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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR WAVE EQUATION

ORLANDO LOPES

Introduction and Statement of the Main Result.

In this paper we deal with the damped nonlinear wave equation:

$$(1) \qquad u_{tt} - \Delta u + cu_t + f(u) = h(t,x), \quad c > 0.$$

We assume $u(t,x)$ and $h(t,x)$ are defined for all $x = (x_1,x_2,x_3)$ in $\mathbb{R}^3$ and are 2π-periodic in each x_i; in other words, we take 2π-periodicity in the spatial variables as boundary condition. For each non-negative integer k and $1 \leq p \leq \infty$, $H_{k,p}(\mathbb{R}^3)$ $(H_k(\mathbb{R}^3)$ for $p = 2)$ denotes the usual Sobolev spaces with the usual norm; $H^{2\pi}_{k,p}(\mathbb{R}^3)$ $(H^{2\pi}_k(\mathbb{R}^3)$ for $p = 2)$ denotes the Sobolev spaces of functions which are 2π-periodic in each variable (of course, the integrals defining the norm are taken over the fundamental cube $[0,2\pi] \times [0,2\pi] \times [0,2\pi]$).

Equation (1) can be viewed as a system

$$(2) \qquad \begin{aligned} u_t &= v \\ v_t &= \Delta u - cu_t - f(u) - h(t,x) \end{aligned}$$

or, more compactly,

$$(3) \qquad \frac{dw}{dt} = Aw + G(t,w), \quad \text{where}$$

$$w = (u,v), \quad Aw = (v,\Delta u), \quad G(t,w) = \begin{bmatrix} 0 \\ -cv - f(u) + h(t,\cdot) \end{bmatrix}.$$

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It is very well known that A generates a strongly continuous semigroup (actually a group) of linear operators in the space $X_k = H^{2\pi}_{k+1} \times H^{2\pi}_{k}$, for each integer $k \geq 0$. As phase space for equation (3) we take the space $X_1 = H^{2\pi}_2 \times H^{2\pi}_1$. Since $L^{2\pi}_\infty \subset H^{2\pi}_2$ continuously, it is easy to see that $w \in X \to G(t,w) \in X_1$ is lipschitzian on bounded sets provided f is C^2. So, if this is the case and $h: \mathbb{R}_+ \to H_1$ is continuous, it follows that local existence and uniqueness of mild solutions of (3) is guaranted in the space X_1; moreover, global existence in time can also be guaranted provided we get an a priori estimate for the norm of the mild solutions in the space X_1. For $t \geq t_0 \geq 0$ and w_0 belonging to X_1, we denote by $w(t,t_0;w_0)$ the solution satisfying $w(t_0,t_0;w_0) = w_0$.

Definition. Equation (3) is uniform ultimately bounded in the space X_1 if there are functions $a(R)$ and $T(R)$ and a constant R such that $|w_0|_{X_1} \leq R$ implies $|w(t,t_0;w_0)|_{X_1} \leq a(R)$ for $t \geq t_0$ and $|w(t,t_0;w_0)|_{X_1} \leq R_0$ for $t \geq t_0 + T(R)$.

Our main result is the following:

Theorem A. Equation (3) is uniform ultimately bounded in the space X_1 provided the following conditions are satisfied.

(i) $f(u)$ is a C^2 function;

(ii) there are constants $k_1 > 0$ and k_2 such that $uf(u) \geq k_1 u^2 + k_2$;

(iii) there are positive constants k_3 and β, $0 \leq \beta < 4$ such that $|f'(u)| \leq k_3(1 + |u|^\beta)$;

(iv) the map $t \in \mathbb{R}_+ \to h(t) \in H^{2\pi}_1$ is continuous and bounded.

Moreover, if the map $t \to h(t)$ is periodic of period $p > 0$, then the Poincaré map $w_0 \to w(p,0;w_0)$ is the sum of a linear map with spectral radius strictly less than one and a compact differentiable map.

