ERGODIC PROPERTIES OF BROWNIAN MOTION ON COVERS OF COMPACT NEGATIVELY-CURVE MANIFOLDS

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Recent works brought a good understanding of global qualitative properties of the potential theory on simply-connected, negatively curved manifold (see [AnS], [AnC], [Ki]). In the case of a compact manifold, averaging by a group of isometries takes place, and we can define asymptotic growth rates of quantities related to the Brownian motion. These numbers are global metric invariants of the compact manifold.

One such invariant is the entropy β, which was defined by Kaimanovich [Kai2], and which is the decay rate of the heat kernel along the Brownian path. Here we introduce a new invariant γ, the growth rate of the volume element (in polar coordinates about some fixed point) along the Brownian path (see Theorem A). Comparison between those invariants — or between the entropy and the volume growth — reflects properties of different measure classes on the ideal boundary, for instance β equals γ if and only if the harmonic measure class and the visibility measure class coincide on the ideal boundary (Theorem B_1).

The idea behind the proof of Theorems A and B is that a typical Brownian path roughly follows a geodesic and that this geodesic itself is typical with respect to some invariant measure under the geodesic flow (see Theorem C). Identifying this measure, we are able to translate its ergodic properties into corresponding properties of the Brownian motion. This shows for instance that

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the harmonic measure is ergodic under the action of the fundamental group on the boundary. Using the same idea, we prove in [KL] a result on the Hausdorff dimension of the harmonic measure.

The paper by Kaimanovich [Kai2] was extremly influencial. I also owe the understanding of most important technical points to conversations with A. Ancona, Y. Kifer and W. Veech.

1. Notations and statement of results

1.1. *Setting*

Throughout the paper, we consider a compact connected n-dimensional Riemannian manifold M with negative sectional curvatures. We denote:

. $-b^2$ $(-a^2)$ the minimum (maximum) of the sectional curvatures,

. SM the spherical bundle of M, endowed with the natural metric and the projection $p: SM \to M$,

. $\Omega = C(\mathbb{R}_+, M)$ the space of continuous paths,

. $\{\mathbb{P}_x, x \in M\}$ the family of probability measures on Ω, which describe the Brownian motion on M (see e.g. [Pi]),

. m the normalized Lebesgue measure on M,

. $\mathbb{P} = m(d\omega_0)\mathbb{P}_{\omega_0}(d\omega)$ the stationary Brownian motion on M, and

. $\{\psi_t, t \geq 0\}$ the semi-group of shift transformations on Ω.

Since M is connected, the semi-flow $(\Omega, \mathbb{P}, \{\psi_t, t > 0\})$ is ergodic. We also denote:

. $\tilde M$ the universal cover of M,

. $\pi: \tilde M \to M$ the projection,

. Γ the group of deck transformations on $\tilde M$,

. for γ in Γ, $D\gamma$ the spherical action of $D\gamma$ on the spherical bundle $S\tilde M$,

and

. $\tilde\Omega = C(\mathbb{R}_+, \tilde M)$ the space of continuous paths in $\tilde M$.

For x in $\tilde M$, we consider geodesic polar coordinates about x, i.e. we identify $T_x\tilde M$ with $\mathbb{R}_+^* \times S_x\tilde M \cup \{0\}$ and a point z in $\tilde M$ is described by the polar coordinates of $\exp_x^{-1} z$. For all x in $\tilde M$ and all ω in Ω such that $\omega(0) = \pi x$, there is a unique path $\tilde\omega$ in $\tilde\Omega$ such that

$$\pi\tilde\omega(t) = \omega(t) \quad \text{for all } t \geq 0;$$

we denote $(r(\omega,t), \theta(\omega,t))$ the geodesic polar coordinates about x of the point $\tilde\omega(t)$. Remark that for all x in $\tilde M$, $\mathbb{P}_{\pi x}$-a.e. ω satisfies $\omega(0) = \pi x$, so that (r,θ) is defined $\mathbb{P}_{\pi x}$-a.e.

For all x in $\tilde M$ let $\tilde\lambda_x$ be the Lebesgue measure on $S_x\tilde M$. We denote $A(x,z)$ the function on $\tilde M \times \tilde M$ such that

$$dV(\exp t\xi) = A(x,(t,\xi))dt\,\tilde\lambda_x(d\xi)$$

where dV is the Riemannian volume element on $\tilde M$. We write $V(x,t)$ for the volume of the ball of radius t about x:

$$V(x,t) = \int_0^t \left(\int (A,(s,\xi))\tilde\lambda_x(d\xi) \right) ds.$$

Finally let $p(t,x,z)$ be the heat kernel on $\tilde M$, i.e. the fundamental solution of $\frac{\partial u}{\partial t} = \Delta u$.

Also, the distribution of $\tilde\omega(t)$ under $\mathbb{P}_{\pi x}$ is $p(t,x,z)\, dV(z)$.

1.2. *Asymptotic quantities*

We list in this section the properties at infinity of the quantities introduced in 1.1.

(1.2.1) There exists a number α satisfying $(n-1)a \leq \alpha \leq (n-1)b$ and for all x in $\tilde{M}$, $\mathbb{P}_{\pi x}$-*a.e.* ω, $\lim_{t\to\infty} \frac{1}{t} r(\omega,t) = \alpha$.

(1.2.1) follows readily from the subadditive ergodic theorem and the expression for r (see [Pi]).

(1.2.2) [Pr] For all x in $\tilde{M}$, $\mathbb{P}_{\pi x}$-*a.e.* ω, $\theta(\omega,t)$ converges, as t goes to infinity towards some limit variable $\theta(\omega,\infty)$.

