Gamma_1\Gamma_0\Gamma_1\Gamma_0\Gamma_1\ldots\ldots,$$

or in short:

$$(0011)^\infty,$$

and:

$$(01)^\infty.$$

FIGURES 3-a and 3-b.come here.

See the end of the paper

One can study this problem by considering the first return map T on a small cylinder C surrounding the local unstable manifold of P. C is cut in two parts, say C_0 and C_1, by the local stable manifold of P and the encoding of the orbits of T in terms of C_0 and C_1 obviously corresponds to the encoding of the orbits of the flow in terms of Γ_0 and Γ_1. In some cases, one has to restrict T to a neighborhood $\underline{C}_0 \cup \underline{C}_1$ of $\overline{C}_0 \cap \overline{C}_1$ in C.

In fact, T is a mapping from $\underline{C}_0 \cup \underline{C}_1 \setminus \overline{C}_0 \cap \overline{C}_1$ to $C= C_0 \cup C_1$ which is continuous on $\underline{C}_i \setminus \overline{C}_0 \cap \overline{C}_1$ for $i\in\{0,1\}$.

If then E_i (respectively $\underline{E}_i$) is the disjoint union of C_i (respectively $\underline{C}_i$) and a point F_i which is the limit of any sequence converging to $\overline{C}_0 \cap \overline{C}_1$, T extends uniquely to a map:

$$\underline{f} : \underline{E}_0 \cup \underline{E}_1 \rightarrow E_0 \cup E_1 / (F_0 = F_1),$$

such that the restrictions of $\underline{f}$ to the $\underline{E}_i$ are continuous for the natural topologies.

Typical questions about $\underline{f}$ are to describe all possible codes of orbits and, more particularly, orbits in the ω-limit set of $\underline{f}$. The same problem can be formulated for flows on any smooth manifold with dimension greater than one. In particular, the spiraling in of the Γ_i's in the stable manifold of P (figure 2) is just an example. What is relevant is the existence of a single unstable direction near P, and that the restrictions of $\underline{f}$ to the $\underline{E}_i$ be contractions with :

$$\underline{f}(\underline{E}_0 \cup \underline{E}_1) \subset E_0 \cup E_1 / (F_0 = F_1),$$

This is always the case if C is small enough and if the unique positive eigenvalue of $DX_0(P)$ is closer to the imaginary axis than any other of its eigenvalues. In such case, understanding $\underline{f}$ amounts to understand the dynamics of quasi-contractions. Details can be found in [G,G,T.2], which contains a more complete bibliography (see also [G], [T,S], and a paper to appear [G,G,R,T]). Notice that part of theorem A as been proved in [G,G,T.2] for **k**<1/2. This is enough for most applications to flows. Going from **k**<1/2 to **k**<1 has pushed us to consider more closely the orbits of C_0 and C_1 and F_1. In terms of flows, this corresponds to the description of the possible symbolic dynamics of the two branches of the unstable manifold of P for X close to X_0. We leave to the reader the task of formulating all implications of Theorem A for flows.

6. ABOUT THE SYMBOLIC DYNAMICS OF ROTATIONS.

Let :

$$r_\alpha : \mathbf{T}^1 \to \mathbf{T}^1,$$

denote the rotation with one lift to the universal cover $\mathbf{R}$ of $\mathbf{T}^1$ given by:

$$R_\alpha : \mathbf{R} \to \mathbf{R}, \quad x \to x+\alpha$$

where α is chosen to belong to [0,1] for reasons which will soon become transparent.

Let us now introduce, beside the usual:

$$\mathrm{mod1} : \mathbf{R} \to [0,1[,$$

the unique transformation:

$$\underline{\mathrm{mod1}} : \mathbf{R} \to]0,1],$$

which agrees with mod1 except on $\mathbf{Z}$, and is continuous on]0,1] .

Then to any R_α there correspond canonically two discontinuous maps:

$$R'_\alpha = (\mathrm{mod1}) \circ R_\alpha|_{[0,1]},$$

and:

$$\underline{R}'_\alpha = (\underline{\mathrm{mod1}}) \circ R_\alpha|_{[0,1]},$$

from the unit interval to itself.

R'_α induces a decomposition of [0,1] as the disjoint union:

$$[0,1-\alpha[\cup [1-\alpha, 1] = I_0 \cup I_1,$$

such that $R'_\alpha|_{I_i}$, $i \in \{0,1\}^2$ is continuous. Correspondingly for $\underline{R}'_\alpha$, we will have $[0,1] = \underline{I}_0 \cup \underline{I}_1$, with $\underline{I}_0 = [0,1-\alpha]$ and $\underline{I}_1 =]1-\alpha,1]$.