Before getting into the proof, we believe it is interesting to make the following remarks about Theorem A and its consequences:

1) If we consider the problem of global existence of solutions of the equation $u_{tt} - \Delta u + |u|^\beta u = 0$, with initial condition (u_0, v_0) in $H_2(\mathbb{R}^3)$ $H_1(\mathbb{R}^3)$, then the exponent $\beta = 4$ is critical; in other words, for $\beta < 4$ solutions are globally defined, and for $\beta = 4$ they are globally defined for small initial data ([1], [2]). Assumption (iii) in Theorem A is related to this fact.

2) Some authors have studied the existence of a p-periodic solution and of finite dimensional attractors for equation (3) in the case $h(t)$ is p-periodic in t and the phase space is $H^{2\pi}_1 \times L^{2\pi}_2$ ([3], [4], [5], [6], [7]) (here, finite dimension is understood in the sense of Hausdorff dimension). For that phase space, the critical exponent is $\beta = 2$. For the case we are treating, the existence of a p-periodic solution and of a finite dimensional attractor follows immediately from the decomposition of the Poincaré map given by Theorem A (see [8], [9]).

3) If the wave equation is given in a bounded open set $\Omega \subset \mathbb{R}$ with smooth boundary and we impose, say, Dirichlet boundary, then everything we have done works, except the proof of Lemma 3. This lemma gives an $L_p - L_q$ estimate for the linear wave equation and its proof follows from a similar result for the wave equation in $\mathbb{R}^3$. Almost surely it also holds in the case of Dirichlet boundary; so while we wait for such a proof, we restrict ourselves to the case where periodicity in x is taken as boundary condition. $L_p - L_q$ estimates have been used by several authors to treat the Cauchy problem for nonlinear wave equations in $\mathbb{R}^n$ (see, for instance, [11], [12] and the references therein).

Proof of Theorem A. As a preparation for the proof of Theorem A we start with a $L_p - L_q$ estimate for the problem

(4) $u_{tt} - \Delta u = 0$ in $\mathbb{R}^3$, $u(0,x) = 0$, $u_t(0,x) = f(x)$.

Lemma 1. The map $f \to S(t)f = u(t,\cdot)$ satisfies

$$|S(t)f|_{L_\infty} \le k_4 t^{3\mu-1} |f|_{H_{1,p}(\mathbb{R}^3)}, \quad t > 0,\ p = \frac{1}{1-\mu},\ \mu > 0.$$

Proof. See [10].

Next lemma deals with equation (4) in the case periodicity is taken as boundary condition.

Lemma 2. $|S(t)f|_{L_\infty} \le k_5 (t^{3\mu-1} + t^{3\mu-1+3/p}) |f|_{H^{2\pi}_{1,p}}$, $t > 0$.

Proof. First of all we have to notice that, due to the wave propagation property, Lemma 1 can be restated as $|S(t)f|_{L_\infty(\Omega)} \le$ $\le k_5 t^{3\mu-1} |f|_{H_{1,p}(\Omega(t))}$, where Ω is any subset of $\mathbb{R}^3$ and $\Omega(t)$ is the domain of dependence of Ω. If $0 < t \le 1$ and Ω is the fundamental cube $[0,2\pi] \times [0,2\pi] \times [0,2\pi]$, then the number of cubes necessary to cover $\Omega(t)$ is less or equal to some fixed integer N_0; also, if t is large, the number of such a cubes grows as fast as t^3, and this proves the lemma.

Lemma 3. Consider the equation $u_{tt} - \Delta u + cu_t + \frac{c^2}{4} u = 0$ $u(0,x) = 0$, $u_t(0,x) = f(x)$, and suppose 2π-periodicity in x is taken as boundary conditions. Then $f \to \tilde{S}(t)t = u(t,\cdot)$ satisfies:

$$|\tilde{S}(t)f|_{L_\infty} \le k_6 e^{-ct/2}(t^{3\mu-1} + t^{3\mu-1+3/p})|f|_{H^{2\pi}_{1,p}}, \quad t > 0, \quad p = \frac{1}{1-\mu}.$$

Proof. Defining $U(t) = e^{ct/2}u(t)$, we see that $U_{tt} - \Delta U = 0$, $U(0,x) = 0$, $U_t(0,x) = f(x)$, and the conclusion follows from the previous lemma.