(1.2.3) Theorem A: *There exists a number* γ *such that for all* x *in* $\tilde{M}$, $\mathbb{P}_{\pi x}$-*a.e.* ω,

$$\lim_{t\to\infty} \frac{1}{t} \ell n\, A(x,\tilde{\omega}(t)) = \gamma.$$

(1.2.4) [Kai$_2$] There exists a number β such that for all x in $\tilde{M}$, $\mathbb{P}_{\pi x}$-*a.e.*

$$\lim_{t\to\infty} -\frac{1}{t} \ell n\, p(t,x,\tilde{\omega}(t)) = \beta.$$

(1.2.5) [Ma] There exists a number h such that for all x in $\tilde{M}$

$$\lim_{t\to\infty} \frac{1}{t} \ell n\, V(x,t) = h.$$

The number h is the topological entropy of the geodesic flow on SM.

Proposition 1: *Let* M *be a compact connected manifold with negative curvature. Then:*

(1.2.6) $\beta \leq \gamma$

(1.2.7) $\beta \leq \alpha h$.

See [Kai$_2$] for (1.2.7). The proof of (1.2.6) is analogous. See also section 4.4.

The results in the next section characterize the case of equality in (1.2.6 or 7) by equivalences of measures on the boundary at infinity of $\tilde{M}$.

1.3. *Measure classes at infinity*

Two geodesics γ and γ' on $\tilde{M}$ are said to be equivalent if $\sup_{t\geq 0} d(\gamma(t),\gamma'(t)) < +\infty$.

The space of equivalence classes in called the *ideal boundary* denoted $\tilde{M}(\infty)$ (see e.g. [BGS]). For all $\tilde{X}$ in $S\tilde{M}$, let $\gamma_{\tilde{X}}$ be the geodesic in $\tilde{M}$ defined by $(\gamma_{\tilde{X}}(0),\gamma'_{\tilde{X}}(0)) = \tilde{X}$. We denote by $\tau: S\tilde{M} \to \tilde{M}(\infty)$ the map which associates to $\tilde{X}$ the class of $\gamma_{\tilde{X}}$. For x in $\tilde{M}$, the restriction τ_x of τ to $S_x\tilde{M}$ is a homeomorphism between $S_x\tilde{M}$ and $\tilde{M}(\infty)$.

(1.3.1) For x, z in $\tilde{M}$, the measures $\tilde{\lambda}_x.\tau_x^{-1}$ and $\tilde{\lambda}_z.\tau_z^{-1}$ are equivalent on $\tilde{M}(\infty)$. The measure class they defined is called the *visibility class* on $\tilde{M}(\infty)$ (see section 2.2).

It is known that the visibility class is ergodic under the action of Γ. We even have ergodicity of the product class $\lambda \times \lambda$ on $\tilde{M}(\infty)\times\tilde{M}(\infty)$ under the product action of Γ defined by $\gamma(\xi,\xi') = (\gamma\xi,\gamma\xi')$ (see the discussion in section 3.3).

For x in $\tilde{M}$, let $\tilde{\mu}_x$ be the distribution on $S_x\tilde{M}$ of the limit direction $\theta(\omega,\infty)$ under $\mathbb{P}_{\pi x}$ (see (1.2.2)).

(1.3.2) For (x,z) in $\tilde{M}$, the measures $\tilde{\mu}_x.\tau_x^{-1}$ and $\tilde{\mu}_z.\tau_z^{-1}$ are equivalent on $\tilde{M}(\infty)$. The measure class they define is called the *harmonic class* on $\tilde{M}(\infty)$.

Property (1.3.2) follows from the Markov property and bounded geometry (see e.g. [An], [Su]).

Let $\tilde{X} \in S\tilde{M}$ and $R > 0$. In [H] is defined a distance $\eta^{a}_{\tilde{X},R}$ on $\tilde{M}(\infty) \setminus \tau(-\tilde{X})$ such that:

(1.3.3) [H] For $\tilde{X},\tilde{Z}$ in $S\tilde{M}$, the h/a dimensional spherical measures associated to $\eta^{a}_{\tilde{X},R}$ and $\eta^{a}_{\tilde{Z},R}$ are equivalent.

The measure class they define is called the Bowen-Margulis class.

See section (2.3) for another equivalent description of the Bowen-Margulis class (and the justification of its name!).

Theorem B. *Consider the universal cover of a compact connected negatively curved manifold.*

1º) *We have equality* $\beta = \gamma$ *if and only if the visibility class and the harmonic class coincide.*

2º) *We have equality* $\beta = \alpha h$ *if and only if the harmonic class and the Bowen-Margulis class coincide.*

3º) *The harmonic measure class* μ *is ergodic under the action of* Γ *on* $\tilde{M}(\infty)$. *The product measure class* $\mu \times \mu$ *is ergodic under the product action of* Γ *on* $\tilde{M}(\infty) \times \tilde{M}(\infty)$.

It follows in particular from statement B3 that if neither 1 nor 2 is realized, the harmonic measure is singular with respect to the visibility class and the Bowen-Margulis class. In the case of surfaces, it is known ([Ka_1],[Ka_2], [Le]) that the visibility class coincide with either the harmonic class or the Bowen-Margulis class only in the case of constant curvature. A tool in the proofs is to characterize metrics of constant curvatures as an extreme point for some property. Theorem B tries to go in the same direction for negatively-curved manifolds of higher dimension.

1.4. *Ergodic theory of the harmonic measure*

We first remark that the family $\{\tilde{\mu}_x, x \in \tilde{M}\}$ is Γ-invariant. Therefore it defines a family $\{\mu_y, y \in M\}$ of probability measures on SM such that $\mu_y(S_yM) = 1$ for all y.

We call this family the *spherical harmonic measures*

The *geodesic flow* is the one-parameter group $\{\phi_t, t \in \mathbb{R}\}$ of transformations of SM defined by $\phi_t X = (\gamma_X(t), \gamma'_X(t))$. The main step in the proofs of Theorems A and B is the construction of some ϕ-invariant probability measure μ on SM related to the spherical harmonic measures.