Now, with f standing for a R_α or a $\underline{R}'_\alpha$ and $J_0 \cup J_1$ for the corresponding splitting of [0,1], we will parallel Definition 4 by the following:

Definition 5 : The **f-address** of a point x in [0,1], denoted by $\mathbf{a_f}(x)$, is 0 or 1 according to wether x belongs to J_0 or J_1. The **f-itinerary** of x, denoted by $\mathbf{I_f}(x)$, is the element of **W** defined by:

$$\mathbf{I_f}(x) = (\mathbf{a_f}(x), \mathbf{a_f}(f(x)), \mathbf{a_f}(f^2(x)), \ldots\ldots).$$

Definition 6 : To f we associate the two symbolic sequences:

$$\mathbf{k_0}(f) = 0\mathbf{I_f}(1) \text{ and } \mathbf{k_1}(f) = 1\mathbf{I_f}(0),$$

and we remark that, except for the two first symbols, $\mathbf{k_0}(f)$ and $\mathbf{k_1}(f)$ coincide. The pair $(\mathbf{k_0}(f), \mathbf{k_1}(f))$ is called the **kneading pair** of f.

Definition 7 : As usual we shall denote by σ the (positive) **shift** on **W**, i.e. the endomorphism of **W** defined by:

$$\sigma(a_1,a_2,a_3,...)=(a_2,a_3,a_4,...).$$

Definition 8 : We shall say that a sequence $W\in\mathbf{W}$ is **m-deflatable** if there exists a m-uple $M(W,m)=(N_1,N_2,N_3,...,N_m)$ with $N_i\in\{0,1\}$, such that W is in the domain of $\mathbf{M}(M(W,m))=N_m o...oN_3oN_2oN_1$, and that W is **infinitly deflatable** if it is m-deflatable for all m>0. Defining $M(W,0)=\emptyset$, and $\mathbf{M}(\emptyset)$ as the identity, an infinitly deflatable word W is of **constant type** if and only if there exist $m_0\geq 0$, and $N\in\{0,1\}$ such that for a M(W,m+1) and a M(W,m), one has $\mathbf{M}(M(W,m+1))=No\mathbf{M}(M(W,m))$ for all $m>m_0$. This definition has an obvious extention to pairs.

This definition is made somewhat cumbersome by the ambiguity in the choice of M(W,m). However this ambiguity is not severe, since M(W,1) is uniquely determined by W exept for $W\in\{(01)^\infty, (10)^\infty\}$ (cf. the ambiguity in the representation of a rational number by a continued fraction). Notice that no ambiguity is involved in the case of sequences which are not of constant type (cf. the non-ambiguity in the representation of irrational numbers by a continued fraction).

Our goal in this section is to identify the set of quasi-rotation compatible pairs. This in turn is by now standard material (see e.g. [G] or [P,T,T] for a summary and bibliography). In fact we have the following:

Theorem 1 [G]**:** Quasi-rotation compatible pairs are of the form :

$$\{\mathbf{k_0}(f), \mathbf{k_1}(f')\}$$

where (f,f') belongs to the set **B** defined by:

$$\mathbf{B}=\{(R'_\alpha,R'_\alpha),(R'_\alpha,\underline{R}'_\alpha),(\underline{R}'_\alpha,R'_\alpha),(\underline{R}'_\alpha,\underline{R}'_\alpha),(R'_{p/q},R'_{p'/q'}),\alpha\in[0,1],$$
$$(p/q,p'/q')\in(\mathbb{Q}\cap[0,1])^2,\ |pq'-p'q|=1\}.$$

This result motivates the name chosen, and reduces the understanding of quasi-rotation compatible pairs to the understanding of the symbolic dynamics of rotations, and more precisely, of the R'_α's and $\underline{R}'_\alpha$'s. In fact,thanks to what we already learned in section 4, Theorem 1 simply follows from recognizing that the set of $\mathbf{k_0}$(f)'s and $\mathbf{k_1}$(f)'s consists in the closure of the set $\mathbf{C} = \mathbf{SG}(\mathbf{W}_1)$ of all orbits under **SG** of points in $\mathbf{W}_1$, with:

$$\mathbf{W}_1 = \{0^\infty, 01^\infty, 10^\infty, 1^\infty\}.$$

Finally we resume here the known facts about kneading pairs that are needed in the proof of theorem A. This is the content of the following:

Theorem 2 [G]**:** The set of kneading pairs $(\mathbf{k_0}(f), \mathbf{k_1}(f))$ for some $\alpha\in\,]0,1[$, and $f\in\{R'_\alpha,\underline{R}'_\alpha\}$ is precisely the set of pairs (W,W') in $\mathbf{W}^2$ such that:

- $\sigma^2(W) = \sigma^2(W') = A$, and $W = 01A$, $W' = 10A$,
- (W,W') is infinitely deflatable,
- $\sigma(W)= \sup_{n\geq 0}\sigma^n(W)$; $\sigma(W')= \inf_{n\geq 0}\sigma^n(W')$.

Then α is rational or irrational according to wether W is of constant type or not. Furthermore, in the case when α is irrational, $(\mathbf{k_0}(f), \mathbf{k_1}(f))$ is uniquely determined by α, i.e. do not dependon wether $f= R'_\alpha$ or $f=\underline{R}'_\alpha$.

At last, we have:

$$(\mathbf{k_0}(f), \mathbf{k_1}(f)) = (010^{\infty},10^{\infty}) \quad \text{if} \quad f= R'_0,$$

$$(\mathbf{k_0}(f), \mathbf{k_1}(f)) = (0^{\infty},10^{\infty}) \quad \text{if} \quad f= \underline{R}'_0,$$

$$(\mathbf{k_0}(f), \mathbf{k_1}(f)) = (01^{\infty},1^{\infty}) \quad \text{if} \quad f= R'_1,$$

$$(\mathbf{k_0}(f), \mathbf{k_1}(f)) = (01^{\infty},101^{\infty}) \quad \text{if} \quad f= \underline{R}'_1.$$