Lemma 4. Under the assumptions of Theorem A, equation (3) is uniform ultimately bounded in the space $X_0 = H_1^{2\pi} \times L_2^{2\pi}$ (for initial conditions in $H_2^{2\pi} \times H_1^{2\pi}$, as long as solution exists in this space).

Proof. Recall that the norm in $H_1^{2\pi}$ is defined by $(|\text{grad}\ u|^2_{L_2^{2\pi}} + |u|^2_{L_2^{2\pi}})^{\frac{1}{2}}$. Defining $W(u,v) = \frac{1}{2}|v|^2_{L_2} +$ $+ \frac{1}{2}|\text{grad}\ u|^2 + \frac{c}{2}\langle u,v\rangle_{L_2} + \frac{c^2}{4}|u|^2_{L_2} + \int_Q F(u(x))\,dx$ where $F(u) = \int_0^u f(s)\,ds$, an easy computation shows that

$$\dot{W}(u,v) = -\frac{c}{2}|v|^2_{L_2^{2\pi}} - c|\text{grad}\ u|^2_{L_2^{2\pi}} - \frac{c}{2}\int_Q uf(u)\,dx + \langle v,h(t)\rangle + \langle u,h(t)\rangle$$

and using assumptions (ii) and (iv) we get

$$\dot{W}(u,v) \le -\gamma(|v|^2_{L_2^{2\pi}} + |\text{grad}\ u|^2_{L_2^{2\pi}} + |u|^2_{L_2^{2\pi}}) + M,$$

where $\gamma > 0$ depends on c and k_1 and M depends on γ and $\sup_{t\ge 0}|h(t)|_{L_2^{2\pi}}$. Let $R_1 > 0$ be defined by $-\gamma R_1^2 + M = -1$ and let us call $w(t) = (u(t),v(t))$. Notice that assumptions (ii) and (iii) imply $F(u)$ is bounded below and $|F(u)| \le k_7(1 + |u|^{\beta+1})$; in particular, since $L_6^{2\pi} \subset H_1^{2\pi}$ continuously, we conclude for each R there is a constant $C_1(R)$ such that $\left|\int_Q F(u(x))\,dx\right| \le C_1(R)$ provided $|u|_{H_1^{2\pi}} \le R$. Moreover, if for some t_1 we have $|W(t_1)|_{X_0} = R_1$ and $|W(t)|_{X_0} \ge R_1$ for $t \ge t_1$ then necessarily

$$(5) \qquad W(u(t),v(t)) \le W(u(t_1),v(t_1)) - (t - t_1).$$

This inequality together with the previous remarks proof the lemma.

Lemma 5. Let $u : [t_0,+\infty) \to \mathbb{R}$ be a non negative continuous function satisfying

$$u(t) \le M e^{-c(t-t\ )/2} + M_2 + \varepsilon M_3 \int_{t_0}^{t} e^{-c(t-s)/2} k(t-s)u(s)ds, \quad t_0 \le t,$$

where M_1, M_2, M_3 are non negative constants and $k(s)$ is a nonegative function such that $\varepsilon M_3 \int_0^{+\infty} e^{-cs/4} k(s)ds \le \frac{1}{2}$. Then

$$u(t) < \bar{u}(t) = 2M_1 e^{-c(t-t_0)/4} + 3M_2, \quad t \ge t_0.$$

Proof. Suppose $u(s) < \bar{u}(s)$, $t_0 \le s < t$; then

$$u(t) \le M_1 e^{-c(t-t_0)/2} + $$

$$+ M_2 + \varepsilon M_3 \int_{t_0}^{t} e^{-c(t-s)/2} k(t-s)[2M_1 e^{-c(s-t_0)/4} + 3M_2] ds$$