Theorem C: *Let* M *be a compact connected negatively-curved manifold,* $\{\mu_y, y \in M\}$ *the spherical harmonic measure. There exists a unique* ϕ*-invariant probability measure* μ *on* SM *satisfying the following equivalent properties:*

1. *For all* y *in* M

$$\text{weak}\lim_{t\to+\infty} \frac{1}{t}\int_0^t (\mu_y \cdot \phi_s^{-1})\, ds = \mu.$$

2. *For all* y *in* M

$$\text{weak}\lim_{t\to+\infty} \frac{1}{t}\int_0^t (\mu_y \cdot \phi_{-s}^{-1})\, ds = \mu.$$

3. *For all continuous function* f *on* SM, *all* x *in the universal cover* $\tilde{M}$ *of* M *and* $\mathbb{P}_{\pi x}$*-a.e.* ω *in* $\Omega = C(\mathbb{R}_+, M)$,

$$\int f d\mu = \lim_{t\to\infty} \frac{1}{\alpha t}\int_0^{r(\omega,t)} f[\phi_s(D\pi(x,\theta(\omega,t)))]\, ds.$$

Moreover (SM,ϕ,μ) *is ergodic.*

In section 2, we recall the results from the ergodic theory of geodesic flows on negatively-curved manifold we shall use, in particular the quasi-invariance of Gibbs measures under the stable foliation. In section 3 we show we can find a Hölder continuous function F on SM such that "the equilibrium state of F admits the spherical harmonic measure as transverse measures". As shown in section 4, the results follow from this property. The measure μ in Theorem C will be the equilibrium state of F.

In the case when M is a compact rank-one symmetric space, the measure μ in Theorem C is the Liouville (Haar) measure and properties C. 1.2.3 only express the ergodicity of (SM,ϕ,μ). W. Veech (personal communication) remarked that a stronger property than C1 holds, namely that weak limit $\mu_y \cdot \phi_t^{-1} = \mu$ $t \to +\infty$

for all y in M, and raised the question of the general negative curvature analog of this property. We cannot answer the question, but Theorem C says that the only invariant measure one could find as a limit of images of spherical harmonic measure is our measure μ.

2. Ergodic theory of the geodesic flow

This section is adapted from [BR], to which we refer for all properties below. Of course, [BR] applies since the geodesic flow on a negatively curved manifold is Anosov [A]. The presentation is slightly different, because we insist on the notion of quasi-invariance under the stable foliation.

2.1. *Stable foliation*

For ε small enough and X in SM, define the local stable manifold of X, $W^s(X)$, by:

$$W^s_\varepsilon(X) = \left\{X' : d(\phi_t X', \phi_t X) < \varepsilon \quad \text{for all} \quad t \geq 0\right\}.$$

The set $W^s_\varepsilon(X)$ is a n-dimensional open C^2-disk imbedded in SM, which vary continuously with X.

The fibration of $\widetilde{SM}$ by $\tau: \widetilde{SM} \to \widetilde{M}(\infty)$ is invariant under Γ. The quotient foliation W^s on SM is defined by the equivalence relation

$$X' \in W^s(X) \iff \sup\{d(\phi_t X, \phi_t X'),\ t \geq 0\} < +\infty.$$

The class $W^s(X)$ is called the stable manifold of X, it is a C^2-embedded submanifold of SM and $W^s_\varepsilon(X)$ is a neighborhood of X inside $W^s(X)$. A $(n-1)$-dimensional submanifold T of SM is said to be transversal to the stable foliation if at each Y in T,

$$T_Y T \cap T_Y W^s_\varepsilon(Y) = \{0\}.$$

There are two families of transversals to the stable foliation we shall consider: the spheres $\{S_y M, y \in M\}$ and the local strong unstable manifolds $W^{uu}(X)$ defined for ε small enough, by:

$$W^{uu}_\varepsilon(X) = \left\{X' : d(\phi_{-t} X', \phi_{-t} X) \leq \varepsilon e^{-\varepsilon t} \quad \text{for all} \quad t \geq 0\right\}.$$

If T, T' are close transversals to the stable foliation, the canonical map $\theta: T \to T'$ is defined by $\theta(Y) = T' \cap W^s_\varepsilon(Y)$.

2.2. *Quasi-invariant measures*

A family of Radon measures $\{\mu_T\}$ on transversals is said to be *quasi-invariant* if the canonical maps preserve the negligible sets. Clearly, if suffices to have a quasi-invariant family defined only on some continuous family of transversals which cover SM. For example, a family of measures on $\widetilde{M}(\infty)$, $\{\tilde\rho_x, x \in \widetilde{M}\}$ which is Γ-invariant defines a family of measures $\{\rho_y, y \in M\}$ on the spheres $\{S_y, y \in M\}$. It is quasi-invariant if and only if for any x,z in $\widetilde{M}$ the measures $\tilde\rho_x$ and $\tilde\rho_y$ are equivalent on $M(\infty)$.

Remark that by taking $\tilde\rho_x = \tilde\lambda_x \cdot \tau_x^{-1}$ we just proved (1.3.1) since we know that the Lebesgue measure on transversals is quasi-invariant under the stable foliation ([A], [AS]). Conversely by taking $\tilde{\rho}_x = \tilde{\mu}_x \cdot \tilde{\tau}_x^{-1}$ we see by (1.3.2) that the spherical harmonic measure is quasi-invariant.

Let μ be a measure on SM. It is said to be *quasi-invariant* under the stable *foliation* if there exists a quasi-invariant family $\{\mu_T\}$ of transversal measures such that for all T a subset E of T is μ_T negligible if and only if $\mu(U\{W^S(Y), Y \in E\}) = 0$. The family $\{\mu_T\}$ is called a transverse measure for μ. Clearly if μ is quasi-invariant and T a transversal, one can define μ_T by:

$$\mu_T(A) = \mu(U\{W^S_\varepsilon(Y);\quad Y \in A\}).$$

2.3. *Equilibrium state*

Let F be a Holder continuous function on SM. There exists a unique measure — called the equilibrium state of F — which realizes the maximum of the functional

$$v \to h_v(\phi_1) + \int F\, dv$$

over all ϕ-invariant probability measures on SM. The maximum is called pressure of F and denoted $P(F)$. Let μ be the equilibrium state of F. The flow $(SM, \{\phi_t; t \in \mathbb{R}\}; \mu)$ is ergodic and the measure μ is quasi-invariant under the stable foliation. As a family of transverse measures for μ on local strong unstable manifolds, one can choose the conditional measures on local strong unstable manifolds that are obtained when decomposing μ on the partition $\{W^{uu}_\varepsilon(Z), Z \in W^S_\varepsilon(X_0)\}$ for some fixed X_0.