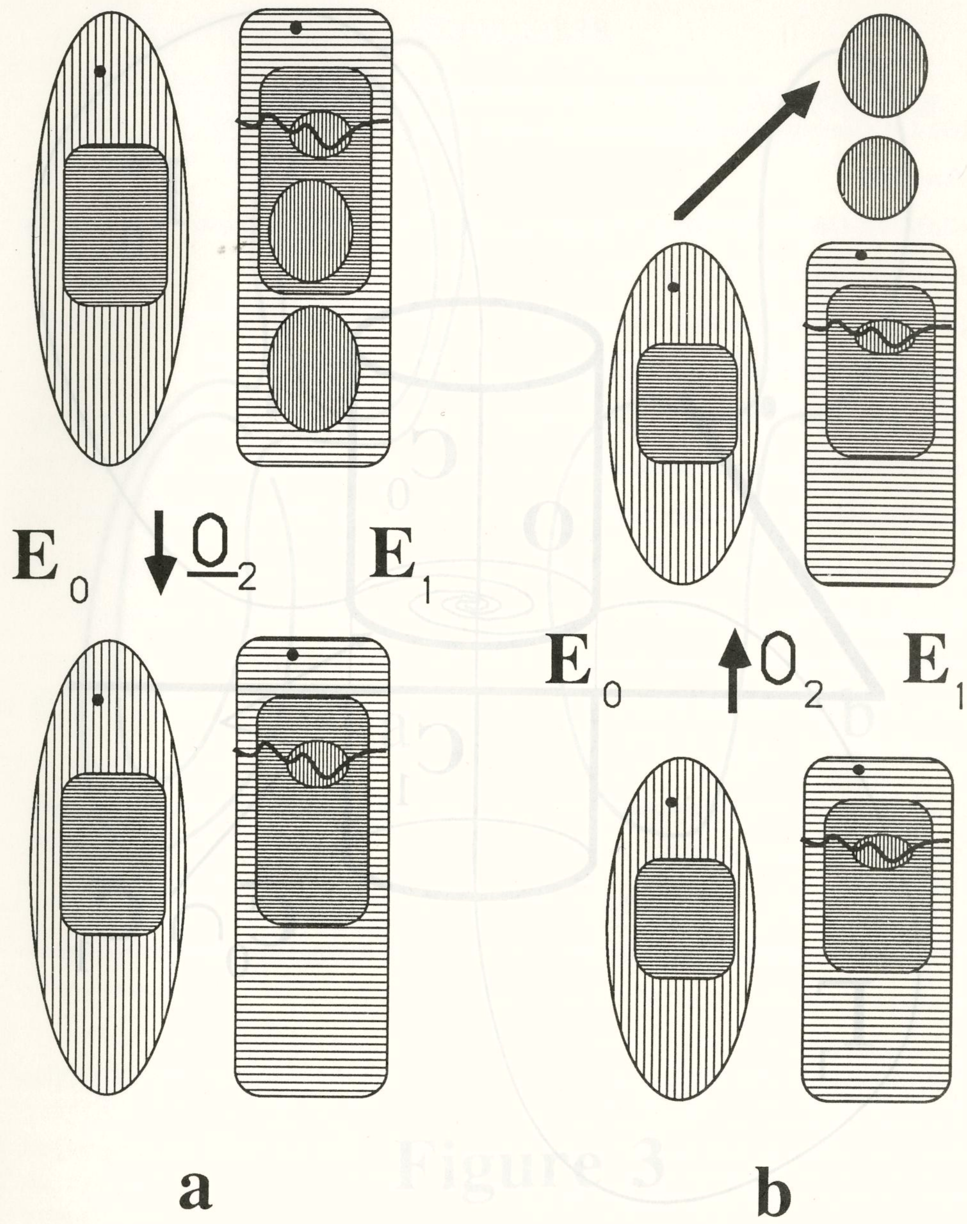

Figure 1

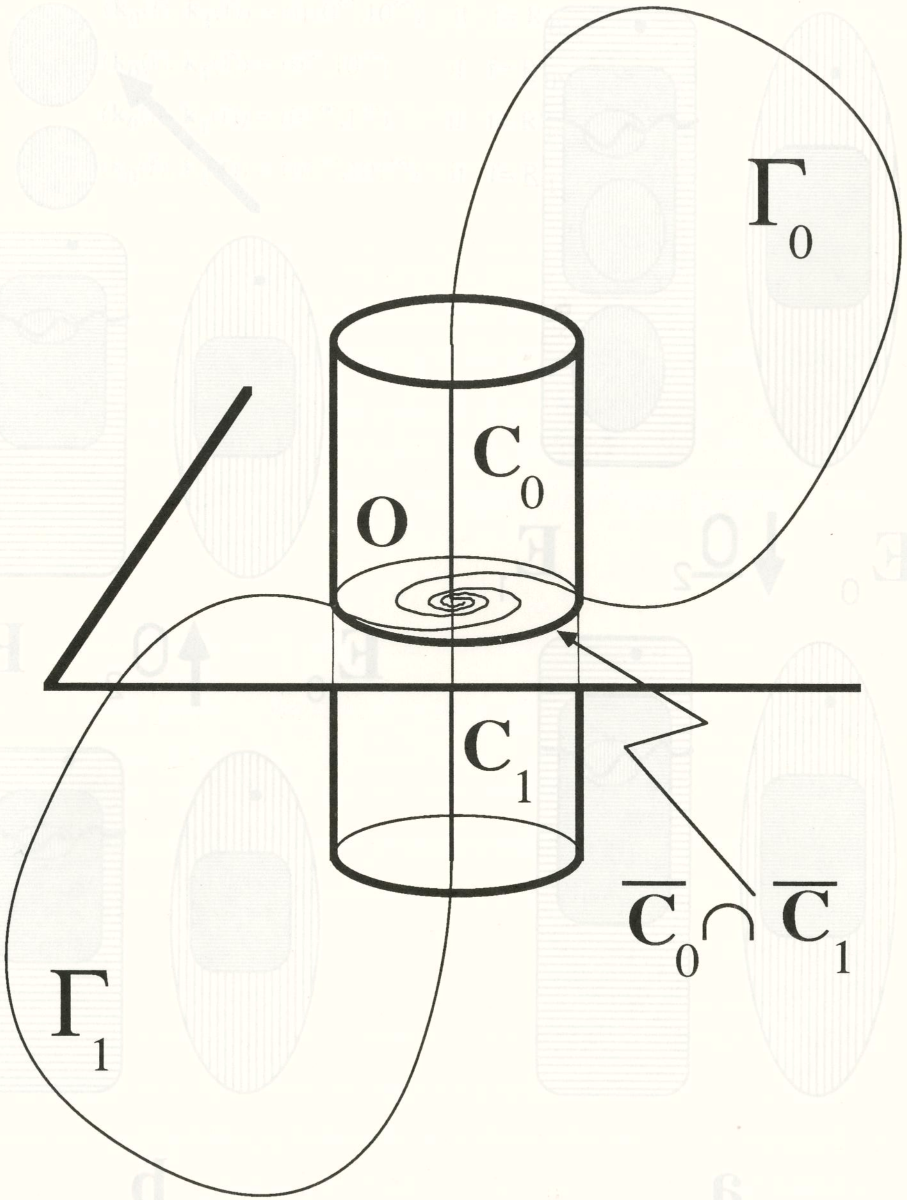

Figure 2

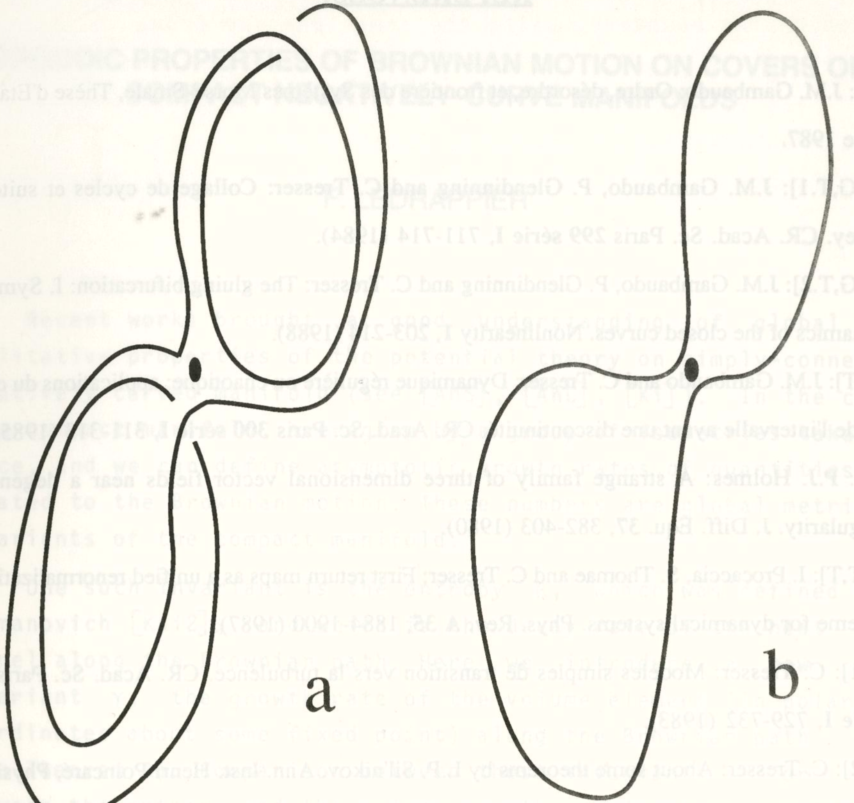

Figure 3

REFERENCES.

[G]: J.M. Gambaudo: Ordre, désordre, et frontière des Systèmes Morse-Smale, Thèse d'Etat, Nice 1987.

[G,G,T.1]: J.M. Gambaudo, P. Glendinning and C. Tresser: Collage de cycles et suites de Farey. CR. Acad. Sc. Paris **299 série I**, 711-714 (1984).

[G,G,T.2]: J.M. Gambaudo, P. Glendinning and C. Tresser: The gluing bifurcation: I. Symbolic dynamics of the closed curves. Nonlinearity **1**, 203-214 (1988).

[G,T]: J.M. Gambaudo and C. Tresser: Dynamique régulière ou chaotique: applications du cercle ou de l'intervalle ayant une discontinuité. CR. Acad. Sc. Paris **300 série I**, 311-313 (1985).

[H]: P.J. Holmes: A strange family of three dimensional vector fields near a degenerate singularity. J. Diff. Equ. **37**, 382-403 (1980).

[P,T,T]: I. Procaccia, S. Thomae and C. Tresser: First return maps as a unified renormalization scheme for dynamical systems. Phys. Rev. **A 35**, 1884-1900 (1987).

[T.1]: C. Tresser: Modèles simples de transition vers la turbulence. CR. Acad. Sc. Paris **296 série I**, 729-732 (1983).

[T.2]: C. Tresser: About some theorems by L.P. Sil'nikov. Ann. Inst. Henri Poincaré, Physique Théorique. **40**, 441-461 (1984).

[T,S]: D. V. Turaev and L.P. Sil'nikov: On bifurcations of a homoclinic "figure eight" for a saddle with a negative saddle value.Dokl. Akad. Nauk SSSR **290**, 1301-1305 (1986).(English translation Soviet Math. Dokl. **34**, 397-401 (1987)).

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