$$= M_1 e^{-c(t-t_0)/2} + M_2 + 2M_1 \varepsilon M_3 e^{-c(t-t_0)/4} \int_{t_0}^{t} e^{-c(t-s)/4} k(t-s)ds +$$

$$+ 3M_2 \varepsilon M_3 \int_{t_0}^{t} e^{-c(t-s)/2} k(t-s)ds \le 2M_1 e^{-c(t-t_0)/4} + \tfrac{5}{2} M_2 < \bar{u}(t)$$

and this proves the lemma.

Proof of Theorem A. We start by rewriting system (2) as

$$u_t = v$$

$$(5) \qquad v_t = \Delta u - cu_t - \frac{c}{4} u - g(u) + h(t),$$

where $g(u) = f(u) - \frac{c^2}{4} u$. Certainly, $g(u)$ also satisfies assumption (iii); so, we can find constants $k_8 \ge 0$, a and b, with $0 < a < 3$, $0 \le b < 1$, such that $a + b = \beta$ and $\frac{|g'(u)|}{1 + |u|^a} \le k_8(1 + |u|^b)$. Then, using that $L_\infty \subset H_2^{2\pi}$ continuously and Lemma 5 with $p = \frac{6}{a+3} > 1$ we get, by the variation of constants formula:

$$|u(t)|_{L_\infty} \le Ke^{-c(t-t_0)/2} |w(t_0)|_{X_1} +$$

$$+ K\int_{t_0}^{t} e^{-c(t-s)/2} [(t-s)^{-\gamma_1} + (t-s)^{\gamma_2}] |g(u(s))|_{H_{1,p}^{2\pi}} ds$$

$0 < \gamma_1 < 1$, $\gamma_2 \ge 0$, $w(t_0) = (u(t_0),v(t_0))$. Using Hölder inequality $|g'(u) \frac{\partial u}{\partial x_i}|_{L_p^{2\pi}} \le |g'(u)|_{L_q^{2\pi}} |\frac{\partial u}{\partial x_i}|_{L_2^{2\pi}}$, $q = \frac{6}{a}$, and the obvious estimate

$$\left|\frac{f'(u)}{1+|u|^a} (1 + |u|^a)\right|_{L_q^{2\pi}} \le \left|\frac{f'(u)}{1 + |u|^a}\right|_{L_\infty} \left|1 + |u|^a\right|_{L_q^{2\pi}} \le$$

$$\le k_8(1 + |u|_{L_\infty}^b) |(1 + |u|^a)|_{L_q^{2\pi}}$$

we get

$$|u(t)|_{L_\infty} \le Ke^{-c(t-t_0)/2} |w(t_0)|_{X_1} +$$

$$K\int_{t_0}^{t} e^{-c(t-s)/2} [(t-s)^{-\gamma_1} + (t-s)^{\gamma_2}] [k_8 |\mathrm{grad}\,u(s)|_{L_2^{2\pi}} (1+|u(s)|^a)_{L_q^{2\pi}} (1+|u(s)|_{L_\infty}^b)$$

$$+ |f(u(s))|_{L_p^{2\pi}}] ds.$$

Now, if $|w(t_0)|_{X_1} = R$ then, according to Lemma 4 there is a $c_2(R)$ such that $|u(s)|_{H_1^{2\pi}} \le c_2(R)$, $s \ge t_0$, in particular, $|1 + |u(s)|^a|_{L_q^{2\pi}} \le c_3(R)$ (we have used the inclusion $L_6^{2\pi} \subset H_1^{2\pi}$), and so

$$|u(t)|_{L_\infty} \le Ke^{-c(t-t_0)/2}|w(t_0)|_{X_1} + \tilde{M}_2(R) + M_3(R)\int_{t_0}^{t} e^{-c(t-s)/2}[(t-s)^{-\gamma_1} + (t-s)^{\gamma_2}]|u(s)|^b_{L_\infty} ds.$$