The classical example is when F is constant and μ is the Bowen-Margulis measure with maximal entropy h on SM. The family v^{uu} of transverse measures on strong unstable manifolds is constructed in [Ma] and satisfies $v^{uu}_X \cdot \phi_t^{-1} = e^{ht} v^{uu}_{\phi_t X}$.

From this family, one infers a family of quasi-invariant transverse measures on spheres. The result of [H] says that the corresponding measures at infinity are equivalent to the h/a-dimensional spherical measures associated to the distances $\eta^a_{X,R'}$ $X \in S\tilde{M}$.

2.4. *Ergodic properties*

In section 3, we shall find a Holder continuous function F such that the family of spherical harmonic measures is a transverse measure for the equilibrium state of F. In the proof of Theorems A, B and C we shall use the following fact:

Proposition 2: *Let F be a Holder continuous function on SM, μ the equilibrium state of F, $\{\mu_T\}$ a family of quasi-invariant transverse measures for μ. Then for all transversal T, for μ_T-a.e. Y in T, we have:*

1. $\displaystyle \text{weak}\lim_{t\to+\infty} \frac{1}{t}\int_0^t \delta_{\phi_s Y}\, ds = \mu$

where δ_z is the Dirac measure at z, and

2. *for ε' small enough,*

$$\lim_{t\to+\infty} -\frac{1}{t}\ln \mu_T B(Y,\varepsilon',t) = h_\mu(\phi),$$

where $B(Y,\varepsilon',t) = \{Y' : d(\phi_s Y, \phi_s Y') \le \varepsilon \text{ for } s,\ 0 \le s \le t\}$.

Proof. The set E of points X in SM such that

$$\text{weak}\lim_{t\to\infty} \frac{1}{t}\int_0^t \delta_{\phi_s X}\, ds = \mu$$

satisfies:

E is of full -measure (by the ergodic theorem) and E is made of W^S leaves (if z belongs to $W^S(X)$ there exists

s_0, t_0 such that $\phi_{s_0} z$ belongs to $W^S_\varepsilon(\phi_{t_0} X)$). Thus E intersect T on a set of full μ_T-measure. This proves 1 and also that for μ_T-$a.e.$ Y in T, $\lim_{t\to\infty} \frac{1}{t}\int_0^t F(\phi_s Y)ds = \int F\, d\mu$. By the volume lemma (see [BR]), for all Y in SM and ε' small enough, $\frac{1}{t}\,\ell n\, \mu B(Y,\varepsilon',t)$ is equivalent to $P(F) - \frac{1}{t}\int_0^t F(\phi_s Y)ds$. So for μ_T-$a.e.$ Y in T, ε' small enough:

$$\lim_{t\to\infty} -\frac{1}{t}\,\ell n\, \mu B(Y,\varepsilon',t) = P(F) - \int F\, d\mu = h_\mu(\phi).$$

We can replace $\mu B(Y,\varepsilon',t)$ by $\mu_T B(Y,\varepsilon',t)$ because the set $B(Y,\varepsilon',t)$ is made of local stable manifolds.

3. Variational principle

3.1. *Harmonic measure*

Proposition 3: *Let M be a compact negatively-curved manifold. There exists a Holder continuous function F_0 such that the spherical harmonic measure can be chosen as a family of transverse measures for the equilibrium state of F_0.*

Proof. We first claim that there exists a family λ^{uu} of measures on strong unstable manifolds $W^{uu}(X)$

$$(W^{uu}(X) = U\{\Phi_t W^{uu}_\varepsilon(\Phi_{-t}X), t \geq 0\})$$

with the following two properties: i) the family λ^{uu} is quasi-invariant, and equivalent to the spherical harmonic measures, under canonical maps, and ii) there exists a Hölder continuous function F_0 on SM such that for all X in SM and λ^{uu}_X-$a.e.$ Z in $W^{uu}(X)$:

$$e^{-F_0(Z)} = \frac{d\Phi_{-1}\lambda^{uu}_{\phi_1 X}}{d\lambda^{uu}_X}(Z).$$

The proof of proposition 3 imitates in a completely parallel way the construction and the characterization of the Bowen - Ruelle measure for Axiom A flows. We only have to replace the Lebesgue measures on strong unstable manifolds by our family λ^{uu}. Namely, we start with a probability λ on SM, carried by some $W^{uu}(X)$ and equivalent to λ^{uu}_X, with a continuous density. Any limit point μ of $\frac{1}{t}\int_0^t \lambda\, \Phi_s^{-1}\, ds$ as t goes to infinity has conditional measures on strong unstable manifolds equivalent to λ^{uu} (by property ii), see [Si 2]). From this follows $h_\mu(\phi_1) = -\int F_0\, d\mu$ (see [Si 1]) and also $h_m(\phi_1) \leq -\int F_0\, dm$ for any other invariant probability measure (see e.g. [LY] section 6). This measure μ is therefore the unique equilibrium state of F_0 and moreover the pressure of F_0 is zero. By property i) of the λ^{uu}, the spherical harmonic measures are a family of transverse measures for μ and this achieves the proof of proposition 3, provided we establish the claim.

For $\tilde{X}$ in $S\tilde{M}$, let $W(\tilde{X})$ be the negative horisphere at $\tilde{X}$ (in particular $\pi\tilde{W}(\tilde{X}) = W^{uu}(\pi\tilde{X})$). We shall construct the family λ^{uu} by constructing a family $\tilde{\lambda}$ of measures on horispheres which is invariant under the action of Γ.

We shall use the fact that there is continuous function k on $\tilde{M}\times\tilde{M}(\infty)$ such that for all x, z in $\tilde{M}$ and $\tilde{\mu}$-$a.e.$ ξ in $\tilde{M}(\infty)$,

$$k(x,z,\xi) = \frac{d\tilde{\mu}_x . \tau_x^{-1}}{d\tilde{\mu}_z . \tau_z^{-1}}(\xi).$$

Moreover for each compact domain $K \subset \tilde{M}$, the function k is Holder continuous on $K\times K\times\tilde{M}(\infty)$ (see [An S] section 6).

Remark that the map τ defines a one-to-one correspondence between $\tilde{W}(\tilde{X})$ and $\tilde{M}(\infty)$ minus the point $\tau(-\tilde{X})$. Fix a point 0 in $\tilde{M}$ and let λ_0 be the measure on $\tilde{W}(\tilde{X})$ such that

$$\lambda_0 \cdot \lambda^{-1} = \tilde{\mu}_0 \cdot \tau_0^{-1}.$$

The measure λ_0 is non-zero since the support of $\tilde{\mu}_0$ is the whole $\tilde{M}(\infty)$ ([An]). We define $\tilde{\lambda}$ on $\tilde{W}(\tilde{X})$ by

$$\frac{d\tilde{\lambda}}{d\lambda_0}(\tilde{Z}) = k(p\tilde{Z},0,\tau\tilde{Z}).$$

The following properties are now easily verified:

The family $\tilde{\lambda}$ we obtain does not depend on the reference point and is Γ invariant.

The family λ^{uu} we obtain on SM by quotienting is equivalent to the spherical harmonic measures under canonical maps.

We have for $\tilde{\lambda}$-a.e. $\tilde{Z}$

$$\frac{d\Phi_{-1}\tilde{\lambda}}{d\tilde{\lambda}}(\tilde{Z}) = \frac{(\gamma_{\tilde{Z}}(1),0,\tau\Phi_1\tilde{Z})}{k(p\tilde{Z},0,\tau\tilde{Z})}$$

$$= k(\gamma_{\tilde{Z}}(1),\gamma_{\tilde{Z}}(0),\tau\tilde{Z})$$

since $\tau\Phi_1\tilde{Z} = \tau\tilde{Z}$.

As $\tilde{Z}$ varies in a compact fundamental domain K for SM, the map $\tau: K \to M(\infty)$ is Holder continuous ([An S]) and the function F_0 defined by

$$F_0(\pi\tilde{Z}) = \ln k(\gamma_{\tilde{Z}}(0), \gamma_{\tilde{Z}}(1), \tau\tilde{Z})$$

is a Holder continuous function on SM.
This achieves the proof of the claim and proposition 3.

Remark that proposition 3 implies the first statement in theorem B3, namely that the harmonic class is ergodic under the action of Γ on $\tilde{M}(\infty)$. For if A is a Γ-invariant subset of $\tilde{M}(\infty)$ and x belongs to $\tilde{M}$, the set $D\pi\tau_x^{-1}A$ is a W^S-invariant subset of $S_{\pi x}M$. Either the set $D\pi\tau_x^{-1}A$ or its complementary set is negligible for the transverse measure.

Consider the function F on SM defined by:

$$F(\pi\tilde{X}) = \frac{d}{dt}\ln k\left[\gamma_{\tilde{X}}(0),\gamma_{\tilde{X}}(t),\tau\tilde{X}\right]_{/t=0}$$

$$= \tilde{X}(\ln k(p\tilde{X},\cdot,\tau\tilde{X})).$$

Since $\int_0^1 F(\phi_s X)ds = F_0(X)$, the function F has the same equilibrium state as F_0. Remark also that

$$P(F) = P(F_0) = 0$$

and

$$\int_0^t F(\phi_s X)ds = \ln k\left[\gamma_{\tilde{X}}(0),\gamma_{\tilde{X}}(t),\tau\tilde{X}\right]$$

for any $\tilde{X}$ with $D\pi\tilde{X} = X$.

We therefore can write from Propositions 2 and 3.

Corollary 1: *Consider the function $F: SM \to \mathbb{R}$ defined by $F(\pi\tilde{X}) = \tilde{X}(\ln k(p\tilde{X},\cdot,\tau\tilde{X}))$ and μ the equilibrium state of F. Then the spherical harmonic measures $\{\mu_y, y \in M\}$ are transverse measures for μ and we have, for all y in M*

1. weak limit $\frac{1}{t}\int_0^t (\mu_y \cdot \phi_s^{-1})ds = \mu$
   $t\to\infty$

2. for μ_y-a.e. X in S_yM; ε' small enough,

$$\lim_{t\to\infty} -\frac{1}{t}\ln \mu_y B(X,\varepsilon',t) = \lim_{t\to\infty}\frac{1}{t}\ln k\left[\gamma_{\tilde{X}}(t),p\tilde{X},\tau\tilde{X}\right] = h_\mu(\phi_1).$$

3.2. *Reversing time*

By Corollary 1 we already know that there is a unique ϕ-invariant measure satisfying property C1. We now show that μ also satisfies property C2. We first apply section 3.1 to the flow $\{\phi_{-t}, t \in \mathbb{R}\}$. The corresponding notion of transverse measure is by projecting along local unstable manifolds. We get the analogous result to Corollary 1:

Corollary 2: *Let* F' *be the function defined on* SM *by* $F'(\pi\tilde{X}) = (-\tilde{X})\ (\ell n\ k(p\tilde{X}, ., \tau(-\tilde{X}))$ *and* μ' *the equilibrium state for* F'. *Then for all* y *in* M, *we have*

$$\underset{t\to+\infty}{\text{weak limit}}\ \frac{1}{t}\int_0^t (\mu_y . \phi_{-s}^{-1})ds = \mu'.$$

We want to show that $\mu = \mu'$, i.e. that the Hölder continuous functions F and F' have the same equilibrium state. By [Li], this will be a direct consequence of the following fact:

Proposition 4: *Let* X *in* SM *generate a periodic orbit: there exists* $p > 0$ *such that* $\phi_p X = X$. *Then,* $\int_0^p F(\phi_x X)ds = \int_0^p F'(\phi_s X)ds.$

Proof of proposition 4: Let $\tilde{X}$ be so that $\tilde{X} = X$ and γ in Γ so that

$$(\gamma_{\tilde{X}}(p), \dot{\gamma}_{\tilde{X}}(p)) = D\gamma\ \tilde{X}.$$

By definition of F and F' we have:

$$\int_0^p F(\phi_s X)ds = \ell n\ k(x, \gamma x, \tau\tilde{X})$$

$$\int_0^p F'(\phi_s X)ds = \ell n\ k(\gamma x, x, \tau(-\tilde{X}))$$

where $x = p\tilde{X}$.