Since $0 < b < 1$, for any $\varepsilon > 0$, there is a $K(\varepsilon)$ such that $|u|^b \le K(\varepsilon) + \varepsilon|u|$, and then

$$|u(t)|_{L_\infty} \le M_1 e^{-c(t-t_0)/2}|w(t_0)|_{X_1} + M_2(R,\varepsilon) + \varepsilon M_3(R)\int_{t_0}^{t} e^{-c(t-s)/2}[(t-s)^{-\gamma_1} + (t-s)^{\gamma_2}]|u(s)|_{L_\infty} ds.$$

If $\varepsilon(R)$ is chosen to satisfy $\varepsilon M_3(R)\int_0^\infty e^{-cs/4}(s^{-\gamma_1}+s^{\gamma_2})ds \le \frac{1}{2}$, then Lemma 5 gives $|u(t)|_{L_\infty} \le c_4(R)$, $t \ge t_0$, for some convenient $c_4(R)$. Defining $U(t,x) = \frac{\partial u}{\partial x_i}(t,x)$ $V(t,x) = \frac{\partial v}{\partial x_i}(t,x)$, $i=1,\dots,n$, we see $(U(t),V(t))$ is a mild solution (in the space $X_0 = H_1^{2\pi} \times L_2^{2\pi}$) of the system

$$U_t = V$$

$$V_t = \Delta U - cV + \frac{c^2}{4}U + f'(u(t))\frac{\partial u}{\partial x_i}(t) - \frac{c^2}{4}\frac{\partial u}{\partial x_i}(t) + \frac{\partial h}{\partial x_i}(t).$$

Since the semigroup generated by the linear part goes to zero exponentially (in the space X_0) and

$$|f'(u(t))\frac{\partial u}{\partial x_i}(t) - \frac{c^2}{4}\frac{\partial u}{\partial x_i}(t) + \frac{\partial}{\partial x_i}h(t)|_{L_2^{2\pi}}, \quad t \ge t_0,$$

is bounded above by some constant $c_5(R)$ (as a consequence of Lemma 4, the previous estimate for $|u(t)|_{L_\infty}$ and assumption (iv)), we conclude $|(U(t),V(t))|_{X_0}$, $t \ge t_0$, is bounded above by some constant $c_6(R)$; hence, $|(u(t),v(t))|_{X_1}$, $t \ge t_0$, is bounded above by some $c_7(R)$. Furthermore, using Lemma 4 again and previous remarks, we know there are $c_8(R_0)$ and $T(R)$ such that

$$|\text{grad } u(s)|_{H_1^{2\pi}} \le c_8(R_0), \quad |1 + |u(s)|^a|_{L_q^{2\pi}} \le c_8(R_0)$$

$$|g(u(s))|_{L_p^{2\pi}} \le c_8(R) \quad \text{for } s \ge t_1(R) = t_0 + T(R);$$

so for $t > t_1(R)$ we have

$$|u(t)|_{L_\infty} \le Ke^{-c(t-t_1(R))/2}|w(t_1(R))|_{X_1} + c_9(R_0)$$

and this implies $|u(t)|_{L_\infty} \le 2c_9(R_0)$ provided $t \ge t_0 + T_1(R)$, for a convenient $T_1(R)$. Using the functions $U(t)$ and $V(t)$ defined above and argueing exactly as before we can see that are $c_{10}(R_0)$ and $T_2(R)$ such that $|(u(t),v(t))|_{X_2} \le c_{10}(R_0)$ for $t \ge t_0 + T_2(R)$ and this proves the uniform ultimate boundedness. The final assertion follows imediately from the fact that the map $u \in H_2^{2\pi} \to g(u) \in H_1^{2\pi}$ is compact and differentiable, and this proves the theorem.

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Universidade Estadual de Campinas
Instituto de Matemática Estatística
e Ciência da Computação
13.081 - Campinas-SP - Brasil