Let $G(x,z)$ be the Green function on $\tilde{M}$, $G(x,z) = \int_0^\infty p(t,x,z)dt$.

The Martin kernel k is given by

$$k(x,z,\xi) = \lim_{y\to\xi} \frac{G(x,y)}{G(z,y)}.$$

In particular, since $\gamma^n x = \gamma_{\tilde{X}}(np)$ goes to $\tau\tilde{X}$ as n goes to $+\infty$ and to $\tau(-\tilde{X})$ as n goes to $-\infty$, we have

$$k(x, \gamma x, \tau\tilde{X}) = \lim_{n\to\infty} \frac{G(x,\gamma^n x)}{G(\gamma x, \gamma^n x)}$$

$$= \lim_{n\to\infty} \frac{G(x,\gamma^n x)}{G(x,\gamma^{n-1} x)} = \lim_{n\to\infty} (G(x,\gamma^n x))^{\frac{1}{n}}$$

and analogously,

$$k(\gamma x, x, \tau(-\tilde{X})) = \lim_{n\to\infty} (G(x,\gamma^{-n} x))^{\frac{1}{n}}.$$

Since γ is an isometry and G is symmetric, $G(x,\gamma^n x) = G(x,\gamma^{-n} x)$ and the proposition is proved.

3.3. *Proof of theorem B3*

The proof is the same as in the case of the visibility measure. We recall it for the convenience of the reader.

Fix 0 in $\tilde{M}$ and for any geodesic γ in $\tilde{M}$, call 0_γ the point of the geodesic which is closest to Q. We define a bijection Σ between $\tilde{M}(\infty) \times \tilde{M}(\infty) \times \mathbb{R}$ and $S\tilde{M}$ by $\Sigma(\xi,\xi',t) = (\gamma(t), \dot{\gamma}(t))$ where γ is the geodesic satisfying: $\gamma(+\infty) = \xi$, $\gamma(-\infty) = \xi'$ and $\gamma(0) = 0_\gamma$.

The action of Γ on $S\tilde{M}$ is transported by Σ into an action of Γ on $\tilde{M}(\infty)\times\tilde{M}(\infty)\times\mathbb{R}$ which commutes with the action of $\mathbb{R}$ given by

$$\psi_s(\xi,\xi',t) = (\xi,\xi',t+s),\ s\in\mathbb{R}.$$

Consider on $S\tilde{M}$ the measure $\tilde{\mu}$ which is Γ-invariant and equals μ on each fundamental domain SM. The measure $\tilde{\mu}$ is $\mathbb{R}$-invariant and since the action of $\mathbb{R}$ is ergodic on the Γ-invariant sets, the action of Γ is ergodic on the $\mathbb{R}$-invariant sets.

The space of $\mathbb{R}$-invariant sets is identified with $\tilde{M}(\infty)\times\tilde{M}(\infty)$. We only have to check that the corresponding measure class is the class of the product of the harmonic measure classes. Locally on $S\tilde{M}$ the coordinates (ξ,ξ',t) can be read by choosing some transversal T to the geodesics. The coordinates ξ and ξ' will be given by $\tau(W^s\cap T)$ and τ-$(W^u\cap T)$ respectively. Since μ is Gibbs, μ is locally equivalent to the product of its conditional measures on $W^s\cap T$, $W^u\cap T$ and of the Lebesgue measure on the geodesics. By the proof of proposition 3 and section 3.2, these conditional measures are equivalent to the harmonic class, so that the measure $\tilde{\mu}$ is equivalent to $\mu\times\mu\times dt$. The action of Γ on $\tilde{M}(\infty)\times\tilde{M}(\infty)$ is ergodic for the class $\mu\times\mu$.

4. Proofs of theorems

4.1. *A Brownian path follows a geodesic*

We recall in this section the basic properties we want to use. We first have a classical comparison lemma:

Lemma 1: *Let $\tilde{M}$ be a simply connected manifold with sectional curvatures bounded from above by $-a^2<0$. Then for all x in $\tilde{M}$, θ,θ' in $S_x\tilde{M}$, $0<t\le u$, we have:*

$$D_t(\theta,\theta') \le D_u(\theta,\theta')\,\frac{\sinh at}{\sinh au}.$$

where $D_u(\theta,\theta')$ is the distance on the sphere $S_u=\{z:d(z,x)=u\}$ of the points (u,θ) and (u,θ').

Proof: Let $\{\gamma_s,\ 0\le s\le 1\}$ be the variation of geodesics in $\tilde{M}$ such that

$$\gamma_0(0)=\gamma_1(0)=x,\quad \gamma_0'(0)=\theta,\quad \gamma_1'(0)=\theta'$$

and $\{\gamma_s(u),\ 0\le s\le 1\}$ is a geodesic on the sphere S_u with uniform speed $D_u(\theta,\theta')$. Write $\phi(t)$ for the length of the curve $\{\gamma_t(s),\ 0\le s\le 1\}$. We have $D_t(\theta,\theta')\le\phi(t)$ and $\phi''(t)\ge -a^2\phi(t)$, since $\phi(t)=\int_0^1\|Y_s(t)\|ds$, where $\{Y_s(t),\ 0\le t\le u\}$ is a Jacobi field along γ_s. The lemma follows.

A consequence of Lemma 1 is

Corollary 3: *Let M be a compact connected negatively curved manifold, f a Hölder continuous function on SM. Then there exists a constant K such that for all x in $\tilde{M}$, all $u>0$ and all θ,θ' in $S_x\tilde{M}$ such that $d((u,\theta),(u,\theta'))\le 1$, we have:*

$$\left|\int_0^u f(\phi_t(D\pi(x,\theta)))dt - \int_0^u f(\phi_t(D\pi(x,\theta')))dt\right|\le K.$$

Corollary 3 follows directly from Lemma 1 if one takes on $S\tilde{M}$ the equivalent distance defined by $d_1(\tilde{X},\tilde{Z})=\sup\{d(\gamma_{\tilde{X}}(s),\gamma_{\tilde{Z}}(s)),\ 0\le s\le 1\}$.

Proposition 5 ([Le], [Kai]): *Let M be a compact connected negatively curved manifold. For all x in M and $\mathbb{P}_x$-a.e. ω in $C(\mathbb{R}_+,M)$,*

$$\lim_{t\to\infty} \frac{1}{t}\, d(\tilde{\omega}(t),\ (r(\omega,t),\theta(\omega,\infty))) = 0.$$

Proof: Fix $\varepsilon > 0$ and define $\tau_0(\omega) = 0$,

$$\tau_n(\omega) = \inf\{t:\ t \geq \tau_{n-1}(\omega), d(\tilde{\omega}(t),\tilde{\omega}(\tau_{n-1}(\omega))) \geq 1\}$$

and $n(\omega,t) = \inf\{n: \tau_n(\omega) \geq \frac{\alpha+\varepsilon}{\alpha-\varepsilon}\, t\}$.

We have, omitting some 's:

$$d(\tilde{\omega}(t),(r(t),\theta(\infty))) \leq d\left[\tilde{\omega}(t),\tilde{\omega}\left[\tau_{n(t)}\right]\right] + d\left[\tilde{\omega}\left[\tau_{n(t)}\right],\left[r(t),\theta\left[\tau_{n(t)}\right]\right]\right]$$

$$+ D_{r(t)}\left[\theta(\tau_{n(t)}),\theta(\infty)\right]$$

$$\leq C + 2d\left[\tilde{\omega}(t),\tilde{\omega}\left(\frac{\alpha+\varepsilon}{\alpha-\varepsilon}\, t\right)\right] + \sum_{m\geq n(t)} D_{r(t)}\left[\theta(\tau_m),\theta(\tau_{m+1})\right].$$

By Lemma 1 the sum can be estimated for t large enough by $F\left[\psi_{\tau_{n(t,\omega)}}(\omega)\right]$ where $F(\omega) = C \sum_{n=0} e^{-a(\alpha-\varepsilon)\tau_n(\omega)}$.

Proposition 5 follows frome the following convergences, true on a set of $\mathbb{P}_x$-probability one:

$$\limsup_{n\to\infty} \frac{1}{n} F(\psi_n(\omega)) = 0$$

$$\lim_{t\to\infty} \frac{1}{t}\tau_{n(t,\omega)} = \frac{\alpha+\varepsilon}{\alpha-\varepsilon}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\, d\left[\tilde{\omega}(t),\tilde{\omega}\left(\frac{\alpha+\varepsilon}{\alpha-\varepsilon}\, t\right)\right] \leq C\varepsilon.$$

The first two estimates come from the fact that the processes $\{\tau_n - \tau_{n-1}, n \geq 0\}$ can be pinched in distribution between two i.i.d. processes with exponential decay of the

distribution (see e.g. [Pi]). The third can be proven for example by comparing with the Brownian motion on the n-dimensional hyperbolic space with constant curvature $-b^2$.

4.2. *Proof of Theorem C*

We already know by section 3.1., 3.2 that there exists a ϕ-invariant ergodic measure μ satisfying C1 and C2. We only have to show that the measure μ satisfies also property C3. We can assume that f is a Holder continuous function on M. Then by Corollary 3 for all x in $\tilde{M}$, all ω in Ω,

$$\left|\int_0^{r,(\omega,t)} f.\phi_s\, D\pi(x,\theta(\omega,t))ds - \int_0^{r(\omega,t)} f.\phi_s\, D\pi(x,\theta(\omega,\infty))ds\right|$$

$$\leq K d(\tilde{\omega}(t),(r(\omega,t),\theta(\omega,\infty))).$$

By Proposition 5 and 1.2.1, we get that for $\mathbb{P}_{\pi x}$-$a.e.$ ω in Ω,

$$\lim_{t\to\infty}\left|\frac{1}{\alpha t}\int_0^{r(\omega,t)} f.\phi_s\, D\pi(x,\theta(\omega,t))ds - \frac{1}{\alpha t}\int_0^{\alpha t} f.\phi_s\, D\pi(x,\theta(\omega,\infty))ds\right| = 0.$$

Property C3 follows then from Proposition 2, since $\tilde{\mu}_x$ is the distribution of $\theta(\omega,\infty)$ and $D\pi(\tilde{\mu}_x) = \mu_x$ is a transverse measure for μ.

4.3. *Proof of Theorem A*

Let x, z be points in $\tilde{M}$, $\tilde{X}$ be such that $\gamma_{\tilde{X}}(0) = x$, $\gamma_{\tilde{X}}(d(x,z)) = z$, $X = \pi\tilde{X}$. We claim that we have following formula

(4.3.1) $$A(x,z) = \left|\det D\phi_{d(x,z)}\ \right|_{T_X S_{pX} M}\Big|$$

In fact, recall that there is a canonical identification (Dp,K) between the space $T_X TM$ and the space $T_{pX}M + T_{pX}M$ such that for all $t \geq 0$ and ξ in $T_X TM$, $D\phi_t \xi$ in $T_{\phi_t X} TM$ is identified with $(Y_\xi(t), Y'_\xi(t))$ in $T_{p\phi_t X}M + T_{p\phi_t X}M$, where $Y_\xi(t)$ is the Jacobi field along γ_X defined by $(Y_\xi(0), Y'_\xi(0)) = (Dp\xi, K\xi)$ (see e.g. [E], Dp and K have their usual signification). Therefore $\det D\phi_t|_{T_X S_{pX}M}$ is given by

$$\|Y_{\xi_1}(t) \wedge Y_{\xi_2}(t) \wedge \ldots \wedge Y_{\xi_{n-1}}(t)\|$$

where ξ_j, $j = 1,\ldots,n-1$ are $n-1$ vectors in $T_{pX}M \ominus X$ with $\|\xi_1 \wedge \ldots \wedge \xi_{n-1}\| = 1$ and Y_{ξ_j} is the Jacobi field along γ_X defined by $(Y_{\xi_j}(0), Y'_{\xi_j}(0)) = (0, \xi_j)$. This is also the formula for $A(\gamma_{\tilde{X}}(0), \gamma_{\tilde{X}}(t))$ and this proves (4.3.1).

Let $E^{uu}(X) = T_X W^{uu}_\varepsilon(X)$ be the expanding subspace at a point X in SM. The space $T_X S_{pX}M$ makes an angle with the stable space $E^s(X) = T_X W^s_\varepsilon(X)$ bounded away from zero, so that the growth rate of $\det D\phi_t|_{T_X S_{pX}M}$ is equivalent to the growth rate of $\det D\phi_t|_{E^{uu}(X)}$. By (4.3.1), we have

$$\lim_{(x,z)\to\infty} \frac{1}{d(x,z)} \left| \ell n\, A(x,z) - \int_0^{d(x,z)} J^u(\phi_s X)\, ds \right| = 0$$

where $J^u(Z) = \left[\frac{d}{dt} \ell n |\det D\phi_t|_{E^{uu}(Z)}| \right]_{t=0}$.

By Theorem C3, we have for all x in $\tilde{M}$, $\mathbb{P}_{\pi x}$-a.e. ω in Ω

$$\lim_{t\to\infty} \frac{1}{t} \ell n\, A(x, \tilde{\omega}(t)) = \alpha \int J^u\, d\mu.$$

This achieves the proof of Theorem A. Also:

Corollary 4: *We have:* $\gamma = \alpha \cdot \int J^u\, d\mu$, *where* $J^u(z) = \left[\frac{d}{dt} \ell n |\det D\phi_t|_{E^{uu}(z)}| \right]_{t=0}$ *and* μ *is the* ϕ*-invariant measure in theorem C. In particular* γ/α *is the sum of the positive Lyapunov exponents of* $(SM, \{\phi_t, t \in \mathbb{R}\}, \mu)$.

4.4. *Proof of Theorem B:*

We first recall another definition of the entropy β (see [Kai2]): For all x in $\tilde{M}$, $t, \delta > 0$ define

$$N(x,t,\delta) = \inf \# \{E : \mathbb{P}_{\pi x}\{d(\tilde{\omega}_t, E) \leq 1\} \geq \delta\}.$$

(4.4.1) For all x in $\tilde{M}$, $0 < \delta < 1$,

$$\beta = \lim_{t\to\infty} \frac{1}{t} \ell n\, N(x,t,\delta).$$

Proposition 6: *Let* M *be a compact connected negatively-curved manifold,* β *defined by (4.4.1),* μ *the* ϕ*-invariant measure on* SM *obtained in Theorem C. We have* $\beta = \alpha h_\mu(\phi)$.

Proof: Fix x in $\tilde{M}$, $0 < \delta < \frac{1}{2}$, $\varepsilon > 0$ *and* $\varepsilon' > 0$ *small, and define for all* $t > 0$:

$$A_t = \left\{ \begin{array}{ll} \omega : d(\tilde{\omega}_t, (\alpha t, \theta(\omega,\infty)) \leq \varepsilon t & \text{and} \\ \tilde{\mu}_x\{\theta' : D_{\alpha t}(\theta', \theta(\omega,\theta)) \leq \varepsilon'\} \leq e^{-\alpha t\, h_\mu(\phi)} e^{\varepsilon t} & \end{array} \right\}$$

By Proposition 5 and property 2 in Corollary 1 we have $\mathbb{P}_{\pi x}(A_t) \geq 2\delta$ for t large enough.

Fix such a t and pick a set E in $\tilde{M}$ such that $\# E = N(x,t,\delta)$ and

$$\mathbb{P}_{\pi x}(d(\tilde{\omega}_t, E) \leq 1\} \geq 1-\delta.$$

The set of directions $\theta(\omega,\infty)$ for ω in $A_t \cap \{d(\tilde{\omega}_t, E) \leq 1\}$ is a set of $\tilde{\mu}_x$-measure at least δ, which can be covered on $S_{\alpha t}$ by less than $N(x,t,\delta)C^{\varepsilon t}$ spheres of radius $\varepsilon'/2$ (C is some geometric constant), and for which the $\tilde{\mu}_x$-measure of spheres of radius ε' is smaller that $e^{-\alpha h_\mu(\phi)t} e^{\varepsilon t}$. This is possible only if

$$N(x,t,\delta) C^{\varepsilon t} e^{-\alpha h_\mu(\phi)t} e^{\varepsilon t} \geq \delta.$$

The estimate $\beta > \alpha h_\mu(\phi)$ follows from the arbitrariness of ε. The proof of the converse inequality is similar (cf. [Le], section IV). Proposition 1 and Theorem B now follow from Proposition 6, Corollary 4 and the variational principle for Anosov flows (see [BR]). For we have $\beta \leq \gamma$ with equality if and only if $h_\mu(\phi) = \int J^u \, d\mu$, which happens if and only if μ is the Liouville measure. Finally the Liouville measure is characterized by the fact that the Lebesgue measures on spheres is a family of transverse measures, which proves Theorem B1. Analogously, we have $\beta \leq \alpha h$ with equality if and only if μ is the measure with maximal entropy. The measure of maximal entropy is characterized by the Bowen-Margulis measures as transverse measures on spheres and this proves Theorem B2.

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